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TRAVELING WAVE SOLUTIONS TO THE HYPERELASTIC ROD EQUATION†

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Abstract. We consider the hyperelastic rod equation describing nonlinear dispersive waves in compressible hyperelastic rods. We investigate the existence of certain traveling wave solutions to this equation. We also determine whether two other equations(the b-family equation and the modified Camassa-Holm equation) have our solution type.

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1. Introduction

This paper is concerned with the hyperelastic rod equation

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \quad t > 0, \quad x \in \mathbb{R}, \tag{1}$$

which was obtained as a model for nonlinear waves in cylindrical hyperelastic rods with $u(t,x)$ representing the radial stretch relative to a pre-stressed state, and the physical parameter γ ranging from -29.4760 to 3.1474 [6, 7, 8]. From the mathematical view point, we regard γ as a real number. Among (1), there are two other important equations.

When $\gamma = 1$, it recovers the standard Camassa-Holm(CH) equation [3]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

arises as a model for the unidirectional propagation of shallow water waves over a flat bottom [3], as well as water waves moving over an underlying shear flow[12]. The CH equation is completely integrable systems with a corresponding Lax pair formulation, a bi-Hamiltonian structure, and an infinite sequence of conservation laws [3]. The CH equation also admits peaked solitary waves or "peakons" [3]:

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$u(t,x) = ce^{-|x-ct|}$, $c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative. These peakons are shown to be stable [5]. Moreover, it admits the multi-peakon solutions (cf. [3]).

If $\gamma = 0$, then (1) becomes the regularized long wave equation, a well-known equation for surface wave in a channel [1]. All solutions are global and the solitary waves are smooth. Despite having a Hamiltonian structure, the equation is not integrable and its solitary waves are not solitons [9].

For general $\gamma \in \mathbb{R}$, mathematical properties of (1) have been studied further in many works. Dai and Huo [8] observed that (1) was formally shown to admit smooth, peaked, and cusped traveling waves. Subsequently, Lenells [13] used a suitable framework for weak solutions to classify all weak traveling waves of the hyperelastic rod equation (1). Constantin and Strauss [6] also investigated the stability of a class of solitary waves for the rod equation (1) on the line.

Note that if $p(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $u = (1-\partial_x^2)^{-1}m = p * m$, where $m := u - u_{xx}$ and $*$ denotes the convolution product on $\mathbb{R}$, given by

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)dy.$$

This formulation allows us to define a weak form of (1) as follows :

$$u_t + \partial_x\Big(\frac{\gamma}{2}u^2 + p * (\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2)\Big) = 0, \quad t > 0, \quad x \in \mathbb{R}. \tag{2}$$

In this paper, we are interested in a special traveling wave (i.e. function of $x - ct$ for some $c > 0$) and in fact that solution will also be a solitary wave solution (i.e. function of $x - ct$, decays to zero as $|x| \to \infty$). Our solution will be bounded in $x \in \mathbb{R}$ for each fixed $t > 0$. In fact, it will be bounded by $\frac{c}{\gamma}$. The solution will be valid for any fixed value of $c > 0$. This will be a weak solution, namely $u(t,x)$ will not be differentiable at the "peak" point. At any other points the solution will have all the derivatives.

We are motivated by the paper [4], where they exhibited a similar solution to the following equation

$$u_t - u_{txx} + (1+2\beta)uu_x = 2u_xu_{xx} + uu_{xxx}$$

where $0 \leq \beta < 1$, but their result is also applicable to

$$u_t - \alpha^2 u_{txx} + (1+2\beta)uu_x = \alpha^2(2u_xu_{xx} + uu_{xxx})$$

for any $\alpha^2 > 0$ and $\beta \in [0,1)$. By using transformation $x \longmapsto cx$, we have

$$u_t - \alpha^2c^2u_{txx} + (1+2\beta)cuu_x = \alpha^2c^3(2u_xu_{xx} + uu_{xxx}).$$

Choose c such that $c = \frac{1}{\alpha}$. Then we obtain

$$u_t - u_{txx} + \frac{(1+2\beta)}{\alpha}uu_x = \frac{1}{\alpha}(2u_xu_{xx} + uu_{xxx}), \qquad \alpha > 0.$$

Our case is then obtained by choosing $\alpha > 0$ such that $\frac{(1+2\beta)}{\alpha} = 3$ (i.e. $1 < \frac{1}{\alpha} \leq 3$). Our case is

$$u_t - u_{txx} + 3uu_x = \frac{3}{1+2\beta}(2u_x u_{xx} + uu_{xxx})$$

with $\beta \in [0, 1)$. In our case, we can use result in [4] to get a solution for our equation

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx})$$

when $\gamma \in (1, 3]$. Solution in [4] covers our equation when $\gamma \in (1, 3]$, but not when $\gamma > 3$. In our case, we also cover $\gamma > 3$. In fact, our solution will be valid when $\gamma > 1$.

This paper is organized as follows. In Section 2, we give the needed results to pursue our goal. In Section 3, we shall show the existence of traveling wave solutions for (2) under the condition $\gamma > 1$, by using an analogous analysis in [4]. Finally, in Section 4 we determine whether the b-family equation [11] and the modified Camassa-holm equation [10] have our solution type.

2. Preliminaries

In this section, we present the following basic technical Lemmas which play a key role to obtain our main results.

Lemma 2.1. *Suppose $\gamma > 1$ and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers defined*

$$\begin{cases} \alpha_1 = 1, \\ \alpha_n = \displaystyle\sum_{i+j=n;\ i,j\geq 1} \left(\frac{\gamma}{2} - \frac{\frac{3-\gamma}{2} + \frac{\gamma}{2}ij}{n^2 - 1} \right) \alpha_i \alpha_j, \qquad n \geq 2. \end{cases} \tag{3}$$

Define

$$\varphi(x) := \sum_{n=1}^{\infty} \alpha_n x^n. \tag{4}$$

Then, there exist $R > 0$ such that $\varphi(x)$ is well-defined on $|x| < R$, where R is the radius of the convergence for $\varphi(x)$.

Proof. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers defined

$$\begin{cases} \xi_1 = 1, \\ \xi_n = \dfrac{\gamma}{2} \displaystyle\sum_{i+j=n;\ i,j\geq 1} \xi_i \xi_j \qquad n \geq 2. \end{cases}$$

Set

$$\varrho(x) := \sum_{n=1}^{\infty} \xi_n x^n.$$

It is obviously that $\alpha_n \leq \xi_n$, for $n = 1, 2, ...$ Note that

$$\varrho(x) = x + \frac{\gamma}{2} \sum_{n=2}^{\infty} \sum_{i+j=n} \xi_i \xi_j x^n = x + \frac{\gamma}{2} \varrho^2(x). \tag{5}$$

Solving the equation (5), we have

$$\varrho(x) = \frac{1}{\gamma}(1 - \sqrt{1 - 2\gamma x}).$$

This implies $\varrho(x)$ is well-defined on $(-\frac{1}{2\gamma}, \frac{1}{2\gamma})$. Hence, $\varphi(x)$ is well-defined, at least, on $(-\frac{1}{2\gamma}, \frac{1}{2\gamma})$, which means there is $R > \frac{1}{2\gamma}$ such that $\varphi(x)$ is well-defined on $|x| < R$. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Consider the following initial value probem(IVP)*

$$(1 - \gamma\varphi)\frac{d^2\varphi}{dz^2} - \varphi = \frac{\gamma}{2}(\frac{d\varphi}{dz})^2 - \frac{3}{2}\varphi^2, \quad \varphi(0) = 0, \quad \varphi'(0) = 1. \tag{6}$$

The solution of (6) can be implicitly expressed as (11).

Proof. Notice that (6) is independent of z. By assuming that $\frac{d\varphi}{dz} := \Psi(\varphi)$, we reach

$$(1 - \gamma\varphi)\Psi(\varphi)\frac{d\Psi}{d\varphi} - \frac{\gamma}{2}\Psi^2(\varphi) = \varphi - \frac{3}{2}\varphi^2. \tag{7}$$

Solving the first-order ordinary differential equation (7), we get

$$\frac{d\varphi}{dz} = \Psi(\varphi) = \sqrt{\frac{1 - \varphi}{1 - \gamma\varphi}}\, \varphi. \tag{8}$$

By (8) and the transformation $z \to x$ with $x := e^z$ used in (15), we obtain

$$\sqrt{\frac{1 - \gamma\varphi}{1 - \varphi}}\, \frac{d\varphi}{\varphi} = dz = \frac{dx}{x}. \tag{9}$$

Using the transformation $\varphi \mapsto \eta$ with $\varphi = \frac{1-\eta^2}{1-\gamma\eta^2}$, we have

$$\frac{2(\gamma - 1)}{(\eta^2 - 1)(\gamma\eta^2 - 1)} d\eta = \frac{1}{x} dx. \tag{10}$$

Solving (10) to obtain an implicit analytic formula for function φ :

$$\frac{4(\sqrt{\gamma} - 1)^{\sqrt{\gamma}-1}}{(\sqrt{\gamma} + 1)^{\sqrt{\gamma}+1}} \left(\frac{\sqrt{\frac{\gamma(\varphi-1)}{\gamma\varphi-1}} + 1}{\sqrt{\frac{\gamma(\varphi-1)}{\gamma\varphi-1}} - 1} \right)^{\sqrt{\gamma}} \left(\frac{\sqrt{\frac{\varphi-1}{\gamma\varphi-1}} - 1}{\sqrt{\frac{\varphi-1}{\gamma\varphi-1}} + 1} \right) = x, \tag{11}$$

for $x > 0$. This completes the proof of Lemma 2.2. □

Under the same assumption of Lemma 2.1, we have the following important result to get (39).

Lemma 2.3. *Assume $\gamma > 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\varphi(x)$ be defined in (3) and (4). Then $\varphi(x)$ has a continuous extension to $x = R$ such that $\varphi(R) = \frac{1}{\gamma}$, or* $\lim_{x\to R^-} \varphi(x) = \frac{1}{\gamma}$.

Proof. By Lemma 2.1, we see tht $\varphi(x)$ is well-defined on $|x| < R$. In order to show that $\varphi(x)$ has a continuous extension such that $\varphi(R) = \frac{1}{\gamma}$, or $\lim_{x\to R^-} \varphi(x) = \frac{1}{\gamma}$, let us define

$$\theta(x) := \sum_{n=2}^{\infty} \left\{ \sum_{i+j=n} \left(\frac{\frac{3-\gamma}{2} + \frac{\gamma}{2} ij}{n^2-1} \right) \alpha_i \alpha_j \right\} x^n.$$

Since

$$\varphi(x) = x + \sum_{n=2}^{\infty} \sum_{i+j=n} \frac{\gamma}{2} \alpha_i \alpha_j x^n - \sum_{n=2}^{\infty} \left\{ \sum_{i+j=n} \left(\frac{\frac{3-\gamma}{2} + \frac{\gamma}{2} ij}{n^2-1} \right) \alpha_i \alpha_j \right\} x^n,$$

we obtain

$$\varphi(x) - x = \frac{\gamma}{2} \varphi^2(x) - \theta(x). \tag{12}$$

Notice that for $|x| < R$

$$\begin{aligned}
\theta(x) &= \frac{3-\gamma}{4} \sum_{n=2} \frac{1}{n-1} \sum_{i+j=n} \alpha_i \alpha_j x^n - \frac{3-\gamma}{4} \sum_{n=2} \frac{1}{n+1} \sum_{i+j=n} \alpha_i \alpha_j x^n \\
&\quad + \frac{\gamma}{4} \sum_{n=2} \frac{1}{n-1} \sum_{i+j=n} ij \alpha_i \alpha_j x^n - \frac{\gamma}{4} \sum_{n=2} \frac{1}{n+1} \sum_{i+j=n} ij \alpha_i \alpha_j x^n \\
&= \frac{(3-\gamma)x}{4} \int_0^x \frac{[\varphi(y)]^2}{y^2} dy - \frac{(3-\gamma)}{4x} \int_0^x [\varphi(y)]^2 dy \\
&\quad + \frac{\gamma x}{4} \int_0^x [\varphi'(y)]^2 dy - \frac{\gamma}{4x} \int_0^x y^2 [\varphi'(y)]^2 dy.
\end{aligned} \tag{13}$$

By (12) and (13), we have

$$\varphi'' + \frac{x\varphi' - \varphi}{x^2} = \frac{\gamma \varphi \varphi'}{x} - \frac{3}{2} \frac{\varphi^2}{x^2} + \frac{\gamma}{2} (\varphi')^2 + \gamma \varphi \varphi''. \tag{14}$$

Since our goal is to show $\lim_{x\to R^-} \varphi(x) = \frac{1}{\gamma}$, we may assume that $x > 0$. Using the transformation $z \mapsto x$ with $x := e^z$, we have

$$\frac{d\varphi}{dx} = \frac{1}{x} \frac{d\varphi}{dz}, \qquad \frac{d^2\varphi}{dx^2} = \frac{1}{x^2} \frac{d^2\varphi}{dz^2} - \frac{1}{x^2} \frac{d\varphi}{dz}. \tag{15}$$

Substituting (15) into (14), we obtain the ordinary differential equation

$$(1 - \gamma\varphi) \frac{d^2\varphi}{dz^2} - \varphi = \frac{\gamma}{2} (\frac{d\varphi}{dz})^2 - \frac{3}{2} \varphi^2, \tag{16}$$

which is independent of z. By Lemma 2.2, we obtain implicitly expression of solution for φ

$$\frac{4(\sqrt{\gamma}-1)^{\sqrt{\gamma}-1}}{(\sqrt{\gamma}+1)^{\sqrt{\gamma}+1}}\left(\frac{\sqrt{\frac{\gamma(\varphi-1)}{\gamma\varphi-1}}+1}{\sqrt{\frac{\gamma(\varphi-1)}{\gamma\varphi-1}}-1}\right)^{\sqrt{\gamma}}\left(\frac{\sqrt{\frac{\varphi-1}{\gamma\varphi-1}}-1}{\sqrt{\frac{\varphi-1}{\gamma\varphi-1}}+1}\right)=x, \tag{17}$$

for $x>0$. Let

$$H(y):=\frac{4(\sqrt{\gamma}-1)^{\sqrt{\gamma}-1}}{(\sqrt{\gamma}+1)^{\sqrt{\gamma}+1}}\left(\frac{\sqrt{\frac{\gamma(\varphi-1)}{\gamma\varphi-1}}+1}{\sqrt{\frac{\gamma(\varphi-1)}{\gamma\varphi-1}}-1}\right)^{\sqrt{\gamma}}\left(\frac{\sqrt{\frac{\varphi-1}{\gamma\varphi-1}}-1}{\sqrt{\frac{\varphi-1}{\gamma\varphi-1}}+1}\right). \tag{18}$$

By direct calculation we have

$$H'(y)=\sqrt{\frac{\gamma y-1}{y-1}}\frac{H(y)}{y}>0 \qquad \text{for} \quad 0<y<\frac{1}{\gamma}. \tag{19}$$

By the Implicit Function Theorem, we conclude that $\varphi(x)$ is well-defined in $|x|<R$, where

$$R:=\frac{4(\sqrt{\gamma}-1)^{\sqrt{\gamma}-1}}{(\sqrt{\gamma}+1)^{\sqrt{\gamma}+1}} \tag{20}$$

and

$$\lim_{x\to R^-}\varphi(x)=\frac{1}{\gamma}. \tag{21}$$

Since $\alpha_n\geq 0,\ n=1,2,...$, it is easy to check that

$$\sum_{n=1}^{\infty}\alpha_n R^n=\varphi(R)=\frac{1}{\gamma}.$$

This completes the proof of Lemma 2.3. □

3. Traveling wave solutions

In this section, we discuss the existence of traveling wave solutions to the hyperelastic rod equation (2) with $\gamma>1$.

Theorem 3.1. *Let $\gamma>1$. For every $c>0$, the functions of the form*

$$\begin{cases} u(t,x)=\displaystyle\sum_{n=1}^{\infty}\beta_n e^{-n|x-ct|}, \\ \beta_n:=cR^n\alpha_n \quad \textit{for} \quad n\geq 1, \end{cases} \tag{22}$$

where α_n and R are defined in (3) and (20), is a solution to (2) in the weak sense.

Proof. We need to show that $u(t,x)$ satisfies (2). Notice that

$$u_t(t,x) = c\,\mathrm{sign}(x-ct)\sum_{n=1}^{\infty} n\beta_n e^{-n|x-ct|}, \tag{23}$$

$$u_x(t,x) = -\mathrm{sign}(x-ct)\sum_{n=1}^{\infty} n\beta_n e^{-n|x-ct|}. \tag{24}$$

Using (23), (24), and $p(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$, we calculate from (2) that

$$\frac{\gamma}{2}u^2(t,x) + p * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(t,x) = K_1 + K_2, \tag{25}$$

where

$$K_1 := \sum_{n=2}^{\infty}\sum_{i+j=n}\frac{\gamma\beta_i\beta_j}{2}e^{-n|x-ct|} \quad \text{and} \quad K_2 := \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{-\infty}^{+\infty} e^{-|x-y|}e^{-n|y-ct|}dy,$$

with $B_n = \sum_{i+j=n}\left(\frac{3-\gamma}{2} + \frac{\gamma}{2}ij\right)\beta_i\beta_j$.

When $x > ct$, by using $\int_{-\infty}^{+\infty} = \int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{+\infty}$, we write K_2 defined in (25) as $K_2 = I_1 + I_2 + I_3$, where

$$I_1 := \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{-\infty}^{ct} e^{-|x-y|}e^{-n|y-ct|}dy$$

$$I_2 := \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{ct}^{x} e^{-|x-y|}e^{-n|y-ct|}dy$$

$$I_3 := \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{x}^{+\infty} e^{-|x-y|}e^{-n|y-ct|}dy. \tag{26}$$

We directly compute I_1 as follows:

$$\begin{aligned} I_1 &= \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{-\infty}^{ct} e^{-(x-y)}e^{n(y-ct)}dy = \frac{1}{2}\sum_{n=2}^{\infty} B_n e^{-(x+nct)} \int_{-\infty}^{ct} e^{(n+1)y}dy \\ &= \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n+1}B_n e^{(ct-x)}. \end{aligned} \tag{27}$$

In a similar manner,

$$I_2 = \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{ct}^{x} e^{-(x-y)}e^{-n(y-ct)}dy = \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n-1}B_n\left(e^{(ct-x)} - e^{n(ct-x)}\right) \tag{28}$$

and

$$I_3 = \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{x}^{+\infty} e^{(x-y)}e^{-n(y-ct)}dy = \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n+1}B_n e^{n(ct-x)}. \tag{29}$$

Plugging (27)-(29) into (26), we deduce that for $x > ct$

$$K_2 = p * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(t,x) = \sum_{n=2}^{\infty} \frac{n}{n^2-1} B_n e^{(ct-x)} - \sum_{n=2}^{\infty} \frac{1}{n^2-1} B_n e^{n(ct-x)}. \tag{30}$$

Using (25) and (30), we obtain for $x > ct$

$$\begin{aligned}
&\partial_x \left(\frac{\gamma}{2}u^2(t,x) + p * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(t,x)\right) \\
&= -\sum_{n=2}^{\infty} n \sum_{i+j=n} \frac{\gamma\beta_i\beta_j}{2} e^{n(ct-x)} - \sum_{n=2}^{\infty} \frac{n}{n^2-1} B_n \Big(e^{(ct-x)} - e^{n(ct-x)}\Big).
\end{aligned} \tag{31}$$

While for the case $x \leq ct$, we split second term(K_2) of (25) into the following three parts:

$$\begin{aligned}
K_2 &= p * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(t,x) \\
&= \frac{1}{2}\sum_{n=2}^{\infty} B_n \left\{\int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{+\infty}\right\} e^{-|x-y|} e^{-n|y-ct|} dy \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{32}$$

For J_1, a direct computation gives rise to

$$\begin{aligned}
J_1 &= \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{-\infty}^{x} e^{-(x-y)} e^{n(y-ct)} dy = \frac{1}{2}\sum_{n=2}^{\infty} B_n e^{-(x+nct)} \int_{-\infty}^{x} e^{(n+1)y} dy \\
&= \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n+1} B_n e^{n(x-ct)}.
\end{aligned} \tag{33}$$

Similarly, one obtains

$$J_2 = \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{x}^{ct} e^{(x-y)} e^{n(y-ct)} dy = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n-1} B_n \Big(e^{(x-ct)} - e^{n(x-ct)}\Big) \tag{34}$$

and

$$J_3 = \frac{1}{2}\sum_{n=2}^{\infty} B_n \int_{ct}^{+\infty} e^{(x-y)} e^{-n(y-ct)} dy = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n+1} B_n e^{(x-ct)}. \tag{35}$$

Using (25) and (33)-(35), we have for $x \leq ct$

$$\begin{aligned}
&\partial_x \left(\frac{\gamma}{2}u^2(t,x) + p * \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(t,x)\right) \\
&= \sum_{n=2}^{\infty} n \sum_{i+j=n} \frac{\gamma\beta_i\beta_j}{2} e^{n(x-ct)} + \sum_{n=2}^{\infty} \frac{n}{n^2-1} B_n \Big(e^{(x-ct)} - e^{n(x-ct)}\Big).
\end{aligned} \tag{36}$$

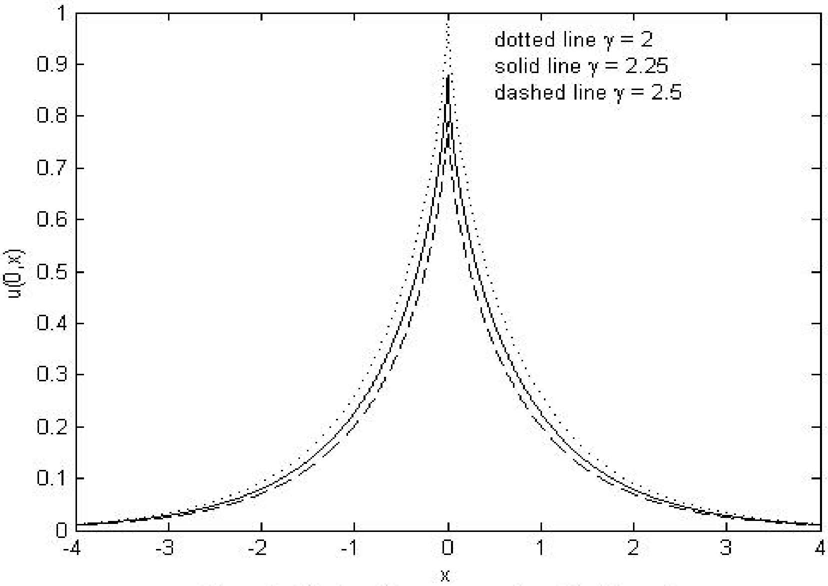

Figure1. The traveling wave profile u(0,x) for c=2.

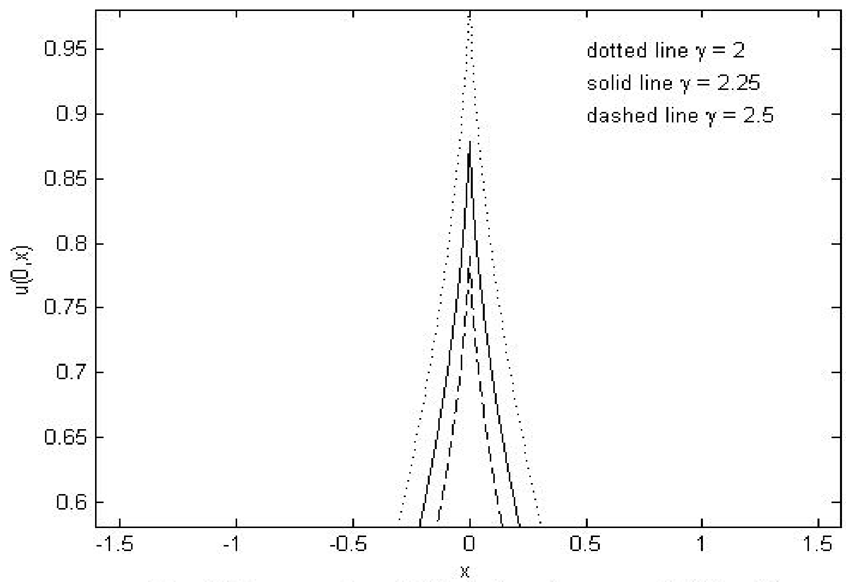

Figure2. The zoom in profile for the traveling wave u(0,x) for c=2.

On the other hand, using (23), we get

$$u_t(t,x)=\begin{cases} c\beta_1 e^{(ct-x)}+c\sum_{n=2}^{\infty} n\beta_n e^{n(ct-x)}, & x>ct,\\ -c\beta_1 e^{(x-ct)}-c\sum_{n=2}^{\infty} n\beta_n e^{n(x-ct)}, & x\le ct. \end{cases} \tag{37}$$

As a result, (2) is equivalent to the recursive sequence of equation given by

$$\begin{cases} c\beta_1=\sum_{n=2}^{\infty}\frac{n}{n^2-1}\sum_{i+j=n}\Big(\frac{3-\gamma}{2}+\frac{\gamma}{2}ij\Big)\beta_i\beta_j,\\ c\beta_n=\sum_{i+j=n}\left(\frac{\gamma}{2}-\frac{\frac{3-\gamma}{2}+\frac{\gamma}{2}ij}{n^2-1}\right)\beta_i\beta_j, \qquad n\ge 2. \end{cases} \tag{38}$$

With the help of (3) and (22), we conclude that (38) holds if

$$\sum_{n=2}^{\infty}\Big[\frac{n}{n^2-1}\sum_{i+j=n}\Big(\frac{3-\gamma}{2}+\frac{\gamma}{2}ij\Big)\alpha_i\alpha_j\Big]R^{n-1}=1. \tag{39}$$

Now, we apply Lemma 2.1 and 2.3 to show that (39) holds. Let us define

$$\Phi(x):=\sum_{n=2}^{\infty}\Big[\frac{n}{n^2-1}\sum_{i+j=n}\Big(\frac{3-\gamma}{2}+\frac{\gamma}{2}ij\Big)\alpha_i\alpha_j\Big]x^{n-1}.$$

From the definition of $\theta(x)$ in Lemma 2.3, we see that $\Phi(x)=\theta'(x)$ for $|x|<R$. By (12), we have $\theta'(x)=1+(\gamma\varphi(x)-1)\varphi'(x)$ for $|x|<R$. By Lemma 2.3, we have $\lim_{x\to R^-}\varphi(x)=\frac{1}{\gamma}$ and hence

$$\lim_{x\to R^-}(\gamma\varphi(x)-1)\varphi'(x)=0.$$

Therefore,

$$\lim_{x\to R^-}\Phi(x)=\lim_{x\to R^-}\theta'(x)=1.$$

Since $\alpha_n\ge 0$ for $n\ge 1$, one can easily to check that

$$\sum_{n=2}^{\infty}\Big[\frac{n}{n^2-1}\sum_{i+j=n}\Big(\frac{3-\gamma}{2}+\frac{\gamma}{2}ij\Big)\alpha_i\alpha_j\Big]R^{n-1}=\Phi(R)=1.$$

This completes the proof of Theorem 3.1. □

4. Remarks

In Sections 2-3 we have found explicit expressions for traveling wave solutions to (2) with $\gamma>1$ that travel in the positive x-direction with speed $c>0$. In the following, we discuss whether two other equations(the b-family equation and the modified Camassa-Holm equation) have our solution type. We expect our non-smooth traveling wave solutions satisfy to the above both equations. Unfortunately, their equations does not have our traveling wave solution type.

Consider the following two other equations which are the b-family equation[11]

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} - uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R} \tag{40}$$

and the modified Camassa-Holm equation[10]

$$m_t + ((u^2 - u_x^2)m)_x = 0, \quad m = u - u_{xx}, \quad t > 0, \quad x \in \mathbb{R}. \tag{41}$$

Moreover, (40) and (41) are equivalent to the following nonlocal form:

$$u_t + \partial_x \left(\frac{1}{2}u^2 + p * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 \right) \right) = 0, \quad t > 0, \quad x \in \mathbb{R} \tag{42}$$

and

$$u_t + \partial_x \left(\frac{1}{3}u^3 + p * \left(\frac{2}{3}u^3 + uu_x^2 \right) \right) - \frac{1}{3}u_x^3 + p * \left(\frac{1}{3}u_x^3 \right) = 0, \quad t > 0, \quad x \in \mathbb{R}. \tag{43}$$

We just apply our solution to the b-family equation. Let us substitute our solution $u(t,x) = \sum_{n=1}^{\infty} \beta_n e^{-n|x-ct|}$ in (22) to (42). In a similar arguments, substituting (23) and (24) into (42) gives us

$$\begin{cases} c\beta_1 = \displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-1} \sum_{i+j=n} \Big(\frac{b}{2} + \frac{3-b}{2} ij \Big) \beta_i \beta_j, \\ c\beta_n = \displaystyle\sum_{i+j=n} \left(\frac{1}{2} - \frac{\frac{b}{2} + (3-b)\frac{ij}{2}}{n^2-1} \right) \beta_i \beta_j, \qquad n \geq 2. \end{cases} \tag{44}$$

Remark 4.1. Since $c > 0$, (44) implies that $\beta_n = 0$ for $n \geq 2$ and $\beta_1 = c$ or 0. Therefore, we conclude that (42) have $u(t,x) = ce^{-|x-ct|}$ or $u(t,x) = 0$. In this case, our solution recovers the single peakon solution.

Next we also apply our solution to the modified Camass-Holm equation. In a similar manner, we can plug our solution $u(t,x) = \sum_{n=1}^{\infty} \beta_n e^{-n|x-ct|}$ into (43). Then we obtain

$$\begin{cases} c\beta_1 = \displaystyle\sum_{n=3}^{\infty} \sum_{i+j+k=n} \left(\frac{n(\frac{2}{3} + jk) + \frac{ijk}{3}}{n^2-1} \right) \beta_i \beta_j \beta_k, \\ c\beta_2 = 0, \\ c\beta_n = \displaystyle\sum_{i+j+k=n} \left(\frac{1}{3} - \frac{\frac{2}{3} + jk + n(\frac{ijk}{3})}{n^2-1} \right) \beta_i \beta_j \beta_k, \quad \text{for} \quad n \geq 3. \end{cases} \tag{45}$$

Remark 4.2. Since $c > 0$, $\beta_1 = \pm\sqrt{\frac{3c}{2}}$ or 0, $\beta_2 = 0$ and $\beta_n = 0$ for $n \geq 3$. Hence (45) have a solution $u(t,x) = \pm\sqrt{\frac{3c}{2}} e^{-|x-ct|}$ or $u(t,x) = 0$. In this case, our solution also recovers single peakon solution which was obtained in [10].

In the following remark, we explain our solutions fit into the classification previously studied traveling wave solutions of (1).

Remark 4.3. In [13], Lenells study our own equation

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx})$$

and categorized all weak traveling wave solutions. Our solution corresponds to his category (f) of Theorem 1 in [13]. He proves the existence of a cuspon solution with decay, which corresponds to our solution such that

$$u(t,x) = c\sum_{n=1}^{\infty} \alpha_n R^n e^{-n|x-ct|} = c\varphi(Re^{-|x-ct|}), \quad c > 0,$$

where α_n and R are defined in (3) and (20).

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