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LIMITING BEHAVIOR OF THE MAXIMUM OF THE PARTIAL SUM FOR NEGATIVELY SUPERADDITIVE DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, some L_p-convergences and complete convergences of the maximum of the partial sum for negatively superadditive dependent random variables are obtained. The proofs of the results are based on a new Rosenthal type inequality concerning negatively superadditive dependent random variables.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The concept of negatively associated(NA) random variables was introduced by Joag-Dev and Proschan(1983) as follows: A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated if for every pair of disjoint subsets $A, B \subset \{1, 2, \cdots, n\}$, $Cov(f(X_i, i \in A)g(X_j, j \in B)) \leq 0$, whenever f and g are coordinatewise nondecreasing functions such that this covariance exists. An infinite family of random variables is NA if every subfamily is NA.

The concept of negatively superadditive dependent(NSD) random variables was introduced by Hu(2000) based on the class of superadditive functions. Superadditive structure functions have important reliability interpretations, which describe whether a system is more series-like or more parallel-like(See Block et al.(1989).)

Definition 1.1. (Kemperman, 1977) A function $\phi : R^n \to R$ is called superadditive if

$$\phi(\mathrm{x} \vee \mathrm{y}) + \phi(\mathrm{x} \wedge \mathrm{y}) \geq \phi(\mathrm{x}) + \phi(\mathrm{y}) \text{ for all } \mathrm{x}, \mathrm{y} \in \mathrm{R}^{\mathrm{n}}, \tag{1.1}$$

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where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

DEFINITION 1.2. (Hu, 2000) A random vector $X = (X_1, X_2, \cdots, X_n)$ is said to be negatively superadditive dependent(NSD) if

$$E\phi(X_1, X_2, \cdots, X_n) \leq E\phi(X_1^*, X_2^*, \cdots, X_n^*), \tag{1.2}$$

where $X_1^*, X_2^*, \cdots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1.2) exist.

Hu(2000) gave an example illustrating that negatively superadditive dependence does not imply negative association, and Hu posed an open problem whether NA implies NSD. Christofiedes and Vaggelatou(2004) solved this open problem and indicated that NA implies NSD. Negatively superadditive dependent structure is an extension of negatively associated structure and sometimes more useful than negatively associated structure.

Moreover, we can get many important probability inequalities for NSD random variables. For example, the structure function of a monotone coherent system can be superadditive(See Block et al.(1989).), so inequalities derived from NSD can give one-side or two-side bounds of the system reliability. The notion of NSD random variables has wide applications in multivariate statistical analysis and reliability theory.

Eghbal et al.(2010) derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables under the assumption that $\{X_i, i \geq 1\}$ is a sequence of nonnegative NSD random variables with $EX_i^r < \infty$ for all $i \geq 1$ and some $r > 1$. Eghbal et al.(2011) provided some Kolmogorov inequality for quadratic forms $T_n = \sum_{1 \leq i < j \leq n} X_i X_j$ and weighted quadratic forms $Q_n = \sum_{1 \leq i < j \leq n} a_{ij} X_i X_j$, where $\{X_i, i \geq 1\}$ is a sequence of nonnegative NSD uniformly bounded random variables. Shen et al.(2013 a) obtained the Khintchine-Kolmogo rov convergence theorem and strong stability for NSD random variables and Shen et al.(2013 b) studied the Marcinkiewicz-type strong law of large numbers and Hajeck-Renyi type inequality for NSD random variables.

In this paper we prove some L_p-convergences and complete convergences of the maximum of the partial sum for NSD random variables by using a new Rosenthal-type inequality for NSD random variables. The results are generalizations of some corresponding ones for NA random variables.

2. Preliminaries

In this section, we will provide some lemmas, which will be used to prove the main results.

LEMMA 2.1. (Hu, 2000) *Let* $(X_1, X_2, \cdots, X_n)$ *be NSD. Then*

(i) $(-X_1, -X_2, \cdots, -X_n)$ *is NSD,*

(ii) $(g_1(X_1), g_2(X_2), \cdots, g_n(X_n))$ *is NSD, where* $g_1, g_2, \cdots, g_n$ *are all nondecreasing functions.*

REMARK 2.2. Let $(X_1, X_2, \cdots, X_n)$ be NSD. Then, $(g_1(-X_1), g_2(-X_2), \cdots, g_n(-X_n))$ is NSD, where $g_1, g_2, \cdots, g_n$ are all nondecreasing functions.

The next lemma is the Rosenthal-type maximal inequality for NSD random variables obtained by Hu(2000) or Wang et al.(2013).

LEMMA 2.3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of NSD random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$ *for all* $n \geq 1$ *some* $p \geq 2$. *Then there exists a positive constant* C_p *depending only on* p *such that for each* $n \geq 1$

$$E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i|^p) \leq C_p\{\sum_{i=1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{p/2}\}. \tag{2.1}$$

The following definition of stochastic domination will play an important role in the paper.

DEFINITION 2.4. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive C such that

$$P(|X_n| > x) \leq CP(|X| > x) \tag{2.2}$$

for all $x \geq 0$ and $n \geq 1$.

By the definition of stochastic domination, we can get the following property for stochastic domination. For the proof, one can refer to Wu(2006) and Shen(2013). Here, we omit the details.

LEMMA 2.5. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables which is stochastically dominated by a random variable* X. *For any* $\alpha > 0$ *and* $b > 0$ *we have*

(i) $E|X_n|^\alpha I(|X_n| \leq b) \leq C_1\{E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\}$,

(ii) $E|X_n|^\alpha I(|X_n| > b) \leq C_2\{E|X|^\alpha I(|X| > b)\}$,

where C_1 *and* C_2 *are positive constants.*

From Lemma 2.4 we also have

$$E|X_n|^\alpha \le CE|X|^\alpha, \tag{2.3}$$

where C is a positive constant.

Finally, we give a lemma which will be used to prove the complete convergence of the maximum of the partial sum for NSD random variables.

Lemma 2.6. (Yuan et al. 2010) *For sequences $\{a_n,\ n \ge 1\}$ and $\{b_n, n \ge 1\}$ of nonnegative real numbers, if*

$$\sup_{n\ge1} n^{-1}\sum_{i=1}^n a_i < \infty \text{ and } \sum_{n=1}^\infty b_n < \infty,$$

then

$$\sum_{i=1}^n a_i b_i \le \Big(\sup_{m\ge1} m^{-1}\sum_{i=1}^m a_i\Big)\sum_{i=1}^n b_i$$

for every $n \ge 1$.

Proof. Let $\{a_i^{'}, 1 \le i \le n\}$ and $\{b_i^{'}, 1 \le i \le n\}$ respectively be the rearrangements of $\{a_i, 1 \le i \le n\}$ and $\{b_i, 1 \le i \le n\}$ satisfying $a_1^{'} \ge a_2^{'} \ge \cdots \ge a_n^{'}$ and $b_1^{'} \ge b_2^{'} \ge \cdots \ge b_n^{'}$. Then $\sum_{i=1}^n a_i b_i \le \sum_{i=1}^n a_i^{'} b_i^{'}$. So without loss of generality, one can assume that $\{a_i, 1 \le i \le n\}$ and $\{b_i, 1 \le i \le n\}$ are nonincreasing. By applying Remark 3(i) in Landers and Rogge(1997), the rest of the proof can be completed if we note the monotonicity of $\{a_n\}$ and $\{b_n\}$. □

3. Results

In Theorem 3.1, we prove L_p-convergence of the maximum of the partial sum for NSD random variables.

Theorem 3.1. *Let $p \ge 2$. Suppose that $\{X_n, n \ge 1\}$ is a sequence of NSD random variables satisfying*

$$\sup_{n\ge1} \frac{1}{n}\sum_{i=1}^n E|X_i|^p < \infty. \tag{3.1}$$

Then for any $\delta > \frac{1}{2}$

$$n^{-\delta}\max_{1\le i\le n}|S_i - ES_i| \to 0 \text{ in } \mathrm{L_p}, \tag{3.2}$$

where $S_n = X_1 + \cdots + X_n$.

Proof. As the proof of Theorem 3.2 of Yuan and Wu(2000) by Lemma 2.2 and the Hölder inequality, we have for $p \geq 2$

$$\begin{aligned}
&E(n^{-\delta}\max_{1\leq i\leq n}|S_i - ES_i|)^p \\
&\ll n^{-p\delta}\sum_{i=1}^{n} E|X_i - EX_i|^p + n^{-p\delta}(\sum_{i=1}^{n} E(X_i - EX_i)^2)^{p/2} \\
&\ll n^{-p\delta}\sum_{i=1}^{n} E|X_i - EX_i|^p + n^{-p\delta}(\sum_{i=1}^{n} EX_i^2)^{p/2} \\
&\leq n^{-p\delta}\sum_{i=1}^{n} E|X_i|^p + n^{-p\delta+(p/2)-1}\sum_{i=1}^{n}(EX_i^2)^{p/2} \\
&\ll n^{-p\delta+(p/2)-1}\sum_{i=1}^{n} E|X_i|^p \\
&\leq n^{-p(\delta-1/2)}\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n} E|X_i|^p \to 0
\end{aligned}$$

by (3.1) and an assumption $\delta > \frac{1}{2}$. Hence, the proof is complete. □

COROLLARY 3.2. *Let $p \geq 2$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of mean zero NSD random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$. Then, for any $\delta > \frac{1}{2}$*

$$n^{-\delta}\max_{1\leq i\leq n}|S_i| \to 0 \text{ in } \mathrm{L_p},$$

where $S_n = X_1 + \cdots + X_n$.

Proof. By (2.1), (2.3), Hölder inequality and assumptions that mean zero and $E|X|^p < \infty$ we have for $p \geq 2$

$$\begin{aligned}
E(n^{-\delta}\max_{1\leq i\leq n}|S_i|)^p &\ll n^{-p\delta}\sum_{i=1}^{n} E|X_i|^p + n^{-p\delta}(\sum_{i=1}^{n} EX_i^2)^{p/2} \\
&\ll n^{-p\delta}\sum_{i=1}^{n} E|X_i|^p + n^{-p\delta}(\sum_{i=1}^{n} E(X_i^2)^{p/2}) \\
&\ll n^{-p\delta}(nE|X|^p) + n^{-p\delta}(nE|X|^p) \\
&= n^{1-p\delta}E|X|^p \to 0
\end{aligned}$$

since $1 - p\delta < 0$. Hence, the proof is complete. □

The concept of complete convergence was introduced by Hsu and Robbins(1947) as follows. A sequence of random variables $\{U_n, n \geq 1\}$

is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|U_n - C| > \epsilon) < \infty$ for all $\epsilon > 0$.

Inspired by Yuan and Wu(2010) we obtain the complete convergence of the maximum of the partial sum for NSD random variables as follows:

THEOREM 3.3. *Let $p \geq 2$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of NSD random variables satisfying*

$$\sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} E|X_i|^p < \infty. \tag{3.3}$$

Then, for any $\delta > \frac{1}{2}$

$$n^{-\delta} \max_{1\leq i\leq n} |S_i - ES_i| \to 0 \text{ completely.} \tag{3.4}$$

Proof. For each $n \geq 1$, let $m = m_n$ be the integer such that

$$2^{m-1} < n \leq 2^m.$$

Observe that

$$\begin{aligned} n^{-\delta} \max_{1\leq i\leq n} |S_i - ES_i| &\leq (2^{m-1})^{-\delta} \max_{1\leq i\leq 2^m} |S_i - ES_i| \\ &\leq (2^{m-1})^{-\delta} \max_{1\leq i\leq 2^m} |S_i - ES_i| \\ &= 2^{\delta} \cdot 2^{-m\delta} \max_{1\leq i\leq 2^m} |S_i - ES_i|. \end{aligned}$$

Hence, it suffices to prove that

$$2^{-m\delta} \max_{1\leq i\leq 2^m} |S_i - ES_i| \to 0 \text{ completely.} \tag{3.5}$$

By Lemma 2.2, Markov inequality, the Hölder inequality and the fact that $(-m) + mp/2 > 0$ we have for all $\epsilon > 0$

$$\begin{aligned} &\sum_{m=0}^{\infty} P(2^{-m\delta} \max_{1\leq i\leq 2^m} |S_i - ES_i| > \epsilon) \\ &\leq \epsilon^{-p} \sum_{m=0}^{\infty} E(2^{-mp\delta} \max_{1\leq i\leq 2^m} |S_i - ES_i|^p) \\ &\ll \sum_{m=0}^{\infty} 2^{-mp\delta} \sum_{i=1}^{2^m} E|X_i - EX_i|^p + \sum_{m=0}^{\infty} 2^{-mp\delta} (\sum_{i=1}^{2^m} E(X_i - EX_i)^2)^{p/2} \\ &\ll \sum_{m=0}^{\infty} 2^{-mp\delta} \sum_{i=1}^{2^m} E|X_i|^p + \sum_{m=0}^{\infty} 2^{-mp\delta} (\sum_{i=1}^{2^m} EX_i^2)^{p/2} \end{aligned}$$

$$\ll \sum_{m=0}^{\infty} 2^{-mp\delta} (\sum_{i=1}^{2^m} EX_i^2)^{p/2}$$
$$\ll \sum_{m=0}^{\infty} 2^{-mp\delta - m + mp/2} \sum_{i=1}^{2^m} E|X_i|^p$$
$$\ll \sum_{i=1}^{\infty} E|X_i|^p \sum_{\{m:2^m \geq i\}} 2^{-mp\delta - m + mp/2}$$
$$\ll \sum_{i=1}^{\infty} i^{-p\delta - 1 + p/2} E|X_i|^p.$$

From Lemma 2.5 we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} i^{-p\delta-1+\frac{p}{2}} E|X_i|^p &\ll \sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} E|X_i|^p \cdot \sum_{n=1}^{\infty} n^{-(p\delta+1-\frac{p}{2})} \\ &\ll \sum_{n=1}^{\infty} n^{-(p\delta+1-\frac{p}{2})} < \infty \end{aligned}$$

since $p\delta + 1 - \frac{p}{2} > 1$. Therefore (3.5) holds and the proof of Theorem 3.3 is complete. □

Corollary 3.4. *Let $p \geq 2$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of mean zero NSD random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$. Then, for any $\delta > \frac{1}{2}$*

$$n^{-\delta} \max_{1\leq i\leq n} |S_i| \to 0 \text{ completely},$$

where $S_n = X_1 + \cdots + X_n$.

Proof. For each $n \geq 1$, let $m = m_n$ be the integer such that

$$2^{m-1} < n \leq 2^m.$$

By Lemma 2.2, (2.3), the Markov inequality and the Hölder inequality we obtain

$$\sum_{m=0}^{\infty} P(2^{-m\delta} \max_{1\leq i\leq 2^m} |S_i| > \epsilon)$$
$$\leq \epsilon^{-p} \sum_{m=0}^{\infty} E(2^{-mp\delta} \max_{1\leq i\leq 2^m} |S_i|^p)$$
$$\ll \sum_{m=0}^{\infty} 2^{-mp\delta} \sum_{i=1}^{2^m} E|X_i|^p + \sum_{m=0}^{\infty} 2^{-mp\delta} (\sum_{i=1}^{2^m} EX_i^2)^{p/2}$$

$$\ll \sum_{m=0}^{\infty} 2^{-mp\delta+\frac{mp}{2}-m} \sum_{i=1}^{2^m} E|X_i|^p$$
$$\ll \sum_{m=0}^{\infty} 2^{-mp\delta+\frac{mp}{2}} 2^m \cdot E|X|^p$$
$$= E|X|^p \sum_{m=0}^{\infty} 2^{-mp(\delta-\frac{1}{2})} < \infty$$

since $(\delta - \frac{1}{2}) > 0$.

Hence, the proof is complete. □

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