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THETA TOPOLOGY AND ITS APPLICATION TO THE FAMILY OF ALL TOPOLOGIES ON X

Jae-Ryong Kim*

Abstract. Topology may described a pattern of existence of elements of a given set X. The family $\tau(X)$ of all topologies given on a set X form a complete lattice. We will give some topologies on this lattice $\tau(X)$ using a topology on X and regard $\tau(X)$ a topological space.

Our purpose of this study is to give new topologies on the family $\tau(X)$ of all topologies induced by old one and its θ topology and to compare them.

1. Introduction

Let X be a set. The family $\tau(X)$ would consist of all topologies on a given *fixed* set X. Here we want to give topologies on the family $\tau(X)$ of all the topologies using the given a topology τ on X and compare the topologies from new one.

The family $\tau(X)$ of all topologies on X form a complete lattice, that is, given any corlection of topologies on X, there is a smallest (respectively largest) topology on X containing(contained in) each member of the corlection. Of course, the partial order $\leq$ on $\tau(X)$ is defined by inclusion $\subseteq$ naturally.

The smallest topology in this lattice $\tau(X)$ is $\{\emptyset, X\}$ and the largest one is $\mathcal{P}(X)$. These topologies will sometimes be denoted by 0 and 1 respectively.

In the sequel, the *closure* and *interior* of A are denoted by $\bar{A}$ and $int(A)$ in a topological space (X,τ). The θ-closure of a subset G of a topological space (X,τ) is defined [12] to be the set of all point $x \in X$ such that every closed neighborhood of x intersect G non-emptily and is denoted by $\bar{G}_\theta$(cf. [1],[6]). Of course for any subset G in X, $G \subset \bar{G} \subset \bar{G}_\theta$

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and $\bar{G}_\theta$ is closed in X. The subset G is called θ-closed if $\bar{G}_\theta = G$. If G is open, the $\bar{G} = \bar{G}_\theta$.

Similarly, the θ-interior of a subset G of a topological space (X, τ) is defined to be the set of all point $x \in X$ for which there exists a closed neighborhood of x contained in G. The θ-interior of G is denoted by $int_\theta G$. Naturally, for any subset G in X, $int_\theta(G) \subset G$. An open set U in (X, τ) is called θ-open if $U = int_\theta(U)$. By the definition of θ-open, the corlection of all θ-open in a topological space (X, τ) form a topology τ_θ on X which will called the θ topology induced by τ which is related to the semi-regular topology on (X, τ).

The semi-regular topology τ_s is the topology having as its base the set of all regular open sets. A subset A of a topological space X is called *regular open* [11] if $A = int\bar{A}$. For any subset A of X, $int(\bar{A})$ is always regular open. The corlection of all regular open subsets of a topological space (X, τ) form a base for a topology τ_s on X coarser than τ, (X, τ_s) is called the semiregularization of (X, τ).

THEOREM 1.1. [6] *Let X be a topological space. If $V \subset X$ is θ-open and $x \in V$ then there exists a regular-open set U such that $x \in U \subset \bar{U} \subset V$.*

Theorem 1.1 implies that in any topological space, $\tau_\theta \leq \tau_s \leq \tau$. The converse need not true [6]. The following theorems are stated in [6].

THEOREM 1.2. *A topological space (X, τ) is regular if and only if $\tau_\theta = \tau$.*

THEOREM 1.3. *Let $A \subset X$ be θ-closed and $x \notin A$. Then there exists a regular-open set which separate x and A.*

THEOREM 1.4. *Let $f : X \to Y$ be continous. If $V \subset Y$ is θ-open, then $f^{-1}(V)$ is θ-open.*

THEOREM 1.5. *Let $f : X \to Y$ be a function from X onto Y that is both open and closed. Then f preserves θ-open sets.*

We should recall the definitions of almost-continuity and θ-continuity: A function $f : X \to Y$ is *almost-continuous*(θ-continuous) if for each $x \in X$ and each regular-open V(open V) containing $f(x)$, there exists a open set U containing x such that $f(U) \subset V$ $(f(\bar{U}) \subset \bar{V})$. It readily follows that continuity $\Rightarrow$ almost-continuity $\Rightarrow$ θ-continuity.

2. Compare of θ topologies defined different topologies

To notice the closure and interior of a subset A in the specific topological space (X, τ), they will be denoted by $\bar{A}^{\tau}$ and $int^{\tau}(A)$ instead of $\bar{A}$ and $int(A)$ respectively. Hence an θ-interior of G in (X, τ) is denoted by $int^{\tau}_{\theta}(G)$ and an θ-open set U in (X, τ) will be denoted by $U = int^{\tau}_{\theta}(U)$.

LEMMA 2.1. *Let ζ, η be topologies on X. If $\zeta \leq \eta$, then $\bar{A}^{\eta} \subset \bar{A}^{\zeta}$ and $int^{\zeta}(A) \subset int^{\eta}(A)$. Hence if A is open and closed in (X, ζ), then A is also open and closed in (X, η) respectively.*

Proof. We will prove only the closed case. The other case follows directly from the definition. Let $x \in \bar{A}^{\eta}$. Then for any open neighborhood G of x in (X, η), $G \cap A \neq \emptyset$. This implies that for any open neighborhood G of x in (X, ζ), $G \cap A \neq \emptyset$. Hence $\bar{A}^{\eta} \subset \bar{A}^{\zeta}$. Consequently if A is closed in (X, ζ) then $A \subset \bar{A}^{\eta} \subset \bar{A}^{\zeta} = A$. Therefore A is closed in (X, η). □

THEOREM 2.2. *Let ζ, η be topologies on X and $\zeta \leq \eta$. Then $\zeta_{\theta} \leq \eta_{\theta}$.*

Proof. We will be sufficient to prove that if U is θ-open in (X, ζ), then it is also θ-open in (X, η). Let U be a θ-open set in (X, ζ). Then $U = int^{\zeta}_{\theta}(U)$. Let $x \in U$. Then there exists a closed neighborhood V of x in (X, ζ) which contained in U. Since $\zeta \leq \eta$, By above Lemma 2.1, V is also a closed neighborhood V of x in (X, η) which contained in U. This implies $U \subset int^{\eta}_{\theta}(U) \subset U$. Hence U is θ-open in (X, η). This completes the proof. □

3. Topologies on the family $\tau(X)$ induced from by a given topology τ

Let (X, τ) be a topological space. We want to give some topologies on $\tau(X)$ induced by the given topology τ and compare these topologies.

DEFINITION 3.1. [5] Let (X, τ) be a topological space, and $G \in \tau$. Let $i(G)$=$\{\zeta{\in}\tau(X) \mid G{\in}\zeta\}$ and denote ϵ=$\{i(G) | G{\in}\tau\}$, a family of subset of $\tau(X)$. Then there is exactly one topology In_{τ} on $\tau(X)$ with ϵ as a subbasis. We will call this topology as inner topology induced by the topology τ. Note that In_1 need not be the discrete topology in $\tau(X)$.

THEOREM 3.2. [4] *$(\tau(X), In_1)$ is T_0 space.*

Let $\zeta \leq \eta$. For all $G \in \zeta$, $G \in \eta$. That is, if $\zeta \in i(G)$, then $i(G) \cap \{\eta\} \neq \emptyset$. This implies $\zeta \in \overline{\{\eta\}}$. Conversely $\zeta \in \overline{\{\eta\}}$ implies

$\zeta \leq \eta$. If this relation holds we say that ζ is a *specialization* of η [7]. For any $\eta \in \tau(X)$ we will denote the subset $\{\zeta \in \tau(X) | \zeta \geq \eta\}$ by $\uparrow(\eta)$. (We shall also use later the notation $\downarrow(\eta)$ for $\{\zeta \in \tau(X) | \zeta \leq \eta\}$. Then since $i(G) = \{ \zeta \in \tau(X) \mid G \in \zeta \}$, $i(G) = \uparrow(\{\emptyset, X, G\})$. Hence $\zeta \in \overline{\{\eta\}}$ iff $\zeta \leq \eta$. Since *Alexandrov topology* Υ on $\tau(X)$ is the corlection of all *upper sets* in $\tau(X)$ (i.e. sets U such that $\eta \in U$ and $\eta \leq \zeta$ imply $\zeta \in U$) [7], $i(G) \in \Upsilon$. Hence we have the following result

THEOREM 3.3. [4] *If* $\tau \leq \zeta \leq 1$, *then* $In_\tau \leq In_\zeta \leq In_1 \leq \Upsilon$.

Combining this theorem and Theorem 1.1 we can have

COROLLARY 3.4. $In_{\tau_\theta} \leq In_{\tau_s} \leq In_\tau \leq In_1 \leq \Upsilon$,

$$(In_\tau)_\theta \leq (In_\tau)_s \leq In_\tau \leq In_1 \leq \Upsilon.$$

Now we will consider the continuity of induced maps. The next theorem was known in [4]:

THEOREM 3.5. *Let* $f:(X, \tau) \to (Y, \eta)$ *be a continuous map. If we define a map* $f_*:(\tau(X), In_\tau) \to (\tau(Y), In_\eta)$ *by* $f_*(w)=\{U \subset Y | f^{-1}(U) \in w\}$, *then the map* f_* *is continuous. If* $\gamma \leq \delta$, *then* $f_*(\gamma) \leq f_*(\delta)$ *and* $f_*(\tau) \geq \eta$. *If, furthermore,* (Z, θ) *is a topological space and* g: $(Y, \eta) \to (Z, \theta)$ *is a map, then*

$$(g \circ f)_* = g_* \circ f_*.$$

Finally, if $f:(X, \tau) \to (X, \tau)$ *is the identity homeomorphism, then so is* f_*.

If we consider In as a map from $\tau(X)$ to $\tau(\tau(X))$ defined by $In(\eta) = In_\eta$, then we have some result:

THEOREM 3.6. [4] $In : (\tau(X), \Upsilon) \to (\tau(\tau(X)), \Upsilon)$ *is continuous.*

Proof. Let $\zeta \in \tau(X)$ and K is a neighborhood of $In(\zeta) = In_\zeta$. Then K is a upper set in $\tau(\tau(X))$. On the other hand the upper set $\uparrow(\zeta)$ in $\tau(X)$ is a neighborhood of ζ. We will show that $In(\uparrow(\zeta)) \subset K$. Let $\delta \in \uparrow(\zeta)$. Then $\delta \geq \zeta$ and $In_\delta \geq In_\zeta$. Hence we have $In_\delta \in K$. □

COROLLARY 3.7. $In : (\tau(X), In_\tau) \to (\tau(\tau(X)), In_{In_\tau})$ *is continuous.*

Proof. Since In_τ is subset of Υ whose elements $i(G)$s are all upper set, it is clear that the restriction function is continuous. □

Let $f : (X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map $f_* : \tau(X) \to \tau(Y)$ by $f_*(w)=\{U \subset Y | f^{-1}(U) \in w\}$, then $f_*(0) = 0$ and $f_*(1) = 1$. Let $\omega \in \tau(X)$. For any subbasic open neighborhood

$i(G)$ of $f_*(\omega)$, $G \in f_*(\omega)$. Thus $f^{-1}(G) \in \omega$. Hence $\omega \in i(f^{-1}(G))$. So that $i(f^{-1}(G))$ is an open neighborhood of ω. Conversely, if $\zeta \in i(f^{-1}(G))$ then $f^{-1}(G) \in \zeta$, $G \in f_*(\zeta)$, $f_*(\zeta) \in i(G)$, and $\zeta \in f_*^{-1}(i(G))$. Consequently we have

$$f_*^{-1}(i(G)) = i(f^{-1}(G)).$$

Note that $i(G) \cup \{0\}$ is also complete sublattice of $\tau(X)$ for a $G \in \tau$. We will denote this sublattice $i(G) \cup \{0\}$ by $i_F(G)$. Then naturally we can restrict domain of definition of f_* to $i_F(H)$ for some open H in X. Hence we can have $f_* : i_F(f^{-1}(G)) \to i_F(G)$ for each open G in (Y, η). Thus we can have:

THEOREM 3.8. [4] *Let* $f:(X,\ \tau) \to (Y,\ \eta)$ *be a continuous bijective map. Then the induce map* $f_* : i_F(f^{-1}(G)) \to i_F(G)$ *is bijective for each open* G *in* (Y, η).

Let X, Y be sets. Then the cardinality of $\tau(X) \times \tau(Y)$ is quite different to the cardinality of $\tau(X \times Y)$. For example, let $X = \{a, b\}, Y = \{1, 2, 3, \}$. Then card($\tau(X)$)= 4, card($\tau(Y)$) = 29. But card($\tau(X \times Y)$) = 209525 [9].

Hence we have $\tau(X \times Y) \ncong \tau(X) \times \tau(Y)$ in general.

Let (X, τ) and (Y, ζ) be topological spaces. We may assume that $\tau(X)$ and $\tau(Y)$ are given the topologies In_τ and In_ζ respectively and assume that $\tau(X \times Y)$ is given topology $In_{\tau\times\zeta}$. The multiplication $\times : \tau(X) \times \tau(Y) \to \tau(X \times Y)$ is defined by $\times(\alpha, \beta) = \alpha \times \beta$ naturally. Then we have

THEOREM 3.9. *The multiplication* $\times : \tau(X) \times \tau(Y) \to \tau(X \times Y)$ *is continuous.*

Proof. Let $(\alpha, \beta) \in \tau(X) \times \tau(Y)$. Then $\alpha \times \beta \in \tau(X \times Y)$. If $i(W)$ is a neighborhood of $\times(\alpha, \beta) = \alpha \times \beta$, where W is open in $(X \times Y, \tau \times \zeta)$. Then we may assume that $W = W_X \times W_Y$ is basic open set in $(\tau(X \times Y), \alpha \times \beta)$. Hence since projection maps are open maps, $\pi_X(W) = W_X$ and $\pi_Y(W) = W_Y$ are also open sets in (X, α) and (Y, β) respectively. Hence $(\alpha, \beta) \in i(W_X) \times i(W_Y)$. Moreover $\times(i(W_X) \times i(W_Y)) \subset i(W)$. In fact, if $\delta \in i(W_X)$ and $\gamma \in i(W_Y)$, then $W_X \in \delta$ and $W_Y \in \gamma$. Hence $W = W_X \times W_Y \in \delta \times \gamma$. This completes the proof. $\square$

Hence we have

THEOREM 3.10. *Let* (X, τ) *and* (Y, ζ) *be topological spaces. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \tau(X)\times\tau(Y) & \xrightarrow{\times} & \tau(X\times Y) \\ \downarrow In\times In & & \downarrow In \\ \tau(\tau(X))\times\tau(\tau(Y)) & \xrightarrow{\times} & \tau(\tau(X\times Y)). \end{array}$$

Proof. It is sufficient to show that $\times(In_\alpha \times In_\beta) = In_{\alpha\times\beta}$ for an element $(\alpha,\beta)\in\tau(X)\times\tau(Y)$. Let $(\alpha,\beta)\in i(U)\times i(V)\in In_\alpha\times In_\beta$. Then U and V are open sets in (X,α) and (Y,β) respectively. Hence $U\times V$ is an open set in $(X\times Y,\alpha\times\beta)$. Hence $U\times V\in(\alpha\times\beta)$, i.e.$(\alpha\times\beta)\in\times(i(U)\times i(V))\subset i(U\times V)$. Hence $\times(In_\alpha\times In_\beta)\subset In_{\alpha\times\beta}$. Conversely if $\delta\times\gamma\in i(W)$ where W is an open in $(X\times Y,\tau\times\zeta)$. Then we may assume that $W = W_X\times W_Y$ is basic open set in $(X\times Y,\tau\times\zeta)$. Hence, the projections $\pi_X(W) = W_X$ and $\pi_Y(W) = W_Y$ are open sets in (X,τ) and (Y,ζ) respectively. Moreover $\times(i(W_X)\times i(W_Y))\subset i(W)$. Consequently we have $\times(In_\alpha\times In_\beta)\supset In_{\alpha\times\beta}$ This completes the proof. □

4. Topology on the family $\tau(X)$ related to the θ topologies on X

Definition 4.1. Let (X,τ) be a topological space, and $G\in\tau$. Let $\theta(G) = \{\zeta\in\tau(X)\mid G \text{ is } \theta-\text{open in } \zeta\}$. And denote $\beta = \{\theta(G)|G\in\tau\}$, a family of subset of $\tau(X)$. Then there is exactly one topology θ_τ on $\tau(X)$ with β as a subbasis. We will call the θ_τ *the θ topology* induced by the topology τ.

Theorem 4.2. *If $\tau\le\zeta\le 1$, then $\theta_\tau\le\theta_\zeta\le\theta_1\le\Upsilon$.*

Proof. For any $G\in\tau\le\zeta$, by the definition of $\theta(G)$, we can naturally have $\theta_\tau\le\theta_\zeta$. Now we will prove that every $\theta(G)$ is upper set in $\tau(X)$. Let $\delta\in\theta(G)$. Then G is an θ-open in (X,δ). Hence $G\in\delta_\theta$. If $\delta\le\gamma$, we have by Theorem 2.1, $G\in\gamma_\theta$. This means G is θ-open in (X,γ). That is $\gamma\in\theta(G)$. Hence $\theta(G)$ is upper set in $\tau(X)$. This completes the proof. □

Now we consider θ as a map from $\tau(X)$ to $\tau(X)$. Then we can define map θ by $\theta(\eta) = \eta_\theta$. Consequently we have next result:

Theorem 4.3. *Let (X,τ) be a topological space. Then the induced map*

$$\theta : (\tau(X),\theta_\tau)\to(\tau(X),\theta_\tau)$$

is continuous.

Proof. Let $\zeta \in \tau(X)$ and $\theta(K)$ is a neighborhood of $\theta(\zeta) = \zeta_\theta$ where $K \in \tau$. Then since $\zeta_\theta = \{U \in \zeta | U : \theta - \text{open in } (\text{X}, \zeta)\}$, K is a θ-open set in (X, ζ_θ). Hence it is also θ-open in (X, ζ). Consequently $\zeta \in \theta(K)$, i.e. $\theta(K)$ is a neighborhood of ζ which satisfied that $\theta(\theta(K)) \subset \theta(K)$. This completes the Theorem. □

The map $\theta : (\tau(X), \theta_\tau) \to (\tau(X), \theta_\tau)$ will be called θ-operator. Moreover this map satisfies that

COROLLARY 4.4. *$\theta(\zeta \wedge \eta) \leq \theta(\zeta) \wedge \theta(\eta)$ and $\theta(\zeta) \vee \theta(\eta) \leq \theta(\zeta \vee \eta)$.*

Proof. This corollary follows from the above definition of map θ and Theorem 4.1. □

Now we want to know the relations between θ_τ and In_{τ_θ}. Let $\eta \in \theta(G) \in \theta_\tau$, then $G \in \eta$, i.e. $\eta \in i(G)$. Hence it is natural that $\theta(G) \subset i(G)$. Let $i(G)$ be a sub basic open in In_{τ_θ}. Then $G \in \tau_\theta$. Hence G is θ-open in τ, that is , $G = int^\tau_\theta(G)$. Hence if $\eta \in i(G)$ and (X, η) is regular, then by above Theorem 1.2, G is also θ-open in (X, η). Hence $\eta \in \theta(G)$, i.e. $i(G) = \theta(G)$. Thus we have the following theorem.

THEOREM 4.5. *Let (X, τ) is a regular space. If we denote $\tau_{reg}(X)$ by the subset of all regular topologies in $\tau(X)$. Then the subspace $\tau_{reg}(X)$ of the space $(\tau(X), \theta_\tau)$ and the subspace $\tau_{reg}(X)$ of the space $(\tau(X), In_{\tau_\theta})$ are identical.*

For a topological space (X, τ), the corlection of all open neighborhoods of p and empty set, that is, $\{V \in \tau | p \in V\} \cup \{\emptyset\}$ becomes a topology on X for any point $p \in X$. We will denote such a topology by τ_p and call *localized topology* of τ at p. Furthermore, we will denote the localized topology of the discrete topology $\mathcal{P}(X)$ on X at p by 1_p.

Denote $\tau_p(X) = \{\eta_p \mid \eta \in \tau(X)\}$ for a point $p \in X$. Since $\tau(X)$ is a complete lattice, we can easily find that $\tau_p(X)$ is a sublattice of $\tau(X)$. The smallest element of this sublattice $\tau_p(X)$ is 0_p=0, the largest element is $\mathcal{P}(X)_p$=1_p≠1. We will call this sublattice $\tau_p(X)$ as *sublattice of all localized topologies* at p in X.

Now we will regard any member τ of $\tau(X)$ as a map from X to $\cup_p \tau_p(X) \subset \tau(X)$ defined by $\tau(p) = \tau_p$. Hence this map τ acts like a vector field on X. Such a map $f : X \to \tau(X)$ defined by $f(p) \in \tau_p(X)$ will be called *topology field* on X [5].

THEOREM 4.6. [5] *Topology field ζ:$(X,\tau) \to (\tau(X), In_\tau)$ is continuous.*

Now we will prove

Theorem 4.7. *If (X,ζ) is a θ topological space, then the topology field ζ:$(X,\tau) \to (\tau(X), \theta_\tau)$ is continuous.*

Proof. Let $p \in X$ and $\theta(G)$ be a subbasic open neighborhood of $\zeta(p)=\zeta_p$. Then G is θ-open in (X,ζ_p). This implies G is θ-open in (X,ζ) because G is open set in (X,ζ) which contains the point p. Moreover since $G \in \tau$, G is a neighborhood of p. Hence if $q \in G$, $\zeta(q)=\zeta_q \in \theta(G)$, so that $\zeta(G) \subset \theta(G)$. This shows that topology field ζ is continuous. □

Corollary 4.8. *If (X,ζ) is a regular topological space, then the topology field ζ:$(X,\tau) \to (\tau(X), \theta_\tau)$ is continuous.*

Let $f : (X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map $f_* : (\tau(X), \theta_\tau) \to (\tau(Y), \theta_\eta)$ by $f_*(w)=\{U \subset Y | f^{-1}(U) \in w\}$, then $f_*(0) = 0$ and $f_*(1) = 1$. Let $\omega \in \tau(X)$. For any subbasic open neighborhood $\theta(G)$ of $f_*(\omega)$ in $(\tau(Y), \theta_\eta)$, where G is open in (Y,η), G is θ-open in $(Y, f_*(\omega))$. By Theorem1.4 $f^{-1}(G)$ is θ-open in (X,ω). Thus $\omega \in \theta(f^{-1}(G))$. Hence $\theta(f^{-1}(G))$ is an open neighborhood of ω in $(\tau(X), \theta_\tau)$.

Now we will prove that $f_*(\theta(f^{-1}(G))) \subset \theta(G)$. Let $\zeta \in \theta(f^{-1}(G))$. Then $f^{-1}(G)$ is θ-open in (X,ζ). Since naturally the map $f : (X,\zeta) \to (Y, f_*(\zeta))$ is continuous, G is θ-open in $(Y, f_*(\zeta))$. This implies that $f_*(\zeta) \in \theta(G)$. Thus we have

Theorem 4.9. *Let f:$(X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map f_*:$(\tau(X), \theta_\tau) \to (\tau(Y), \theta_\eta)$ by $f_*(w)=\{U \subset Y | f^{-1}(U) \in w\}$, then the map f_* is continuous. If $\gamma \leq \delta$, then $f_*(\gamma) \leq f_*(\delta)$ and $f_*(\tau) \geq \eta$. And for any θ topology field ζ, the diagram*

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{f} & (Y,\eta) \\ \downarrow \zeta & & \downarrow f_*(\zeta) \\ (\tau(X),\theta_\tau) & \xrightarrow{f_*} & (\tau(Y),\theta_\eta) \end{array}$$

commutes. Furthermore, if (Z, λ) is a topological space and g: $(Y, \eta) \to (Z, \lambda)$ is a map, then

$$(g \circ f)_* = g_* \circ f_*.$$

Finally, if f:$(X ,\tau) \to (X, \tau)$ is the identity homeomorphism, then so is f_*.

Proof. The continuity of the map f_*:$(\tau(X), \theta_\tau) \to (\tau(Y), \theta_\eta)$ was proved already. And the commutativity of the diagram follows from the next fact.

$$\begin{aligned} f_*(\zeta_p) &= \{U|f^{-1}(U) \in \zeta_p\} \\ &= \{U|p \in f^{-1}(U) \in \zeta\} \\ &= \{U|f(p) \in U, f^{-1}(U) \in \zeta\} \\ &= \{U|U \in f_*(\zeta), f(p) \in U\} \\ &= f_*(\zeta)_{f(p)}. \end{aligned}$$

All other statements follow directly from the definitions. □

Additionally, if f is open and closed and $\omega \in \theta(f^{-1}(G))$, then $f^{-1}(G)$ is θ-open in (X, ω). By the Theorem 1.5, G is θ-open in $(X, f_*(\omega))$, i.e. $f_*(\omega) \in \theta(G)$. That is, $\omega \in f_*^{-1}(\theta(G))$. Consequently we have the following theorem.

Theorem 4.10. *If $f : (X, \tau) \to (Y, \eta)$ is a continuous and open and closed surjective map, then for any open G in Y*

$$f_*^{-1}(\theta(G)) = \theta(f^{-1}(G)).$$

Let (X, τ) and (Y, ζ) be topological spaces. We may assume that $\tau(X)$ and $\tau(Y)$ are given the topologies θ_τ and θ_ζ respectively and assume that $\tau(X \times Y)$ is given topology $\theta_{\tau\times\zeta}$. Next theorem is the result.

Theorem 4.11. *The multiplication $\times : \tau(X) \times \tau(Y) \to \tau(X \times Y)$ is continuous.*

Proof. Let $(\alpha, \beta) \in \tau(X) \times \tau(Y)$. Then $\alpha \times \beta \in \tau(X \times Y)$. If $\theta(W)$ is a neighborhood of $\times(\alpha, \beta) = \alpha \times \beta$, where W is open in $(X \times Y, \tau \times \zeta)$. Then we may assume that $W = W_X \times W_Y$ is basic open set in $(\tau(X \times Y), \tau \times \zeta)$. Since projection maps are open maps, $\pi_X(W) = W_X$ and $\pi_Y(W) = W_Y$ are also open sets in (X, τ) and (Y, ζ) respectively. Since W is θ-open in $(\tau(X \times Y), \alpha \times \beta)$, projection maps W_X and W_Y are θ-opens in (X, α) and (Y, β) respectively. Hence $(\alpha, \beta) \in \theta(W_X) \times \theta(W_Y)$. Moreover $\times(\theta(W_X) \times \theta(W_Y)) \subset \theta(W)$. In fact, if $\delta \in \theta(W_X)$ and $\gamma \in \theta(W_Y)$, then W_X is θ-open in (X, δ) and W_Y is θ-open in (Y, γ). Since the product of θ-opens is θ-open [6], $W = W_X \times W_Y$ is θ-open in $(X \times Y, \delta \times \gamma)$. Therefore $\delta \times \gamma \in \theta(W)$. This completes the proof. □

Hence we have

Theorem 4.12. *Let (X, τ) and (Y, ζ) be topological spaces. Then*

$$\tau_\theta \times \zeta_\theta = (\tau \times \zeta)_\theta.$$

Consequently we have the following commutative diagram:

$$\begin{array}{ccc} \tau(X)\times\tau(Y) & \xrightarrow{\times} & \tau(X\times Y) \\ \downarrow\theta\times\theta & & \downarrow\theta \\ \tau(X)\times\tau(Y) & \xrightarrow{\times} & \tau(X\times Y). \end{array}$$

Proof. Let $U\times V\in\tau_\theta\times\zeta_\theta$. Then U, V are θ-open sets in (X,τ), (Y,ζ) respectively. By the Theorem 5 in [6], $U\times V$ is a θ-open set in $(X\times Y,\tau\times\zeta)$. Hence $U\times V\in(\tau\times\zeta)_\theta$. Conversely if W is a θ-open in $(X\times Y,\tau\times\zeta)$. Then projection $\pi_X(W)$ and $\pi_Y(W)$ are θ-opens in (X,τ) and (Y,ζ) respectively. This completes the proof. □

Again we consider Θ as a map from $\tau(X)$ to $\tau(\tau(X))$ defined by $\Theta(\eta)=\theta_\eta$, then we have next result:

Theorem 4.13. *Let (X,τ) be a topological space. Then the induced map*

$$\Theta:(\tau(X),\Upsilon)\to(\tau(\tau(X)),\Upsilon)$$

is continuous.

Proof. Let $\zeta\in\tau(X)$ and K is a neighborhood of $\Theta(\zeta)=\theta_\zeta$ where K is upper set in $\tau(\tau(X))$. On the other hand the upper set $\uparrow(\zeta)$ in $\tau(X)$ is a neighborhood of ζ. We will show that $\Theta(\uparrow(\zeta))\subset K$. Let $\delta\in\uparrow(\zeta)$. Then $\delta\geq\zeta$ and $\theta_\delta\geq\theta_\zeta$. Hence we have $\theta_\delta\in K$. □

The map $\Theta:(\tau(X),\theta_\tau)\to(\tau(X),\theta_\tau)$ will be called Θ-operator from $\tau(X)$ to $\tau(\tau(X)$.

Corollary 4.14. *Let (X,τ) be a topological space. Then the induced map*

$$\Theta:(\tau(X),\theta_\tau)\to(\tau(\tau(X)),\theta_{\theta_\tau})$$

is continuous.

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Department of Mathematics
Kookmin University
Seoul 136-702, Republic of Korea
E-mail: kimjr@kookmin.ac.kr