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ON CONSTRUCTING A HIGHER-ORDER EXTENSION OF DOUBLE NEWTON'S METHOD USING A SIMPLE BIVARIATE POLYNOMIAL WEIGHT FUNCTION

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Abstract. In this paper, we have suggested an extended double Newton's method with sixth-order convergence by considering a control parameter γ and a weight function $H(s,u)$. We have determined forms of γ and $H(s,u)$ in order to induce the greatest order of convergence and established the main theorem utilizing related properties. The developed theory is ensured by numerical experiments with high-precision computation for a number of test functions.

1. Introduction

We can find non-linear equations governing phenomena of natural sciences at various fields: for example, shapes of leaves, snow crystals, scales of fishes, orbits of satellites, trajectories of ballistic missiles, location chase for a missing person by GPS and so on. High-precision computation facilitates obtaining accurate solutions of such non-linear problems. In order to process high-precision computation efficiently, it demands a development of an algorithm locating the desired numerical solution as precisely as possible. It is called double Newton's method (two-substep newton's method) [1] that is obtained by repeating classical Newton's method [1, 2, 3, 4, 10] twice. It is well known as a way to resolve numerical solutions of many non-linear equations $f(x) = 0$. Its convergence is well known when an initial value x_0 is chosen near the theoretical solution α of $f(x) = 0$.

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The corresponding algorithm for double Newton's method can be represented as follows:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}, n = 0, 1, 2, \cdots, \end{cases} \tag{1.1}$$

Double Newton's method converges to the desired solution with fourth-order convergence [1, 11], as a result of repeating the classical Newton's method, because convergence order of Newton's method in each step of (1.1) is two. A number of variants of two-step Newton's method can be found in [5, 6, 7, 8, 10].

Our main objective of this paper is to propose an extended double Newton's method by introducing a bivariate weight function H and a control parameter γ as follows:

$$\begin{cases} y_n = x_n - \gamma \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} = y_n - H(s,u) \frac{f(y_n)}{f'(y_n)}, n = 0, 1, 2, \cdots, \end{cases} \tag{1.2}$$

In Section 2 we will pursue the required methodology and analysis to determine forms of γ and $H(s,u)$ for maximal order of convergence. In addition, we will derive the theoretical asymptotic error constant, order of convergence and the error equation. Observe that (1.2) has four functional evaluations per iteration, being exactly the same as double Newton's method and its efficiency index [11] can be improved as large as possible. Our goal is to improve the convergence order which may be optimal in the sense of the conjecture of Kung-Traub [8]. The advantage of proposed method (1.2) is that we can choose various forms of $H(s,u)$ containing selectable parameters. Section 3 will discuss numerical experiments with high-precision Mathematica programming [12] for various test functions in order to verify the developed theory.

2. Methodology and convergence analysis

This section deals with methodology and analysis for developing a new family of sixth-order numerical methods via extension of double Newton's method. To simplify the form of the bivariate weight function $H(s,u)$ in (1.2), we employ a second-order bivariate polynomial of the form:

$$H(s,u) = a_0 + (a_1 + a_2 s)u + (a_3 + a_4 s)u^2 \tag{2.1}$$

where $s = \frac{f'(y_n)}{f'(x_n)}$ and $u = \frac{f(y_n)}{f(x_n)}$; $a_0,\ a_1,\ a_2,\ a_3,\ a_4$ are parameters to be determined for maximal order of convergence. Let $f : \mathbb{R} \to \mathbb{R}$ have a simple root $\alpha \in \mathbb{R}$ and be sufficiently smooth in a small neighborhood of α. Expanding Taylor series [4] for $f(x_n)$ and $f'(x_n)$ about α, we obtain:

$$f(x_n) = f'(\alpha)(e_n + \sum_{j=2}^{6} c_j e_n^j + O(e_n^7)), \tag{2.2}$$

$$f'(x_n) = f'(\alpha)(1 + \sum_{j=2}^{7} j c_j e_n^{j-1} + O(e_n^7)), \tag{2.3}$$

where $e_n = x_n - \alpha (n = 1, 2, \cdots)$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)} (k = 1, 2, \cdots)$. Using (2.2) and (2.3) with the aid of symbolic computation of Mathematica, we find $h = \frac{f(x_n)}{f'(x_n)}$ and $y_n = x_n - \gamma h$ in (1.2) as follows:

$$\begin{aligned} h = \frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 \\ &+(8c_2^4 - 20c_2^2 c3 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 + (-16c_2^5 - 52c_2^3 c_3 \\ &-33c_2c_3^2 - 28c_2^2c_4 + 17c_3c_4 + 13c_2c_5 - 5c_6)e_n^6 + O(e_n^7). \end{aligned} \tag{2.4}$$

$$\begin{aligned} y_n = x - \gamma h &= \alpha + (1-\gamma)e_n + c_2\gamma e_n^2 + 2(-c_2^2 + c_3)\gamma e_n^3 + (4c_2^3 - 7c_2c_3 \\ &+3c_4)\gamma e_n^4 - 2((4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)\gamma)e_n^5 + (16c_2^5 \\ &-52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)\gamma e_n^6 + O(e_n^7). \end{aligned} \tag{2.5}$$

Thus we are able to find $f(y_n), f'(y_n)$ and describe s, u, $\frac{f(y_n)}{f'(y_n)}$ as well as $H(s, u)$ shown below:

$$s = 1 - 2(c_2\gamma)e_n + 3(2c_2^2 + c_3(\gamma - 2))\gamma e_n^2 + \sum_{i=3}^{6} \xi_i e_n^i + O(e_n^7), \tag{2.6}$$

$$u = (1-\gamma) + c_2\gamma^2 e_n - (3c_2^2 + c_3(\gamma - 3))\gamma^2 e_n^2 + \sum_{i=3}^{5} \zeta_i e_n^i + O(e_n^6), \tag{2.7}$$

$$\frac{f(y_n)}{f'(y_n)} = (1-\gamma)e_n - c_2(1 - 3\gamma + \gamma^2)e_n^2 + \sum_{i=3}^{6} \Omega_i e_n^i + O(e_n^7), \tag{2.8}$$

$$\begin{aligned} H(s,u) &= (a_0 + (a_1 + a_2)(1-\gamma) + (a_3 + a_4)(1-\gamma)^2) \\ &+(-2a_2c_2(1-\gamma)\gamma - 2a_4c_2(1-\gamma)^2\gamma + (a_1 + a_2)c_2\gamma^2 \\ &+2(a_3 + a_4)c_2(1-\gamma)\gamma^2)e_n + \sum_{i=2}^{5} \Psi_i e_n^i + O(e_n)^6, \end{aligned} \tag{2.9}$$

where $\xi_i, \zeta_i, \Omega_i, \Psi_i$ are multi-variate polynomials dependent upon parameters $\gamma, a_0, a_1, a_2, a_3, a_4, a_5$ and constants $c_2, c_3, c_4, c_5, c_6,\ c_7$. Using (2.2)–(2.9), we are able to express

$$x_{n+1} = x_n - \gamma \frac{f(x_n)}{f'(x_n)} - H(s,u)\frac{f(y_n)}{f'(y_n)},$$

the last equation in (1.3) in terms of a univariate polynomial of e_n described below:

$$\begin{aligned} x_{n+1} &= \alpha + (1 - (a_0 - (a_1 + a_2)(-1+\gamma) \\ &+ (a_3 + a_4)(-1+\gamma)^2)(1-\gamma) - \gamma)e_n + \sum_{i=2}^{6} \psi_i e_n^i + O(e_n)^7, \end{aligned} \tag{2.10}$$

where, coefficients ψ_i are dependent upon $\gamma, a_0, a_1, a_2, a_3, a_4, a_5$ and constants $c_2, c_3, c_4, c_5, c_6,\ c_7$ for $2 \le i \le 6$. Equating the coefficient of e_n in (2.10) to zero and solving for a_0 yields

$$a_0 = 1 - a_3 - a_4 + a_1(\gamma - 1) + a_2(\gamma - 1) - (a_3 + a_4)(\gamma - 2)\gamma. \tag{2.11}$$

Substituting a_0 into $\psi_2 = 0$, and solving for a_1 gives

$$a_1 = a_2(-3 + \frac{2}{\gamma}) + (\gamma - 1)\frac{(-1 - 2a_4\gamma + 2(a_3 + 2a_4)\gamma^2)}{\gamma^2}. \tag{2.12}$$

Substituting a_0 and a_1 into $\psi_3 = 0$, and simplifying leads us to the following equation:

$$\begin{aligned} \psi_3 = (-1+\gamma)(c_3 + c_3\gamma^2(-1 + a_2 + a_4(-1+\gamma)^2 - a_2\gamma) + c_2\gamma) \\ + c_2^2(-2 + \gamma^2 - 2(a_2 + 2a_4)\gamma^3 + (a_3 + 5a_4)\gamma^4)) = 0. \end{aligned} \tag{2.13}$$

The resulting equation $\psi_3 = 0$ immediately yields $\gamma = 1$ or

$$\begin{aligned} (c_3 + c_3\gamma^2(-1 + a_2 + a_4(-1+\gamma)^2 - a_2\gamma) + c_2\gamma) \\ + c_2^2(-2 + \gamma^2 - 2(a_2 + 2a_4)\gamma^3 + (a_3 + 5a_4)\gamma^4)) = 0 \end{aligned}$$

In the latter case, i.e., $\gamma \neq 1$ gives $\psi_4 \neq 0$, which states method (1.2) is of only fourth-order convergence. Hence we consider only the former case when $\gamma = 1$. Substituting $\gamma = 1, a_0$ and a_1 into $\psi_4 = 0$, and solving for a_3 yields

$$a_3 = 1 + 2a_2 - a_4. \tag{2.14}$$

Substituting $\gamma = 1, a_0, a = 1$ and a_3 into $\psi_5 = 0$, and solving for a_2 and a_4 regardlessly of c_2 and c_3, we get

$$a_2 = a_4 = -2, \tag{2.15}$$

which further simplifies $a_0 = 1, a_1 = 2$ and $a_3 = -1$.

Substituting finally $\gamma = 1, a_0 = 1, a_1 = 2, a_2 = a_4 = -2,\ a_3 = -1$ into ψ_6, and simplifying yields $\psi_6 = c_2^2(14c_2^3 - 9c_2c_3 + c_4)$. In addition, the bivariate weight function $H(s,u)$ in (2.1) finally becomes:

$$H(s,u) = 1 + 2(1-s)u - (1+2s)u^2. \tag{2.16}$$

As a consequence of the above analysis having been done thus far, we establish the following theorem.

THEOREM 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ have a simple root $\alpha \in \mathbb{N}$ and be sufficiently smooth in a small neighborhood of α. Let $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k \in \mathbb{N}$, $s = \frac{f'(y)}{f'(x)}$, $u = \frac{f(y)}{f(x)}$ and $y = x - \frac{f(x)}{f'(x)}$. Then selected parameters $a_0 = 1, a_1 = 2, a_2 = -2, a_3 = -1, a_4 = -2, \gamma = 1$ give the desired $H(s,u) = 1 + 2(1-s)u - (1+2s)u^2$ and enable us to guarantee that our proposed method (1.2) converges with sixth-order convergence and the asymptotic error constant $\eta = c_2^2(14c_2^3 - 9c_2c_3 + c_4)$, provided that an initial value is chosen in a sufficiently small neighborhood of α.*

3. Numerical results and conclusion

Throughout the numerical experiments, the minimum number of precision digits was chosen as 300 (by means of Mathematica command $\$MinPrecision = 300$), being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constants requiring small-number divisions. The zero α, however, was separately computed with 350 digits of precision to have 300 significant digits, whenever its exact value is not known. The error bound $\epsilon = 10^{-250}$ was used for moderately accurate computation. The values of initial guess x_0 were selected closely to α to guarantee the convergence.

To verify the validity of Theorem 2.1, numerical experiments with high-precision Mathematica programming have been implemented for selected three test functions denoted by f_1, f_2, f_3 below:

$$\begin{cases} f_1(x) = x\log(x+1) + \sin x,\ x_0 = 0.01,\ \alpha = 0, \\ f_2(x) = e^{x^2} + \cos\frac{\pi}{2x} - 2,\ x_0 = 0.9,\ \alpha \approx 0.887425501228536, \\ f_3(x) = \sin x^2 - x^2 + 3, x_0 = 1.9,\ \alpha \approx 1.96311538301723. \end{cases}$$

Listed in Table 1 are numerical results, where abbreviations MT, **DN**, **IB** stand for methods, double Newton's method and improved bivariate weight function method (1.2), respectively. Table 1 lists $f(x)$, methods, iteration indexes n, approximate zeros x_n, residual errors $|f(x_n)|$, errors $|e_n| = |x_n - \alpha|$ and computational asymptotic error constants $|\frac{e_n}{{e_{n-1}}^p}|$

f	MT	n	x_n	$\lvert f(x_n)\rvert$	e_n	e_{n+1}/e_n^p	η
f_1	**DN**	1	9.35×10^{-9}	9.35×10^{-9}	9.35×10^{-9}	0.9359591609	1.000000000
		2	7.67×10^{-33}	7.67×10^{-33}	7.67×10^{-33}	0.9999999376	
		3	3.46×10^{-129}	3.46×10^{-129}	3.46×10^{-129}	1.000000000	
		4	1.44×10^{-514}	1.44×10^{-514}	1.44×10^{-514}		
	IB	1	1.74×10^{-11}	1.74×10^{-11}	1.74×10^{-11}	17.48541827	20.33333333
		2	5.81×10^{-64}	5.81×10^{-64}	5.81×10^{-64}	20.33333333	
		3	-2.05×10^{-376}	2.05×10^{-376}	2.05×10^{-376}		
f_2	**DN**	1	0.887425501228536	4.36×10^{-8}	7.45×10^{-9}	0.2983261491	0.2842418676
		2	0.887425493769975	5.15×10^{-33}	8.79×10^{-34}	0.2842418756	
		3	0.887425493769975	9.96×10^{-133}	1.70×10^{-133}	0.2842418674	
		4	0.887425493769975	$0. \times 10^{-299}$	$0. \times 10^{-300}$		
	IB	1	0.887425493768326	9.65×10^{-12}	1.64×10^{-12}	0.4170754337	0.3871369876
		2	0.887425493769975	4.55×10^{-71}	7.77×10^{-72}	0.3871369876	
		3	0.887425493769975	$0. \times 10^{-299}$	$0. \times 10^{-300}$		
f_3	**DN**	1	1.96311631638413	4.32×10^{-6}	9.33×10^{-7}	0.05881816994	0.05007966223
		2	1.96311538301723	1.76×10^{-25}	3.80×10^{-26}	0.05007954173	
		3	1.96311538301723	4.84×10^{-103}	1.04×10^{-103}	0.05007966223	
		4	1.96311538301723	$0. \times 10^{-299}$	$0. \times 10^{-299}$		
	IB	1	1.96311539511041	5.60×10^{-8}	1.20×10^{-8}	0.1913062750	0.1341666381
		2	1.96311538301723	1.94×10^{-48}	4.19×10^{-49}	0.1341666291	
		3	1.96311538301723	3.39×10^{-291}	7.32×10^{-292}		

TABLE 1. Convergence behavior of **DN** and **IB** for f_1, f_2, f_3

as well as the theoretical asymptotic error constant η. The computed asymptotic error constant agrees up to 10 significant digits with the theoretical one. The computed zero is accurate up to 300 significant digits, although the first 15 digits are displayed.

As can be seen in Table 1, our proposed numerical scheme (1.2) has been verified to be accurate enough by observing the numerical results that converge to theoretical asymptotic error constants. According to the fact that the exponent of the n^{th} iterate error e_n increases by a factor of 6 as n increases, our theory well confirms the theoretical order of convergence. In addition, we find that approximate numerical solutions have been clearly approaching the theoretical solutions, as iteration index n increases. The efficiency index [9, 11] defined by $EI = p^{1/d}$, with p as the order of convergence and d the number of new functional evaluations are given by $4^{1/4} \approx 1.41421$ for **DN** and $6^{1/4} \approx 1.56508$ for **IB**, respectively. Therefore, **IB** is found to be more efficient than **DN**.

It is our belief that the bivariate weight function approach shown in this paper would play an essential role to develop high-order numerical methods in the near future.

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