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PERIODICAL EXPANSIVENESS FOR C^1-GENERIC DIFFEOMORPHISMS

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Abstract. C^1-generically, if a transitive diffeomorphism f is periodically expansive, then it is hyperbolic.

1. Introduction

Let M be a $n(\geq 2)$-dimensional closed C^∞ Riemannian manifold without boundary, and $\mathrm{Diff}(M)$ be a space of diffeomorphisms of M. Let $f \in \mathrm{Diff(M)}$, and Λ be a closed f-invariant set in M. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. We say that Λ is *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then f is called Anosov. It is well-known that if Λ is hyperbolic, then it admits a dominated splitting.

We denote $Orb_f(x)$ by the orbit of x under f, $P(f)$ by the set of periodic points for f, and $P_h(f)$ by the set of hyperbolic periodic points for f.

For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}(-\infty \leq a < b \leq \infty)$ in M is called a *δ-pseudo orbit* of f if $d(f(x_i),\ x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. Given $f \in \mathrm{Diff}(M)$, a closed f-invariant set $\Lambda \subset M$ is said to be *chain*

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transitive if for any points $x, y \in \Lambda$ and $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a_\delta}^{b_\delta} \subset \Lambda$ of f such that $x_{a_\delta} = x$ and $x_{b_\delta} = y$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a}^{b} \subset \Lambda(a < b)$ of f such that $x_a = x$ and $x_b = y$. Write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set of points $\{x \in M : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. The relation $\leftrightsquigarrow$ on $\mathcal{R}(f)$ induces an equivalence relation, whose classes are called *chain components* of f. Every chain component of f is a closed f-invariant set and is denoted by $C_f(p)$ for some hyperbolic periodic point p.

It is well known that if p is a hyperbolic periodic point of f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \quad \text{as} \quad n \to \infty\}, \text{ and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \quad \text{as} \quad n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. We denote the index of $p = \dim W^s(p)$.

Every point in the transversal intersection $W^s(p) \pitchfork W^u(p)$ of $W^s(p)$ and $W^u(p)$ is called the *homoclinic point of f associated to p*. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f and it is denoted by $H_f(p)$. If $p \in P(f)$ is either a source or a sink, then $H_f(p) = Orb_f(p)$.

Note that the homoclinic class $H_f(p)$ is a subset of the chain component $C_f(p)$ of f containing p.

A compact invariant set $\Lambda \subset M$ is called *transitive* for f if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x) = \{y \in M : f^{n_i}(x) \to y \text{ as } n_i \to +\infty\}$ which is called the *ω-limit set of x*. Moreover, we say that f is a *transitive diffeomorphism* if M is transitive for f.

We say that $p, q \in P_h(f)$ are *homoclinically related* if $W^s(p)$ has a transversal intersection with $W^u(p)$ and $W^u(p)$ has a transversal intersection with $W^s(q)$. Then we denote this by $p \sim q$.

In the middle of the twenty century, the notion of expansiveness was introduced by Utz [8]. A diffeomorphism f is called *expansive* if there is $\delta > 0$ such that for any distinct $x, y \in M$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \delta$. In dynamical systems, it has been studied of many types of expansiveness such as entropy expansive, G-expansive, continuum-wise expansive, pointwise expansive, measure expansive, n-expansive, measure sensitive and so on. In this paper, we introduce the *periodically expansiveness* by Fakhari [4] as follows.

Definition 1.1. Let $p \in P_h(f)$. The homoclinic class $H_f(p)$ of a diffeomorphism f is *periodically expansive* if there is $\delta > 0$ such that for any q homoclinically related to p and any $x \in H_f(p)$, if $d(f^n(q), f^n(x)) < \delta$ for all $n \in \mathbb{Z}$, then $q = x$. And the constant δ is called the *expansive constant.*

If f is an expansive homeomorphism on $H_f(p)$, then f is clearly periodically expansive on $H_f(p)$ and hence any hyperbolic homoclinic class is periodically expansive.

We say that a subset $\mathcal{R} \subset$ Diff(M) is a residual subset if it contains a countable intersection of open dense sets. The finite intersection of residual subsets is a residual subset. Since Diff(M) is a Baire space when it is equipped with the C^1-topology, any residual subset of Diff(M) is dense. We will say that a property holds *generically* if there exists a residual subset $\mathcal{R}$ such that any $f \in \mathcal{R}$ has that property. Sometimes, we will say that a diffeomorphism f is *generic* when we refer that f could be taken in a residual subset. We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$.

Now, we introduce the main theorem of this paper.

Theorem 1.2. *For C^1-generic f, if a transitive diffeomorphism f is periodically expansive, then it is Anosov.*

2. Proof of main theorem 1.2

To prove Theorem 1.2, we need some lemmas.

Lemma 2.1. *For a transitive diffeomorphism f, M is equal to the chain component $C_f(p)$ of f for all $p \in P_h(f)$.*

Proof. Let $C_f(p)$ be the chain component of a hyperbolic periodic point p for f. Clearly $C_f(p) \subset M$, it is enough to show that $M \subset C_f(p)$. Let $y \in M$. Since M is transitive, for any $\delta > 0$, there is $x \in M$ and $n_1 > 0, n_2 > 0$ such that $d(f(y), f^{-n_1}(x)) < \delta$ and $d(f^{n_2}(x), p) < \delta$. Then we can construct a δ-pseudo orbit ξ_1 from y to p as follows:

$$\xi_1 = \{y, f^{-n_1}(x), f^{-(n_1-1)}(x), \cdots, f^{-1}(x), x, f(x), \cdots, f^{n_2-1}(x), p\}.$$

Similarly, we can construct a δ-pseudo orbit ξ_2 from p to y. So that $y \in C_f(p)$. □

Lemma 2.2. [3] *There is a residual set $\mathcal{G}_1 \subset$ Diff(M) such that for $f \in \mathcal{G}_1$, every chain component $C_f(p)$ of f is equal to the homoclinic class $H_f(p)$ of f, where p is a hyperbolic periodic point of f.*

From above two lemmas, we know that $M = C_f(p) = H_f(p)$ for a C^1-generic transitive diffeomorphism f and $p \in P_h(f)$.

Lemma 2.3. *There is a residual set $\mathcal{G}_2 \subset \mathrm{Diff}(M)$ such that for a periodically expansive transitive diffeomorphism $f \in \mathcal{G}_2$, every point q in $H_f(p) \cap P(f)$ is hyperbolic.*

Proof. There is a residual set $\mathcal{G}_2$ of $\mathrm{Diff}(M)$ such that for every transitive diffeomorsphism $f \in \mathrm{Diff}(M)$, every periodic point of f is hyperbolic and all their invariant manifolds are intersect transversely (Kupka-Smale). Since $H_f(p) = M$, $H_f(p) \cap P(f) = M \cap P(f) = P(f)$. Then by the Kupka-Smale property, every $q \in P(f)$ is hyperbolic. □

Remark 2.4. *Let f be a diffeomorphism on M and p be a hyperbolic periodic point of f. We know that $H_f(p)$ is eqaul to the set $\overline{\{q \in P_h(f) : q \sim p\}}$. So it is also well-known that $H_f(p) = \overline{P(f) \cap H_f(p)}$.*

Now, we recall some definitions and notions. The support of a measure μ is denoted by $\mathrm{supp}(\mu)$. Let

$$\mathcal{M}_f(M) = \{\mu : \mu \text{ is an } f\text{-invariant Borel probability measures on } M \text{ such that } \mathrm{Supp}(\mu) \subset \Lambda\},$$

endowed with the weak topology.

Let $Orb_f(p)$ be a periodic orbit of f and let $p \in Orb_f(p)$ be a periodic point of f of with a period $\pi(p)$. Then its associated *ergodic measure* μ_p is defined by

$$\mu_p = \frac{1}{\pi(p)} \sum_{i=0}^{\pi(p)-1} \delta_f^i(p).$$

Lemma 2.5 (Mañé's Ergodic General Density Theorem, [5]). *For any C^1-generic diffeomorphism f, the convex hull of periodic measures is dense in $\mathcal{M}_f(M)$.*

Lemma 2.6. [6] *Let Λ be a compact invariant set of $f \in \mathrm{Diff}(M)$ and E be a continuous invariant subbundle. If there is $m > 0$ such that*

$$\int \log \|(Df^m)|_E\| d\mu < 0$$

for any f-invariant ergodic measure μ, then E is contraction.

Remark 2.7. *There is a residual set $\mathcal{G}_3 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_3$ and any hyperbolic periodic point p of f, for any ergodic measure μ of f, there is a sequence of periodic points p_n such that $\mu_{p_n} \to \mu$ in the weak topology and $Orb_f(p_n) \to \mathrm{Supp}(\mu)$ in Housdorff metric.*

To end the proof of Theorem 1.2, it is enough to show the following lemma.

LEMMA 2.8. *There is a residual set $\mathcal{G}_4 \subset \text{Diff}(M)$ such that for any transitive diffeomorphism $f \in \mathcal{G}_4$, if f is a periodically expansive diffeomorphism, then f is transitive Anosov.*

Proof. Let $\mathcal{G}_4 = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3$. Then we see that for a C^1-generic transitive diffeomorphism $f \in \mathcal{G}_4$, M becomes the homoclinic class $H_f(p)$ for some $p \in P_h(f)$.

First of all, we show that any periodic point belongs to a periodically expansive homoclinic class $H_f(p)$ of $f \in \mathcal{G}_4$ has the constant index as the index of p. Note that for any periodic point $q \in H_f(p)$ with $\text{index}(q) = \text{index}(p)$, we have $q \sim p$. Since $H_f(p)$ is periodically expansive, $\text{index}(q) = \text{index}(p)$ for all $q \in H_f(p)$.

From Lemma 2.3, for any periodically expansive diffeomorphisms f, M has a dominated splitting $E \oplus F$.

Let μ be an ergodic measure supported on $H_f(p)$. Let p_n be the sequence of periodic points given by Remark 2.7. Then, we have

$$\int \|Df|_E\| d\mu = \lim_{n\to\infty} \int \|Df|_E\| d\mu_{p_n} < 0.$$

Thus E is contraction. Similarly, the bundle F is expansion, and we complete the proof. □

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