

COMMON FIXED POINT THEOREMS FOR TWO MAPPINGS WITH ψ - ϕ -CONTRACTIVE OR EXPANSIVE TYPE CONDITIONS ON COMPLEX-VALUED METRIC SPACES

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ABSTRACT. A continuous and non-decreasing function ψ and another continuous function ϕ with $\phi(z) = 0 \iff z = 0$ defined on $\mathbb{C}^+ = \{x + yi : x, y \geq 0\}$ are introduced, the ψ - ϕ -contractive or expansive type conditions are considered, and the existence theorems of common fixed points for two mappings defined on a complex valued metric space are obtained. Also, Banach contraction principle and a fixed point theorem for a I-expansive type mapping are given on complex valued metric spaces.

1. Introduction and preliminaries

Real metric spaces have been widely generalized and improved. For example, cone metric spaces([4]), topological vector space-valued cone metric spaces([5]) and cone metric type space([3]) and so on. A number of authors have discussed and obtained some fixed point and common fixed point theorems in these spaces, see ([3, 4, 5, 6, 7, 10, 11, 16]). These obtained results greatly generalized and improved some corresponding conclusions.

In 2011, Azam([1]) introduced a partial order \preceq on the set \mathbb{C} of complex numbers, used the idea in ([3, 4, 5]) to define a complex metric d on a nonempty set X and a complex-valued metric space (X, d) , and gave coincidence point theorems and common fixed point theorems for

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two mappings satisfying a contractive type condition in this space. And the authors in ([9, 12, 13, 14, 15]) generalized and improved the corresponding conclusions in ([1]). Recently, the author in ([17]) use two real continuous functions to obtain the unique common fixed point theorems for two mappings defined on complex valued metric spaces and the author in ([8]) obtained the Cauchy principle and discussed the existence problems of coincidence points and common fixed points for two mappings with expansive conditions. These results also generalize and improve the fixed point theory.

Here, we introduce the concept of non-decreasing functions defined on the subset \mathbb{C}^+ of \mathbb{C} (the set of complex numbers) and discuss the existence problems of unique common fixed points for ψ - ϕ -contractive or expansive mappings on complex valued metric spaces.

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion.

Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$$z_1 \preceq z_2 \Leftrightarrow [\operatorname{Re}(z_1) \leq \operatorname{Re}(z_2)] \wedge [\operatorname{Im}(z_1) \leq \operatorname{Im}(z_2)].$$

Consequently, $z_1 \preceq z_2$ if and only if one of the following conditions is satisfied:

- (C1) $\operatorname{Re}(z_1) = \operatorname{Re}z_2, \operatorname{Im}z_1 = \operatorname{Im}z_2$;
- (C2) $\operatorname{Re}(z_1) < \operatorname{Re}z_2, \operatorname{Im}z_1 = \operatorname{Im}z_2$;
- (C3) $\operatorname{Re}(z_1) = \operatorname{Re}z_2, \operatorname{Im}z_1 < \operatorname{Im}z_2$;
- (C4) $\operatorname{Re}(z_1) < \operatorname{Re}z_2, \operatorname{Im}z_1 < \operatorname{Im}z_2$.

In particular, we write $z_1 \prec z_2$ if only (C4) is satisfied.

Obviously, the following statements hold:

- (i) If $b \geq a \geq 0$, then $az \preceq bz$ for any $z \in \mathbb{C}$ with $0 \preceq z$;
- (ii) if $0 \preceq z_1 \prec z_2$, then $|z_1| < |z_2|$;
- (iii) if $z_1 \preceq z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$;
- (iv) if $z_1 \preceq z_2$ and $z \in \mathbb{C}$, then $z + z_1 \preceq z + z_2$.

DEFINITION 1.1. ([1, 9, 12, 13]) Let X be a nonempty set. If a mapping $d : X \times X \rightarrow \mathbb{C}$ satisfies the following conditions:

- (i) $0 \preceq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X , and (X, d) is called a complex valued metric space.

EXAMPLE 1.2. ([12]) Let $X = \mathbb{C}$. Define a mapping $d : X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|, \quad \forall z_1, z_2 \in X,$$

where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.

EXAMPLE 1.3. Let $X = \{a, b, c\}$. Define a mapping $d : X \times X \rightarrow \mathbb{C}$ by

$$d(a, a) = d(b, b) = d(c, c) = 0,$$

$$d(a, b) = d(b, a) = 3+4i, \quad d(a, c) = d(c, a) = 2+3i, \quad d(b, c) = d(c, b) = 4+5i.$$

Obviously, (X, d) is also a complex valued metric space.

DEFINITION 1.4. ([1, 9, 12, 13]) Let (X, d) be a complex valued metric space, $\{x_n\}_{n \geq 1}$ a sequence in X and $x \in X$.

- (i) If for any $c \in \mathbb{C}$ with $0 < c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < c$ for all $n > n_0$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to $x \in X$ and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.
- (ii) If for any $c \in \mathbb{C}$ with $0 < c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and any $m \in \mathbb{N}$, $d(x_n, x_{n+m}) < c$, then $\{x_n\}$ is said to be a Cauchy sequence.
- (iii) If every Cauchy sequence in X is convergent, then X is said to be complete.

DEFINITION 1.5. ([2]) Let X be a nonempty set, $f, g : X \rightarrow X$ two mappings. f and g are called weakly compatible if $x \in X$ and $fx = gx$, then $fgx = gfx$.

DEFINITION 1.6. ([2]) Let $f, g : X \rightarrow X$ be two mappings. If there exist $w, x \in X$ such that $w = fx = gx$, then x is called a coincidence point of f and g , w is called a point of coincidence of f and g .

The following result is the famous Banach contraction principle:

THEOREM 1.7. Let (X, d) be a complete real metric space and $f : X \rightarrow X$ a self-mapping. If for each $x, y \in X$,

$$d(fx, fy) \leq h d(x, y),$$

where $h \in [0, 1)$, then f has a unique fixed point in X .

The next result is the fixed point theorem for a map with a I-expansive type condition([18]):

THEOREM 1.8. *Let (X, d) be a complete real metric space and $f : X \rightarrow X$ an onto mapping. If for each $x, y \in X$,*

$$d(fx, fy) \geq h d(x, y),$$

where $h > 1$, then f has a unique fixed point in X .

LEMMA 1.9. ([8, 17]) *If a sequence $\{x_n\}$ in a complex valued metric space (X, d) is convergent, then its limit point is unique.*

LEMMA 1.10. ([8, 17]) *If (X, d) is a complex valued metric space, $\{x_n\}$ converges to $x \in X$, $\{y_n\}$ converges to $y \in X$. Then*

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y); \quad \lim_{n \rightarrow \infty} |d(x_n, y_n)| = |d(x, y)|.$$

Specially, for any fixed element $z \in X$, the following holds

$$\lim_{n \rightarrow \infty} d(x_n, z) = d(x, z); \quad \lim_{n \rightarrow \infty} |d(x_n, z)| = |d(x, z)|.$$

LEMMA 1.11. ([8](Cauchy Principle)) *Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d) . If there exists $0 \leq h < 1$ such that for all $n \in \mathbb{N}$,*

$$d(x_{n+1}, x_n) \preceq h d(x_n, x_{n-1}).$$

Then $\{x_n\}$ is a Cauchy sequence.

LEMMA 1.12. ([2]) *Let $f, g : X \rightarrow X$ be weakly compatible. If f and g have a unique point of coincidence, that is, there exist an element $x \in X$ and a unique element $w \in X$ satisfying $w = fx = gx$, then w is the unique common fixed point of f and g .*

2. Points of coincidence and common fixed points

Let $\mathbb{C}^+ = \{x + yi : x, y \geq 0\}$.

DEFINITION 2.1. A function $\psi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is said to be non-decreasing if $u \preceq v$ if and only if $\psi(u) \preceq \psi(v)$ for each $u, v \in \mathbb{C}^+$.

EXAMPLE 2.2. Let $\psi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ by $\psi(z) = kz + z_0, \forall z \in \mathbb{C}^+$, where $k > 0$ and $z_0 \in \mathbb{C}^+$ are fixed elements. Then obviously ψ is a non-decreasing function.

EXAMPLE 2.3. Let $\psi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ by

$$\psi(z) = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 i, \quad \forall z \in \mathbb{C}^+.$$

Then ψ is a non-decreasing function. In fact, let $u = \alpha_1 + \beta_2 i$ and $v = \alpha_2 + \beta_2 i$ be two elements in \mathbb{C}^+ , then

$$u \preceq v \iff 0 \leq \alpha_1 \leq \alpha_2, 0 \leq \beta_1 \leq \beta_2 \iff (\alpha_1)^2 + (\beta_1)^2 i \preceq (\alpha_2)^2 + (\beta_2)^2 i$$

that is, $u \preceq v \iff \psi(u) \preceq \psi(v)$. Hence ψ is a non-decreasing function.

THEOREM 2.4. *Let (X, d) be a complex valued metric space and $f, g : X \rightarrow X$ two mappings with $gX \subset fX$. Suppose that for each $x, y \in X$,*

$$(2.1) \quad \psi(d(gx, gy)) \preceq \psi(d(fx, fy)) - \phi(d(fx, fy)),$$

where $\psi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is a non-decreasing and continuous function, $\phi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is a continuous function and $\phi(z) = 0$ if and only if $z = 0$. If gX or fX is complete, then f and g have a unique point of coincidence. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take any element $x_0 \in X$. Using the condition $gX \subset fX$, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$y_n = gx_n = fx_{n+1}, \quad n = 0, 1, 2, \dots$$

For any fixed $n = 1, 2, \dots$,

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(gx_n, gx_{n+1})) \\ &\preceq \psi(d(fx_n, fx_{n+1})) - \phi(d(fx_n, fx_{n+1})) \\ &= \psi(d(y_{n-1}, y_n)) - \phi(d(y_{n-1}, y_n)) \\ &\preceq \psi(d(y_{n-1}, y_n)). \end{aligned}$$

Hence using the monotonicity of ψ , we obtain

$$(2.2) \quad d(y_n, y_{n+1}) \preceq d(y_{n-1}, y_n), \quad n = 1, 2, \dots$$

Let $d(y_n, y_{n+1}) = \alpha_n + \beta_n i$ for all $n = 1, 2, \dots$, then by (2.2),

$$0 \leq \alpha_n \leq \alpha_{n-1}, \quad 0 \leq \beta_n \leq \beta_{n-1}, \quad n = 1, 2, \dots$$

Hence there exist $\alpha, \beta \geq 0$ such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ and $\lim_{n \rightarrow \infty} \beta_n = \beta$, therefore,

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = \lim_{n \rightarrow \infty} (\alpha_n + \beta_n i) = \alpha + \beta i.$$

Let $n \rightarrow \infty$, then using the above, we obtain

$$\psi(\alpha + \beta i) \preceq \psi(\alpha + \beta i) - \phi(\alpha + \beta i) \preceq \psi(\alpha + \beta i),$$

hence $\phi(\alpha + \beta i) = 0$, therefore $\alpha + \beta i = 0$. This shows that

$$(2.3) \quad \lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = \lim_{n \rightarrow \infty} (\alpha_n + \beta_n i) = 0.$$

In what follows, we will prove that $\{y_n\}$ is a Cauchy sequence. Otherwise, there exists $c \in \mathbb{C}$ with $0 < c$ such that for each $k \in \mathbb{N}$, there exist $n(k), m(k) \in \mathbb{N}$ satisfying the following conditions:

$$n(k) > m(k) > k, \quad d(y_{n(k)}, y_{m(k)}) > c.$$

Let $n(k)$ be the smallest natural number satisfying the condition $n(k) > m(k) > k$ for all k , then the following holds:

$$d(y_{n(k)-1}, y_{m(k)}) \preceq c, \forall k \in \mathbb{N}.$$

Since

$c < d(y_{n(k)}, y_{m(k)}) \preceq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \preceq d(y_{n(k)}, y_{n(k)-1}) + c$,
using (2.3), we obtain

$$(2.4) \quad \lim_{k \rightarrow \infty} d(y_{n(k)}, y_{m(k)}) = c.$$

Obviously, the following two statements hold:

$$d(y_{n(k)-1}, y_{m(k)-1}) \preceq d(y_{n(k)-1}, y_{n(k)}) + d(y_{n(k)}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1})$$

and

$$d(y_{n(k)}, y_{m(k)}) \preceq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}).$$

Hence

$$\begin{aligned} & -[d(y_{n(k)}, y_{n(k)-1}) + d(y_{m(k)-1}, y_{m(k)})] \\ & \preceq d(y_{n(k)}, y_{m(k)}) - d(y_{n(k)-1}, y_{m(k)-1}) \\ & \preceq [d(y_{n(k)}, y_{n(k)-1}) + d(y_{m(k)-1}, y_{m(k)})]. \end{aligned}$$

Therefore using (2.3) again, we obtain

$$(2.5) \quad \lim_{k \rightarrow \infty} d(y_{n(k)}, y_{m(k)}) = \lim_{k \rightarrow \infty} d(y_{n(k)-1}, y_{m(k)-1}) = c.$$

By (2.1),

$$\begin{aligned} \psi(d(y_{n(k)}, y_{m(k)})) &= \psi(d(gx_{n(k)}, gx_{m(k)})) \\ &\preceq \psi(d(fx_{n(k)}, fx_{m(k)})) - \phi(d(fx_{n(k)}, fx_{m(k)})) \\ &= \psi(d(y_{n(k)-1}, y_{m(k)-1})) - \phi(d(y_{n(k)-1}, y_{m(k)-1})) \\ &\preceq \psi(d(y_{n(k)-1}, y_{m(k)-1})). \end{aligned}$$

Let $k \rightarrow \infty$, then $\psi(c) \preceq \psi(c) - \phi(c) \preceq \psi(c)$ by (2.5), hence $\phi(c) = 0$, and so $c = 0$ which is a contradiction with $0 < c$. This shows that $\{y_n\}$ is a Cauchy sequence.

Suppose that fX is complete. Since $y_n = fx_{n+1} \in fX$, there exist $u, v \in X$ such that $y_n \rightarrow v = fu$ as $n \rightarrow \infty$.

By (2.1),

$$\begin{aligned} \psi(d(y_n, gu)) &= \psi(d(gx_n, gu)) \\ &\preceq \psi(d(fx_n, fu)) - \phi(d(fx_n, fu)) = \psi(d(y_{n-1}, fu)) - \phi(d(y_{n-1}, fu)) \\ &\preceq \psi(d(y_{n-1}, fu)), \end{aligned}$$

hence $d(y_n, gu) \preceq d(y_{n-1}, fu)$ by the monotonicity of ψ .

Since $y_n \rightarrow fu$ as $n \rightarrow \infty$, so for each $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that $d(y_{n-1}, fu) \prec c$ for all $n > N$, hence $d(y_n, gu) \prec c$ for all $n > N$. This shows that $\{y_n\} \rightarrow gu$ as $n \rightarrow \infty$, hence $v = fu = gu$ by Lemma 1.9. Therefore u is the coincidence point of f and g , v is the point of coincidence of f and g .

Suppose that gX is complete. Since $y_n = gx_n \in gX \subset fX$, there exist $u, v, w \in X$ such that $y_n \rightarrow v = gw = fu$ as $n \rightarrow \infty$. The rest of the argument is similar to the proof for the case that fX is complete.

If v_1 is another point of coincidence of f and g , then there exists $u_1 \in X$ such that $v_1 = fu_1 = gu_1$. Since

$$\begin{aligned} \psi(d(v, v_1)) &= \psi(d(gu, gu_1)) \\ &\preceq \psi(d(fu, fu_1)) - \phi(d(fu, fu_1)) \preceq \psi(d(fu, fu_1)), \end{aligned}$$

that is,

$$\psi(d(v, v_1)) \preceq \psi(d(v, v_1)) - \phi(d(v, v_1)) \preceq \psi(d(v, v_1)),$$

hence $\phi(d(v, v_1)) = 0$, which implies $d(v, v_1) = 0$, i.e., $v = v_1$. Therefore v is the unique point of coincidence of f and g . Finally, if f and g are weakly compatible, then v is the unique common fixed point of f and g by Lemma 1.12. \square

Using Theorem 2.4, we can give many particular results. Because of the limitation of length, we give only one here.

THEOREM 2.5. *Let (X, d) be a complex valued metric space and $f, g : X \rightarrow X$ two mappings with $gX \subset fX$. Suppose that for each $x, y \in X$,*

$$d(gx, gy) \preceq h d(fx, fy),$$

where $h \in [0, 1)$. If gX or fX is complete, and f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $\psi = 1_{\mathbb{C}^+}$ and $\phi = (1 - h)\psi$, then all conditions of Theorem 2.4 are satisfied. So the conclusion follows from Theorem 2.4. \square

EXAMPLE 2.6. *Consider the complex valued metric space (X, d) in Example 1.3. Define two mappings as follows:*

$$fa = a, fb = c, fc = b; ga = a, gb = a, gc = c.$$

Then $fX = X$ is complete and $gX \subset fX$. Obviously, f and g are weakly compatible. Take $h = 0.8$. Since

$$\begin{aligned} d(ga, gc) &= d(a, c) = 2 + 3i \preceq 0.8(3 + 4i) = 0.8d(a, b) = 0.8d(fa, fc); \\ d(gb, gc) &= d(a, c) = 2 + 3i \preceq 0.8(4 + 5i) = 0.8d(c, b) = 0.8d(fb, fc). \end{aligned}$$

Hence f , g and h satisfy all conditions of Theorem 2.5, so f and g have a unique common fixed point. In fact, a is the unique common fixed point of f and g .

Using Theorem 2.5, we can obtain the next two fixed point theorems:

THEOREM 2.7. *Let (X, d) be a complex valued metric space and $g : X \rightarrow X$ a mapping. Suppose that for each $x, y \in X$,*

$$d(gx, gy) \preceq h d(x, y),$$

where $h \in [0, 1)$. If gX is complete, then g has a unique fixed point.

Proof. Let $f = 1_X$ in Theorem 2.5. The rest proof is trivial. \square

THEOREM 2.8. *Let (X, d) be a complete complex valued metric space and $f : X \rightarrow X$ a surjective mapping. Suppose that for each $x, y \in X$,*

$$d(fx, fy) \succeq k d(x, y),$$

where $k > 1$. Then f has a unique fixed point.

Proof. We have that $fX = X$ is complete and we obtain that

$$d(x, y) \preceq h d(fx, fy) \iff d(fx, fy) \succeq k d(x, y),$$

where $k = \frac{1}{h}$. Obviously $k > 1 \iff h < 1$. Let $g = 1_X$, then the conclusion follows from Theorem 2.5. \square

REMARK 2.9. Theorem 2.7 and Theorem 2.8 are the versions of Banach's contraction principle and the fixed point theorem for a I-expansive mapping([18]) respectively on complex valued metric spaces.

REMARK 2.10. The condition (1) in Theorem 2.4 can be replaced by the next condition without affecting its conclusion(The proof is almost similar to the proof of Theorem 2.4):

$$(2.6) \quad \psi(d(gx, gy)) \preceq \psi(d(fx, fy)) - \phi(d(gx, gy)).$$

REMARK 2.11. Let $\psi = 1_X, \phi = (k-1)\psi$, where $k > 1$, then Theorem 2.4 with (2.6) instead of (2.1) also implies Theorem 2.5(Take $h = \frac{1}{k}$).

Now, we give another type common fixed point theorem for two mappings with an expansive condition.

THEOREM 2.12. *Let (X, d) be a complete complex valued metric space and $f, g : X \rightarrow X$ two surjective mappings. Suppose that for each $x, y \in X$,*

$$(2.7) \quad \psi(d(fx, gy)) \succeq \psi(d(x, y)) + \phi(d(x, y)),$$

where $\psi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is a non-decreasing and continuous function, $\phi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is a continuous function and $\phi(z) = 0$ if and only if $z = 0$. Then f and g have a unique common fixed point.

Proof. Take an element $x_0 \in X$ and construct a sequence $\{x_n\}$ as follows

$$x_{2n} = fx_{2n+1}, \quad x_{2n+1} = gx_{2n+2}, \quad n = 0, 1, 2, \dots$$

For any fixed n , by (2.7),

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(fx_{2n+3}, gx_{2n+2})) \\ &\succeq \psi(d(x_{2n+3}, x_{2n+2})) + \phi(d(x_{2n+3}, x_{2n+2})) \\ &\succeq \psi(d(x_{2n+3}, x_{2n+2})), \end{aligned}$$

hence

$$(2.8) \quad d(x_{2n+2}, x_{2n+3}) \preceq d(x_{2n+1}, x_{2n+2}).$$

Similarly,

$$\begin{aligned} \psi(d(x_{2n}, x_{2n+1})) &= \psi(d(fx_{2n+1}, gx_{2n+2})) \\ &\succeq \psi(d(x_{2n+1}, x_{2n+2})) + \phi(d(x_{2n+1}, x_{2n+2})) \\ &\succeq \psi(d(x_{2n+1}, x_{2n+2})), \end{aligned}$$

hence

$$(2.9) \quad d(x_{2n+1}, x_{2n+2}) \preceq d(x_{2n}, x_{2n+1}).$$

Therefore combining (2.8) and (2.9), we obtain

$$(2.10) \quad d(x_{n+1}, x_{n+2}) \preceq d(x_n, x_{n+1}), \quad n = 0, 1, 2, \dots$$

Let $d_n = d(x_n, x_{n+1})$, $n = 0, 1, 2, \dots$, then just as the proof of theorem 2.4, it is easy to check $\{d_n\} \rightarrow r$ for some $r \in \mathbb{C}^+$. Since

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &\succeq \psi(d(x_{2n+3}, x_{2n+2})) + \phi(d(x_{2n+3}, x_{2n+2})) \\ &\succeq \psi(d(x_{2n+3}, x_{2n+2})), \end{aligned}$$

let $n \rightarrow \infty$, then we obtain

$$\psi(r) \succeq \psi(r) + \phi(r) \succeq \psi(r).$$

Hence $\phi(r) = 0$, which implies that $r = 0$, that is,

$$(2.11) \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$, in order to prove that $\{x_n\}$ is Cauchy, we need only show that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > n > N$, $d(x_m, x_n) < \epsilon$, where the parity of m and n is different. Suppose that

$\{x_n\}$ is not a Cauchy sequence, then there exists a constant $\epsilon > 0$ such that for each positive integer k , there exists two integers $m(k)$ and $n(k)$ with $m(k) > n(k)$ and the parity of $m(k)$ and $n(k)$ is different such that

$$d(x_{m(k)}, x_{n(k)}) > \epsilon.$$

For k , let $m(k)$ denotes the smallest integer exceeding $n(k)$ and satisfying the above, then

$$(2.12) \quad d(x_{m(k)}, x_{n(k)}) > \epsilon, \quad d(x_{m(k)-2}, x_{n(k)}) \leq \epsilon, \quad \forall k = 1, 2, \dots.$$

Note that

$$(2.13) \quad d(x_{m(k)}, x_{n(k)}) \leq d(x_{n(k)}, x_{m(k)-2}) + d_{m(k)-2} + d_{m(k)-1};$$

$$(2.14) \quad -d_{n(k)} \leq d(x_{m(k)}, x_{n(k)+1}) - d(x_{m(k)}, x_{n(k)}) \leq d_{n(k)};$$

$$(2.15) \quad -d_{m(k)} \leq d(x_{m(k)+1}, x_{n(k)+1}) - d(x_{m(k)}, x_{n(k)+1}) \leq d_{m(k)};$$

$$(2.16) \quad -d_{m(k)} \leq d(x_{m(k)+1}, x_{n(k)}) - d(x_{m(k)}, x_{n(k)}) \leq d_{m(k)}.$$

By (2.11)–(2.16), we obtain

$$\begin{aligned} \epsilon &= \lim_{k \rightarrow \infty} d(x_{m(k)-2}, x_{n(k)}) = \lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k \rightarrow \infty} d(x_{m(k)+1}, x_{n(k)}) \\ &= \lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)+1}) = \lim_{k \rightarrow \infty} d(x_{m(k)+1}, x_{n(k)+1}). \end{aligned}$$

If $m(k)$ is even and $n(k)$ is odd, then using (2.7), we obtain

$$\begin{aligned} \psi(d(x_{m(k)}, x_{n(k)})) &= \psi(d(fx_{m(k)+1}, gx_{n(k)+1})) \\ &\succeq \psi(d(x_{m(k)+1}, x_{n(k)+1})) + \phi(d(x_{m(k)+1}, x_{n(k)+1})) \\ &\succeq \psi(d(x_{m(k)+1}, x_{n(k)+1})). \end{aligned}$$

Let $k \rightarrow \infty$, then the above becomes

$$\psi(\epsilon) \succeq \psi(\epsilon) + \phi(\epsilon) \succeq \psi(\epsilon),$$

so $\phi(\epsilon) = 0$, which implies $\epsilon = 0$, this is a contradiction.

Similarly, if $m(k)$ is odd and $n(k)$ is even, then

$$\begin{aligned} \psi(d(x_{m(k)}, x_{n(k)})) &= \psi(d(gx_{m(k)+1}, fx_{n(k)+1})) \\ &\succeq \psi(d(x_{m(k)+1}, x_{n(k)+1})) + \phi(d(x_{m(k)+1}, x_{n(k)+1})) \\ &\succeq \psi(d(x_{m(k)+1}, x_{n(k)+1})). \end{aligned}$$

Let $k \rightarrow \infty$, then the above becomes

$$\psi(\epsilon) \succeq \psi(\epsilon) + \phi(\epsilon) \succeq \psi(\epsilon),$$

so $\phi(\epsilon) = 0$, which implies $\epsilon = 0$, this is also a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Let $x_n \rightarrow x \in X$ by the completeness of X , then there exist $u, v \in X$ such that $x = fu = gv$.

By (2.7),

$$\begin{aligned}\psi(d(x, x_{2n+1})) &= \psi(d(fu, gx_{2n+2})) \\ &\succeq \psi(d(u, x_{2n+2})) + \phi(d(u, x_{2n+2})) \succeq \psi(d(u, x_{2n+2})),\end{aligned}$$

so

$$d(u, x_{2n+2}) \preceq d(x, x_{2n+1}), \quad n = 0, 1, 2, \dots$$

Hence $x_{2n+2} \rightarrow u$ as $n \rightarrow \infty$, therefore $u = x = fu$ by Lemma 1.9. Similarly, we obtain $v = x = gv$, so $x = fx = gx$, i.e., x is a common fixed point of f and g . If y is another common fixed point of f and g , then by (2.7),

$$\psi(d(x, y)) = \psi(d(fx, gy)) \succeq \psi(d(x, y)) + \phi(d(x, y)) \succeq \psi(d(x, y)),$$

hence $\phi(d(x, y)) = 0 \Rightarrow d(x, y) = 0 \Leftrightarrow x = y$, i.e., x is the unique common fixed point of f and g . \square

REMARK 2.13. The condition (2.7) in Theorem 2.12 can be replaced by the following condition without affecting its conclusion (The proof is almost similar to the proof of Theorem 3.12):

$$\psi(d(fx, gy)) \succeq \psi(d(x, y)) + \phi(d(fx, gy)).$$

REMARK 2.14. Using theorem 2.12, we can give many particular fixed point and common fixed point theorems, but we omit them here.

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