

MEASURE EXPANSIVENESS OF CIRCLE HOMEOMORPHISMS

SANGJIN KIM* AND WOOCHUL JUNG**

ABSTRACT. In this paper we introduce the notions of expansiveness and measure expansiveness for homeomorphisms on a compact metric space, and we study the measure expansiveness of circle homeomorphisms.

1. Introduction

In the middle of the twenty century, the notion of expansiveness was introduced by Utz [?]. Roughly speaking a system is expansive if two orbits cannot remain close to each other under the action of the system.

This notion is very important in the context of the theory of dynamical systems. For instance, it is responsible for many chaotic properties for homeomorphisms defined on compact spaces.

As pointed out by Morales [?], in light of the rich consequences of expansiveness in the dynamics of a system, it is natural to consider other notions of expansiveness. In this paper we introduce a notion generalizing the usual concept of expansiveness which is called *measure expansive*. Moreover, we check a variety of the theorem of expansive measures for homeomorphisms and examples in S^1 .

Let X be a compact metric space with a metric d , and let f be a homeomorphism from X to X . First of all, we recall the notion of expansiveness.

DEFINITION 1.1. A homeomorphism $f : X \rightarrow X$ is called *expansive* if there is $\delta > 0$ such that for any distinct points $x, y \in X$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \delta$.

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Correspondence should be addressed to Sangjin Kim, sjkim0508@cnu.ac.kr.

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Given $x \in X$ and $\delta > 0$, we define the *dynamical δ -ball at x* ,

$$\Gamma_\delta^f(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \in \mathbb{Z}\}.$$

(Denote $\Gamma_\delta(x)$ for simplicity if there is no confusion.) Then we see that f is expansive if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for all $x \in X$.

Let β be the Borel σ -algebra on X . Denote by $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with weak* topology. We say that $\mu \in \mathcal{M}(X)$ is *atomic* if there exists a point $x \in X$ such that $\mu(\{x\}) > 0$. Let $\mathcal{M}^*(X) = \{\mu \in \mathcal{M}(X) : \mu \text{ is nonatomic}\}$, and let $\mathcal{M}_f^*(X) = \{\mu \in \mathcal{M}^*(X) : \mu \text{ is } f\text{-invariant}\}$.

DEFINITION 1.2. Let $\mu \in \mathcal{M}^*(X)$. A homeomorphism $f : X \rightarrow X$ is said to be *μ -expansive* (or μ is expansive for f) if there is $\delta > 0$ (Here, δ is called an expansive constant of μ with respect to f) such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in X$.

DEFINITION 1.3. A homeomorphism $f : X \rightarrow X$ is said to be *measure expansive* if there is $\delta > 0$ such that f is μ -expansive for all $\mu \in \mathcal{M}^*(X)$.

As we know, the expansivity implies the measure expansivity of a homeomorphism of a non atomic metric space. We can observe that Denjoy map has expansive measures but it is not expansive on S^1 .

2. Basic properties on circle homeomorphisms

In this section, we start to discuss the orientation preserving homeomorphism of the circle. So, we can define $f : S^1 \rightarrow S^1$ to be an orientation preserving homeomorphism if any lifting map F of homeomorphism f to the covering space \mathbb{R} is strictly increasing. By using the fact that the circle goes on itself, we can explain dynamical systems which have a unpredictable behavior. An important quantity which could make this determination possible is the rotation number. This number measures the average amount that a point is rotated by the homeomorphism. When this has the rational number p/q on S^1 , the homeomorphism has a periodic point with period q . When this has the irrational number, there are no periodic orbits.

DEFINITION 2.1. let $f : S^1 \rightarrow S^1$ be an orientation preserving homeomorphism. Then there is a homeomorphism $F : \mathbb{R} \rightarrow \mathbb{R}$ which is called a lifting map of f such that $\pi \circ F = f \circ \pi$.

For an irrational rotation number, it is important to consider whether $f : S^1 \rightarrow S^1$ is either transitive or nontransitive.

DEFINITION 2.2. A homeomorphism f is said to be transitive if it has a dense positive orbit in the circle.

Here, there is a significant theorem which is called the Poincaré Classification Theorem.

THEOREM 2.3. ([?], Theorem 11.2.7) Let $f : S^1 \rightarrow S^1$ be an orientation preserving homeomorphism with an irrational rotation number τ .

- (1) If f is transitive, then f is conjugate to the rotation $\mathbf{R}_\tau(f)$.
- (2) If f is not transitive, then f is semi-conjugate to the the rotation $\mathbf{R}_\tau(f)$ as a surjective continuous map $h : S^1 \rightarrow S^1$.

DEFINITION 2.4. A map $f : S^1 \rightarrow S^1$ is called a *Denjoy map* if f is nontransitive homeomorphism with an irrational rotation number $\rho(f)$.

$$\text{Here, } \rho(f, x) = \lim_{n \rightarrow \infty} \frac{1}{n}(F^n(x) - x) \text{ and}$$

$$F : \mathbb{R} \rightarrow \mathbb{R} \text{ is a lifting map of } f.$$

Recently, C. A. Morales [?] showed the following properties.

LEMMA 2.5. ([?], Theorem 1.35) A circle homeomorphism is expansive measures if and only if it is Denjoy.

COROLLARY 2.6. *There are no expansive homeomorphisms of S^1 .*

To explain the expansivity of S^1 , we need the following result that there are no expansive homeomorphisms of a compact interval in theorem 4 [?]. Next property is motivated to every homeomorphisms f of a compact metric space X one has that $supp(\mu) \subset \Omega(f)$ for all invariant Borel probability measures μ of f . By using the property that a homeomorphism of S^1 has no expansive measures supported on S^1 , we obtain the following one that there are no measure-expansive homeomorphisms of S^1 . This implies immediately that there are no expansive homeomorphisms of S^1 .

3. Measure expansiveness of circle homeomorphisms

DEFINITION 3.1. For any homeomorphisms $f : X \rightarrow X$ and $x \in X$, the α -limit set of x is defined by

$$\alpha(x) = \{y \in X \mid \text{There exists a sequence } (n_i) \text{ such that } f^{-n_i} \rightarrow y \text{ as } i \rightarrow \infty\}.$$

DEFINITION 3.2. A homeomorphism $f : X \rightarrow X$ of a metric space X is proximal if $\inf_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) = 0$ for every $x, y \in X$.

EXAMPLE 3.3. A homeomorphism $f : S^1 \rightarrow S^1$ is proximal.

Let $A_1 = [(\cos 0, \sin 0), (\cos \frac{\pi}{2}, \sin \frac{\pi}{2})]$ and $A_2 = [(\cos \frac{\pi}{2}, \sin \frac{\pi}{2}), (\cos(\frac{\pi}{2} + \frac{\pi}{4}), \sin(\frac{\pi}{2} + \frac{\pi}{4}))]$ be subsets of S^1 . Define $f_1 : A_1 \rightarrow A_2$ with $(\cdot) \mapsto (\cos(\frac{t}{2} + \frac{\pi}{2}), \sin(\frac{t}{2} + \frac{\pi}{2}))$. Then f_1 is homeomorphism on A_1 .

Let $A_n : [(\cos(\sum_{k=1}^{n-1} \frac{\pi}{2^k}), \sin(\sum_{k=1}^{n-1} \frac{\pi}{2^k})), (\cos(\sum_{k=1}^n \frac{\pi}{2^k}), \sin(\sum_{k=1}^n \frac{\pi}{2^k}))]$ for all $n \geq 1$, and $f_n : A_n \rightarrow A_{n+1}$ defined by $(\cos t, \sin t) \mapsto (\cos(\frac{t}{2} + \frac{\pi}{2}), \sin(\frac{t}{2} + \frac{\pi}{2}))$ for all $n \geq 2$. It is fact that f_n is a homeomorphism of A_n for all $n \in N$.

Define a homeomorphism $f : S^1 \rightarrow S^1$ by $f(x) = f_n(x)$ for all $n \in N$, for all $x \in A_n$. Then $f(A_n)$ converges to one point set $(1, 0)$, this implies that $d(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow \infty$. Thus, $f : S^1 \rightarrow S^1$ is a proximal homeomorphism of compact metric spaces without expansive measures.

THEOREM 3.4. If $f : S^1 \rightarrow S^1$ is a local homeomorphism of the circle S^1 , then the Lebesgue measure Leb on S^1 is expansive for f if and only if f is expansive.

Proof. It is enough to show that the expansiveness of a local homeomorphism f on S^1 satisfies the expansiveness for f . Then there exists a $\delta_1 > 0$ such that $Leb(\Gamma_{\delta_1}(x)) = 0$ for all $x \in S^1$. Since f is locally homeomorphism, let $\epsilon = 1/2$ there exists a $\delta_2 > 0$ such that $d(x, y) < \delta_2$ then $d(f(x), f(y)) < 1/2$. Suppose $S^1 - \{x, y\}$ consists of two connected component A and B. Then $Arc(x, y) = A$ if $L(A) < L(B)$ and $Arc(x, y) = B$ if $L(A) > L(B)$, where $L(A)$ and $L(B)$ are a length of connected component A and B, respectively.

So, $Diam(f(Arc(x, y))) = Arc(f(x), f(y)) < 1/2$. Let $\delta = \min\{\delta_1, \delta_2\}$, our claim is that $\Gamma_\delta(x) = \{x\}$, for all $x \in S^1$. Suppose $y \in \Gamma_\delta(x)$ (with $x \neq y$) i.e., $d(f^n(x), f^n(y)) < \delta < \delta_1$ for all $n \geq 0$. Let $z \in Arc(x, y)$, then $Diam(f(x, y)) < 1/2$, it satisfied $d(f(x), f(z)) < d(f(y), f(x)) < \delta$. Hence, $d(f(x), f(z)) < d(f(y), f(x)) < \delta$.

Similarly, $d(f^n(z), f^n(x)) < \delta$ then $z \in \Gamma_\delta(x)$. We know that $Arc(x, y) \subset \Gamma_\delta(x)$, then this means $Leb(\Gamma_\delta(x)) > 0$. Thus, $Leb(\Gamma_{\delta_1}(x)) > 0$, as $\Gamma_\delta(x) \subset \Gamma_{\delta_1}(x)$. This is contradiction to $Leb(\Gamma_{\delta_1}(x)) = 0$. \square

References

- [1] B. Hasselblatt and A. Katok, *Introduction to the Mordern Theory of Dynamical Systems*: With a supplementary chapter by Katok and Leonardo Mendoza, Encyclopedia of Mathematic and its Applycations **54**, Cambridge University Press, Cambridge, 1995.
- [2] C. A. Morales, *Measure-expansive systems*, preprint.

- [3] B. F. Bryant, *Expansive Self-Homeomorphisms of a Compact Metric Space*, Amer. Math. Monthly **69** (1962), no. 5, 386-391.
- [4] W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc. **1** (1950), 769-774.

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: sjkim0508@cnu.ac.kr

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: wcjung@cnu.ac.kr