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VECTOR OPTIMIZATION INVOLVING GENERALIZED SEMILOCALLY PRE-INVEX FUNCTIONS

SUDHA GUPTA, VANI SHARMA AND MAMTA CHAUDHARY*

ABSTRACT. In this paper, a vector optimization problem over cones is considered, where the functions involved are η-semidifferentiable. Necessary and sufficient optimality conditions are obtained. A dual is formulated and duality results are proved using the concepts of cone ρ-semilocally preinvex, cone ρ-semilocally quasi-preinvex and cone ρ-semilocally pseudo-preinvex functions.

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1. Introduction

Ewing [1] introduced the concept of semilocally convex functions. It was further extended to semilocally quasiconvex, semilocally pseudoconvex functions by Kaul and Kaur [2]. Necessary and sufficient optimality conditions were derived by Kaul and Kaur [3, 4], and Suneja and Gupta [8].

Weir and Mond [13] considered preinvex functions for multiple objective optimization. Further Weir and Jeyakumar [12] introduced the class of cone-preinvex functions and obtained optimality conditions and duality theorems for a scalar and vector valued programs. Weir [11] introduced cone-semilocally convex functions and studied optimality and duality theorems for vector optimization problems over cones. Preda and Stancu-Minasian [5, 6, 7] studied optimality and duality results for a fractional programming problem where the functions involved were semilocally preinvex.

In the recent years Suneja et al. [9] introduced the concepts of ρ-semilocally preinvex and related functions and obtained optimality and duality for multiobjective non-linear programming problem, Suneja and Bhatia [10] defined

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*Corresponding author.

cone-semilocally preinvex and related functions. They obtained necessary and sufficient optimality conditions for a vector optimization problem over cones. In this paper, we have defined cone ρ-semilocally preinvex, cone ρ-semilocally quasipreinvex, cone ρ-semilocally pseudopreinvex functions and established necessary and sufficient optimality conditions for a vector optimization problem over cones.

2. Definitions and Preliminaries

Let $S \subseteq R^n$ and $\eta : S \times S \to R^n$ and $\theta : S \times S \to R^n$ be two vector valued functions.

Definition 2.1. The set $S \subseteq R^n$ is said to be η-locally star shaped set at $x^* \in S$ if for each $x \in S$ there exists a positive number $a_\eta(x, x^*) \leq 1$ such that $x^* + \lambda\eta(x, x^*) \in S$, for $0 \leq \lambda \leq a_\eta(x, x^*)$.

Definition 2.2 ([10])**.** Let $S \subseteq R^n$ be an η-locally star shaped set at $x^* \in S$ and $K \subseteq R^m$ be a closed convex cone with non-empty interior. A vector valued function $f : S \to R^m$ is said to be K-semilocally preinvex (K-Slpi) at x^* with respect to η if corresponding to x^* and each $x \in S$, there exist a positive number $d_\eta(x, x^*) \leq a_\eta(x, x^*) \leq 1$ such that

$$\lambda f(x) + (1-\lambda)f(x^*) - f(x^* + \lambda\eta(x, x^*)) \in K, \quad \text{for } 0 < \lambda < d_\eta(x, x^*).$$

We now introduce ρ semilocally preinvex functions over cones.

Definition 2.3. Let $S \subseteq R^n$ be an η-locally star shaped set at $x^* \in S$, $\rho \in R^m$ and $K \subseteq R^m$ be a closed convex cone with nonempty interior. A vector valued function $f : S \to R^m$ is said to be ρ-semilocally preinvex over $K(k\rho$-Slpi) at $x^* \in S$ with respect to η if corresponding to x^* and each $x \in S$, there exists a positive number $d_\eta(x, x^*) \leq a_\eta(x, x^*) \leq 1$ such that

$$\lambda f(x) + (1-\lambda)f(x^*) - f(x^* + \lambda\eta(x, x^*)) - \rho\lambda(1-\lambda)\|\theta(x, x^*)\|^2 \in K, \quad \text{for } 0 < \lambda < d_\eta(x, x^*).$$

Remark 2.1. If $\rho = 0$ the definition of $K\rho$-Slpi function reduces to that of K-slpi function given by Suneja and Meetu [10].

If $K = R^+$, the definition of $K\rho$-slpi function reduces to that of ρ-slpi function given by Suneja et al. [9]. In addition if $\eta(x, x^*) = x - x^*$ then $K\rho$-semilocally preinvex functions reduces to K-semilocally convex functions defined by Weir [11].

We now give an example of a function which is $K\rho$-slpi but fails to be ρ-slpi.

Example 2.1. We consider the following η-locally star shaped set as given by Suneja and Meetu [10]. Let $S = R \setminus E$, where

$$E = \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \{2\}$$

$$\eta(x,x^*) = \begin{cases} x - x^*,\ x, x^* > \dfrac{1}{2},\ x \neq 2,\ x^* \neq 2,\ \text{or } x, x^* < -\dfrac{1}{2} \\ x^* - x,\ x > \dfrac{1}{2},\ x \neq 2,\ x^* < -\dfrac{1}{2} \text{ or } x^* > \dfrac{1}{2},\ x^* \neq 2,\ x < -\dfrac{1}{2} \end{cases}$$

$$a_\eta(x,x^*) = \begin{cases} \left|\dfrac{2-x^*}{x-x^*}\right|, & \text{if } \dfrac{1}{2} < x^* < 2,\ 2 < x \text{ or } \dfrac{1}{2} < x^* < 2,\ x < -\dfrac{1}{2} \\ \dfrac{x^*-2}{x^*-x}, & \text{if } 2 < x^*, \dfrac{1}{2} < x < 2 \\ 1, & \text{otherwise.} \end{cases}$$

$$\theta(x,x^*) = x - x^*$$

Consider the function $f : S \to R^2$ defined by

$$f(x) = \begin{cases} (x,0), & x > \dfrac{1}{2} \\ (0,-x), & x < -\dfrac{1}{2}. \end{cases}$$

Let $\rho = (-1,-1)$ and $K = \{(x,y) : x \geq 0, y \leq x\}$.
Then f is $K\rho$-slpi at $x^* = -1$. But f is not ρ-slpi because for $x = 1$, $\lambda = \dfrac{1}{2}$,

$$\lambda f(x) + (1-\lambda)f(x^*) - f(x^* + \lambda\eta(x,x^*)) - \rho\lambda(1-\lambda)\|\theta(x,x^*)\|^2 = \left(\frac{3}{2}, -\frac{1}{2}\right) \ngeq (0,0).$$

Definition 2.4. The function $f : S \to R^m$ is said to be η-semidifferentiable at $x^* \in S$ if

$$(df)^+(x^*, \eta(x,x^*)) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}[f(x^* + \lambda\eta(x,x^*)) - f(x^*)]$$

exists for each $x \in S$.

Theorem 2.1. *If f is $K\rho$-Slpi at x^* then*

$$f(x) - f(x^*) - (df)^+(x^*, \eta(x,x^*)) - \rho\|\theta(x,x^*)\|^2 \in K, \quad \text{for all } x \in S.$$

Proof. Since the function f is $K\rho$-slpi at x^* with respect to η therefore corresponding to each $x \in S$ there exists a positive number

$$d_\eta(x,x^*) \leq a_\eta(x,x^*) \leq 1$$

such that

$$\lambda f(x) + (1-\lambda)f(x^*) - f(x^* + \lambda\eta(x,x^*)) - \rho\lambda(1-\lambda)\|\theta(x,x^*)\|^2 \in K, \quad \text{for } 0 < \lambda < d_\eta(x,x^*),$$

which implies

$$f(x) - f(x^*) - \frac{1}{\lambda}[f(x^* + \lambda\eta(x,x^*)) - f(x^*)] - \rho(1-\lambda)\|\theta(x,x^*)\|^2 \in K, \quad \text{for } 0 < \lambda < d_\eta(x,x^*).$$

Since K is a closed cone, therefore by taking limit as $\lambda \to 0^+$, we get

$$f(x) - f(x^*) - (df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 \in K, \quad \text{for all } x \in S. \quad \square$$

We now introduce $K\rho$-semilocally naturally quasi preinvex ($K\rho$-slnqpi) over cones.

Definition 2.5. The function f is said to be $K\rho$-semilocally naturally quasi preinvex ($K\rho$-Slnqpi) at x^* with respect to η if

$$-(f(x) - f(x^*)) \in K \Rightarrow -(df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 \in K\,.$$

Theorem 2.2. *If f is $K\rho$-slpi at $x^* \in S$ with respect to η then f is $K\rho$-slnqpi at x^* with respect to same η.*

Proof. Let f be $K\rho$-slpi at x^*, then there exists a positive number $d_\eta(x, x^*) \leq a_\eta(x, x^*)$ such that

$$\lambda f(x) + (1-\lambda) f(x^*) - f(x^* + \lambda\eta(x, x^*)) - \rho\lambda(1-\lambda)\|\theta(x, x^*)\|^2 \in K, \quad \text{for } 0 < \lambda < d_\eta(x, x^*). \tag{2.1}$$

Suppose that

$$-(f(x) - f(x^*)) \in K$$

then

$$-\lambda(f(x) - f(x^*)) \in K, \text{ for } \lambda > 0. \tag{2.2}$$

Adding (2.1) and (2.2) we get

$$-\left[f(x^* + \lambda\eta(x, x^*)) - f(x^*)\right] - \rho\lambda(1-\lambda)\|\theta(x, x^*)\|^2 \in K, \text{ for } 0 < \lambda < d_\eta(x, x^*).$$

$$\Rightarrow -\frac{1}{\lambda}\left[f(x^* + \lambda\eta(x, x^*)) - f(x^*)\right] - \rho(1-\lambda)\|\theta(x, x^*)\|^2 \in K, \text{ for } 0 < \lambda < d_\eta(x, x^*).$$

Since K is a closed cone, therefore taking limit as $\lambda \to 0^+$, we get

$$-(df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 \in K.$$

Thus

$$\begin{aligned} & -(f(x) - f(x^*)) \in K \\ \Rightarrow \quad & -(df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 \in K, \quad \text{for } x \in S. \end{aligned} \quad \square$$

But the converse is not true as shown in the following example.

Example 2.2. Consider set $S = R/E$, where $E = \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \{2\}$. Then as discussed in Example 2.1, S is η-locally star shaped.
Consider the function $f : S \to R^2$ defined by

$$f(x) = \begin{cases} (-x^2, 0), & x < -\frac{1}{2} \\ (0, -x), & x > \frac{1}{2}. \end{cases}$$

$$\theta(x, x^*) = x - x^* .$$

Then function f is $K\rho$-slnqpi at $x^* = -2$, for $\rho = (1, 0)$, where

$$k = \{(x, y) | y \le 0,\ y \ge x\},$$

because

$$\begin{aligned} & -(f(x) - f(x^*)) \in K \Rightarrow -2 \le x < -\frac{1}{2} \\ \Rightarrow \quad & -(df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 = (-4(x+2) - (x+2)^2, 0) \in K . \end{aligned}$$

But the function f fails to be $k\rho$-slpi at $x^* = -2$ by Theorem 2.1 because for $x = 1$,

$$f(x) - f(x^*) - (df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 = (7, -1) \notin K .$$

Definition 2.6. The function $f : S \to R^m$ is said to be $K\rho$-semilocally quasi preinvex ($K\rho$-slqpi) at x^* with respect to η if

$$f(x) - f(x^*) \notin \operatorname{int} K \Rightarrow -(df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 \in K, \quad \text{for } x \in S .$$

Remark 2.2. The following diagram illustrates the relation among $K\rho$-slpi function, $K\rho$-slnqpi and $K\rho$-slqpi functions.

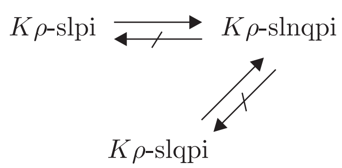

FIGURE 1

We now give an example of a function which is $K\rho$-slnqpi but fails to be $k\rho$-slqpi.

Example 2.3. The function f considered in Example 2.2 is $K\rho$-slnqpi at $x^* = -2$. But fails to be $K\rho$-slqpi at $x^* = -2$ because for $x = 1$

$$f(x) - f(x^*) = (4, -1) \notin \operatorname{int} K,$$

but

$$-(df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 = (3, 0) \notin K .$$

The next definition introduces cone semilocally pseudo preinvex functions over cone.

Definition 2.7. The function $f : S \to R^m$ is said to be $K\rho$-semilocally pseudo preinvex ($K\rho$-slppi) at x^*, with respect to η if

$$-(df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 \notin \operatorname{int} K \Rightarrow -(f(x) - f(x^*)) \notin \operatorname{int} K.$$

3. Optimality Conditions

Consider the following Vector Optimization Problem

$$\text{(VOP)} \quad K\text{-minimize } f(x)$$
$$\text{subject to } -g(x) \in Q$$

where $f : S \to R^m$ and $g : S \to R^p$ are η-semidifferentiable functions with respect to same η and $S \subseteq R^n$ is a nonempty η-locally star shaped set.

Let $K \subseteq R^m$ and $Q \subseteq R^p$ be closed convex cones having non-empty interior and let $X = \{x \in S : -g(x) \in Q\}$ be the set of all feasible solutions of (VOP).

Definition 3.1. A point $x^* \in X$ is called

(i) a weak minimum of (VOP), if for all $x \in X$, $f(x^*) - f(x) \notin \operatorname{int} K$.

(ii) a minimum of (VOP), if for all $x \in X$, $f(x^*) - f(x) \notin K \setminus \{0\}$.

(iii) a strong minimum of (VOP), if for all $x \in X$, $f(x) - f(x^*) \in K$.

We will use the following Alternative Theorem given by Weir and Jeyakumar [12].

Theorem 3.1. *Let X, Y be real normed linear spaces and K be a closed convex cone in Y with nonempty interior, let $S \subseteq X$. Suppose that $f : S \to Y$ be K-preinvex. Then exactly one of the following holds:*

(i) *there exists $x \in S$ such that $-f(x) \in \operatorname{int} K$,*

(ii) *there exists $0 \neq p \in K^*$ such that $(p^T f)(S) \subseteq R_+$,*

where int *denotes interior and K^* is the dual cone of K.*

We now establish the necessary optimality conditions for (VOP).

Theorem 3.2 (Fritz John Type Necessary Optimality Conditions)**.** *Let $x^* \in X$ be a weak minimum of* (VOP) *and suppose $(df)^+(x^*, \eta(x, x^*))$ and $(dg)^+(x^*, \eta(x, x^*))$ are K-preinvex and Q-preinvex functions of x respectively with respect to same $\eta(x, x^*)$ and $\eta(x^*, x^*) = 0$ then there exists $\tau^* \in K^*$, $\mu^* \in Q^*$ such that*

$$\tau^{*T}(df)^+(x^*, \eta(x, x^*)) + \mu^{*T}(dg)^+(x^*, \eta(x, x^*)) \geq 0, \quad \textit{for all } x \in S. \tag{3.1}$$

$$\mu^{*T} g(x^*) = 0. \tag{3.2}$$

Proof. We assert that the system

$$-F(x) \in \operatorname{int}(K \times Q) \tag{3.3}$$

has no solution $x \in S$, where

$$F(x) = ((df)^+(x^*, \eta(x, x^*)), (dg)^+(x^*, \eta(x, x^*)) + g(x^*)).$$

If possible, let there be a solution $x^0 \in S$ of (3.3). Then

$$-F(x^0) \in \operatorname{int}(K \times Q) \;\Rightarrow\; -(df)^+(x^*, \eta(x^0, x^*)) \in \operatorname{int} K$$

and

$$-(dg)^+(x^*, \eta(x^0, x^*)) - g(x^*) \in \operatorname{int} Q.$$

Since S is locally star shaped and $x^*, x^0 \in S$, therefore we can find $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$,

$$x^* + \lambda\eta(x^0, x^*) \in S.$$

By definition of $(df)^+(x^*, \eta(x, x^*))$ and $(dg)^+(x^*, \eta(x, x^*))$, it follows that

$$-[f(x^* + \lambda\eta(x^0, x^*)) - f(x^*)] \in \operatorname{int} K$$

and

$$- [g(x^* + \lambda\eta(x^0, x^*)) - g(x^*)] - g(x^*) \in \operatorname{int} Q.$$

$$\Rightarrow \quad f(x^*) - f(x^* + \lambda\eta(x^0, x^*)) \in \operatorname{int} K$$

and

$$-g(x^* + \lambda\eta(x^0, x^*)) \in \operatorname{int} Q, \quad \text{for } \lambda \in (0, \lambda_0),$$

which is a contradiction as x^* is a weak minimum of (VOP). Hence the system (3.3) has no solution $x \in S$.

Also F is $(K \times Q)$ preinvex on S as $(df)^+(x^*, \eta(x, x^*))$ and $(dg)^+(x^*, \eta(x, x^*))$ are K-preinvex and Q-preinvex on S respectively. Therefore, by Theorem 3.1, there exists $\tau^* \in K^*$ and $\mu^* \in Q^*$ not both zero such that

$$\tau^{*T}(df)^+(x^*, \eta(x, x^*)) + \mu^{*T}((dg)^+(x^*, \eta(x, x^*)) + g(x^*)) \geq 0, \quad \text{for all } x \in S. \tag{3.4}$$

Taking $x = x^*$, we get

$$\mu^{*T} g(x^*) \geq 0. \tag{3.5}$$

Also $\mu^* \in Q^*$ and $-g(x^*) \in Q$, implies that

$$\mu^{*T} g(x^*) \leq 0. \tag{3.6}$$

From (3.5) and (3.6), we get

$$\mu^{*T} g(x^*) = 0.$$

From (3.4), we get

$$\tau^{*T}(df)^+(x^*, \eta(x, x^*)) + \mu^{*T}(dg)^+(x^*, \eta(x, x^*)) \geq 0, \quad \text{for all } x \in S. \qquad \square$$

We use the following Slater type constraint qualification to prove the Kuhn-Tucker type necessary optimality conditions for (VOP).

Definition 3.2. The function g is said to satisfy Slater type constraint qualification at x^* if g is Q-preinvex at x^* and there exists $\hat{x} \in S$ such that $-g(\hat{x}) \in \operatorname{int} Q$.

Theorem 3.3 (Kuhn Tucker Type Necessary Optimality Conditions)**.** *Let $x^* \in X$ be a weak minimum of* (VOP) *and suppose $(df)^+(x^*, \eta(x, x^*))$ and $(dg)^+(x^*, \eta(x, x^*))$ are K-preinvex and Q-preinvex functions of x respectively with respect to the same $\eta(x, x^*)$. Suppose that g is Q-slpi at x^* and g satisfies Slater type constraint qualification at x^* and $\eta(x^*, x^*) = 0$, then there exists $0 \neq \tau^* \in K^*$, $\mu^* \in Q^*$ such that* (3.1) *and* (3.2) *hold.*

Proof. Since x^* is a weak minimum of (VOP), therefore by Theorem 3.2, there exist $\tau^* \in K^*$, $\mu^* \in Q^*$ such that (3.1) and (3.2) hold.
If possible, let $\tau^* = 0$, then from (3.1), we get

$$\mu^{*T}(dg)^+(x^*, \eta(x, x^*)) \geq 0, \quad \text{for all } x \in S. \tag{3.7}$$

Since g is Q-slpi at x^*, therefore we have

$$g(x) - g(x^*) - (dg)^+(x^*, \eta(x, x^*)) \in Q, \quad \text{for all } x \in S.$$

$$\Rightarrow \quad \mu^{*T}(g(x) - g(x^*) - (dg)^+(x^*, \eta(x, x^*))) \geq 0, \quad \text{for all } x \in S. \tag{3.8}$$

Adding (3.7) and (3.8) and using (3.2), we get

$$\mu^{*T} g(x) \geq 0, \quad \text{for all } x \in S. \tag{3.9}$$

Again by Slater type constraint qualification, there exists $\hat{x} \in S$ such that

$$-\,g(\hat{x}) \in \operatorname{int} Q \quad \Rightarrow \quad \mu^{*T} g(\hat{x}) < 0,$$

which is a contradiction to (3.9). Hence $\tau^* \neq 0$. □

Now we will establish some sufficient conditions for (VOP).

Theorem 3.4. *If $x^* \in X$, f is $K\rho$-slpi and g is $Q\sigma$-slpi at x^* and there exist $0 \neq \tau^* \in K^*$ and $\mu^* \in Q^*$ satisfying the conditions* (3.1) *and* (3.2)*, then x^* is a weak minimum of* (VOP) *provided*

$$\tau^{*T}\rho + \mu^{*T}\sigma \geq 0.$$

Proof. Suppose that x^* is not a weak minimum of (VOP), then there exists $x \in X$ such that

$$f(x^*) - f(x) \in \operatorname{int} K\,.$$

Since $0 \neq \tau^* \in K^*$, it follows that

$$\tau^{*T}(f(x^*) - f(x)) > 0\,. \tag{3.10}$$

Since f is $K\rho$-slpi and g is $Q\sigma$-slpi at x^*, therefore

$$f(x) - f(x^*) - (df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 \in K$$

and

$$g(x) - g(x^*) - (dg)^+(x^*, \eta(x, x^*)) - \sigma\|\theta(x, x^*)\|^2 \in Q.$$

$$\begin{aligned}
\Rightarrow \quad & \tau^{*T}(f(x) - f(x^*)) \\
& \geq \tau^{*T}(df)^+(x^*, \eta(x, x^*)) + \tau^{*T}\rho\|\theta(x, x^*)\|^2 \\
& \geq -\mu^{*T}(dg)^+(x^*, \eta(x, x^*)) + \tau^{*T}\rho\|\theta(x, x^*)\|^2 \\
& \geq -\mu^{*T}(dg)^+(x^*, \eta(x, x^*)) - \mu^{*T}\sigma\|\theta(x, x^*)\|^2 \\
& \geq -\mu^{*T}(g(x) - g(x^*)) \\
& = -\mu^{*T} g(x) \\
& \geq 0,
\end{aligned}$$

which contradicts (3.10). □

Theorem 3.5. *Let* $x \in X$. *If there exist* $0 \neq \tau^* \in K^*$, $\mu^* \in Q^*$ *satisfying the conditions* (3.1) *and* (3.2), g *is* $Q\sigma$*-slqpi at* x^* *and* f *is* $K\rho$*-slppi at* x^* *then* x^* *is a weak minimum of* (VOP) *provided*

$$\tau^{*T}\rho + \mu^{*T}\sigma \geq 0\,.$$

Proof. Let $x \in X$ and suppose $\mu^* \neq 0$. Then $-g(x) \in Q$ implies that

$$\mu^{*T} g(x) \leq 0.$$

From condition (3.2), it follows that

$$\mu^{*T}(g(x) - g(x^*)) \leq 0,$$

which gives that

$$g(x) - g(x^*) \notin \operatorname{int} Q.$$

Also g is $Q\sigma$-slqpi at x^*, therefore, we get

$$\begin{aligned} & -(dg)^+(x^*, \eta(x, x^*)) - \sigma\|\theta(x, x^*)\|^2 \in Q, \\ \Rightarrow \quad & \mu^{*T}(dg)^+(x^*, \eta(x, x^*)) + \mu^{*T}\sigma\|\theta(x, x^*)\|^2 \leq 0. \\ \Rightarrow \quad & \mu^{*T}\sigma\|\theta(x, x^*)\|^2 \leq -\mu^{*T}(dg)^+(x^*, \eta(x, x^*)). \end{aligned}$$

If $\mu^* = 0$, then the above inequality holds trivially.

On using (3.1), we have

$$\begin{aligned} & \tau^{*T}(df)^+(x^*, \eta(x, x^*)) \geq \mu^{*T}\sigma\|\theta(x, x^*)\|^2 \geq -\tau^{*T}\rho\|\theta(x, x^*)\|^2. \\ \Rightarrow \quad & -\tau^{*T}((df)^+(x^*, \eta(x, x^*)) + \rho\|\theta(x, x^*)\|^2) \leq 0. \\ \Rightarrow \quad & -(df)^+(x^*, \eta(x, x^*)) - \rho\|\theta(x, x^*)\|^2 \notin \operatorname{int} K. \end{aligned}$$

Since f is $K\rho$-slppi at x^*, we get

$$-(f(x) - f(x^*)) \notin \operatorname{int} K \quad \Rightarrow \quad f(x^*) - f(x) \notin \operatorname{int} K.$$

Thus x^* is a weak minimum of (VOP). □

4. Duality

We associate the following Mond-Weir type dual with (VOP),

(VOD) K-maximize $f(u)$

subject to

$$\tau^T(df)^+(u, \eta(x, u)) + \mu^T(dg)^+(u, \eta(x, u)) \geq 0, \text{ for all } x \in X, \quad (4.1)$$

$$\mu^T g(u) \geq 0,$$

$$u \in S,\ 0 \neq \tau \in K^*,\ \mu \in Q^*.$$

Theorem 4.1 (Weak Duality). *Let $x \in X$ and (u, τ, μ) be dual feasible, suppose f is $K\rho$-slppi and g is $Q\sigma$-slqpi at u then*

$$f(u) - f(x) \notin \operatorname{int} K,$$

provided $\tau\rho + \mu\sigma \geq 0$.

Proof. Since $x \in X$ and (u, τ, μ) is dual feasible, therefore, we get

$$\mu^T(g(x) - g(u)) \leq 0.$$

If $\mu \neq 0$, then the above inequality gives

$$g(x) - g(u) \notin \operatorname{int} Q.$$

Since g is $Q\sigma$-slqpi at u, we get

$$-(dg)^+(u, \eta(x, u)) - \sigma\|\theta(x, u)\|^2 \in Q.$$
$$\Rightarrow \quad \mu^T(dg)^+(u, \eta(x, u)) + \mu^T\sigma\|\theta(x, u)\|^2 \leq 0\,.$$

If $\mu = 0$, then the above inequality holds trivially. Now using (4.1), we get

$$\mu^T\sigma\|\theta(x, u)\|^2 \leq -\mu^T(dg)^+(u, \eta(x, u)) \leq \tau^T(df)^+(u, \eta(x, u))$$
$$\Rightarrow \tau^T(df)^+(u, \eta(x, u)) \geq \mu^T\sigma\|\theta(x, u)\|^2 \geq -\tau^T\rho\|\theta(x, u)\|^2.$$
$$\Rightarrow -\tau^T(df)^+(u, \eta(x, u)) + \rho\|\theta(x, u)\|^2 \leq 0.$$
$$\Rightarrow -(df)^+(u, \eta(x, u)) - \rho\|\theta(x, u)\|^2 \notin \operatorname{int} K.$$

Since f is $K\rho$-slppi at u, we get

$$-(f(x) - f(u)) \notin \operatorname{int} K \quad \Rightarrow \quad (f(u) - f(x)) \notin \operatorname{int} K.$$

Thus u is a weak minimum of (VOD). □

Theorem 4.2 (Strong Duality). *Let x^* be a weak minimum of* (VOP), *$(df)^+(u, \eta(x, u))$ be K-preinvex and $(dg)^+(u, \eta(x, u))$ be Q-preinvex functions on S. Suppose slater type constraint qualification holds at x^*. Then there exist $0 \neq \tau^* \in K^*$, $\mu^* \in Q^*$ such that (x^*, τ^*, μ^*) is feasible for* (VOD). *Moreover, if for each feasible (u, τ, μ) of* (VOD), *hypothesis of above theorem holds then (x^*, τ^*, μ^*) is a weak maximum of* (VOD).

Proof. Since all the conditions of Theorem 3.3 hold, therefore, there exist $0 \neq \tau^* \in K^*$, $\mu^* \in Q^*$ such that (3.1) and (3.2) hold. This implies that (x^*, τ^*, μ^*) is feasible for (VOD). If possible let (x^*, τ^*, μ^*) be not a weak maximum of (VOD), then there exists (u, τ, μ) feasible for (VOD) such that

$$f(u) - f(x^*) \in \operatorname{int} K.$$

But this is a contradiction to weak duality result as $x^* \in X$ and (u, τ, μ) is feasible for (VOD). Hence (x^*, τ^*, μ^*) must be a weak maximum of (VOD). □

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Sudha Gupta has completed her Ph.D in 1998 and is working in Laxmibhai College, university of Delhi. Her area of interest include vector optimization and generalized convexity.

Department of Mathematics, Laxmibai College (University of Delhi), Ashok Vihar,
Delhi 110052, India.
e-mail: vrindagupta88@gmail.com

Vani Sharma has completed her Ph.D in 2008 and is working in Satyawati College, University of Delhi. Her area of interest include vector optimization and generalized convexity.

Department of Mathematics, Satyawati College (Morning) (University of Delhi),
Ashok Vihar, Delhi 110052, India.
e-mail: vani5@rediffmail.com

Mamta Chaudhary is working in Satyawati College, university of Delhi, Delhi. Currently, she is perusing her Ph.D. in Mathematics. Her area of interest include vector optimization and generalized convexity.

Department of Mathematics, Satyawati College (Morning) (University of Delhi),
Ashok Vihar, Delhi 110052, India.
e-mail: mam.gupta18@gmail.com