KYUNGPOOK Math. J. 55(2015), 715-730
http://dx.doi.org/10.5666/KMJ.2015.55.3.715
pISSN 1225-6951 eISSN 0454-8124

Geometry of Energy and Bienergy Variations between Riemannian Manifolds

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Abstract. In this note, we extend the definition of harmonic and biharmonic maps via the variation of energy and bienergy between two Riemannian manifolds. In particular we present some new properties for the generalized stress energy tensor and the divergence of the generalized stress bienergy.

1. Introduction

Harmonic maps are critical points of the energy functional defined on the space of smooth maps between Riemannian manifolds. There are many studies on harmonic maps. Also, p-harmonic maps and exponentially harmonic maps have been developed by J. Eells and J. H. Sampson [10],[9] and biharmonic maps G. Y. Jiang [12], and E. Loubeau and C. Oniciuc [13]. The F-harmonic map has dveloped by F. Ara [1], recently the notion of f-harmonic maps has introduced by N. Course [5] and M. Djaa and S. Ouakkas [15],[8] and developed by Y. J. Chiang [3], Y. L. Ou [16], S. Feng [11] and other.

The goal in this work is the characterization of the Euler-Lagrange equations in the general case of Riemannian manifolds. First we establish the first energy variation (Theorem 3.1) and the second energy variation (Theorem 4.1). Our results are extensions of harmonic maps, p-harmonic and exponentially harmonic maps,

* Corresponding Author.
Received November 15, 2014; accepted May 13, 2015.
2010 Mathematics Subject Classification: 53A45, 53C20, 58E20.
Key words and phrases: Energy variation, L-harmonic maps, L-biharmonic maps, Stress L-bi-energy tensors.

biharmonic maps, F-harmonic maps, f-harmonic maps and and bi-f-harmonic maps. In particular we present some new properties for the generalized stress energy tensor (Theorem 5.1) and the divergence of the generalized stress bienergy (Theorem 5.2).

2. L-Harmonic Maps

Consider a smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds,

$$L : M \times N \times \mathbb{R} \to (0, \infty), \quad (x, y, r) \mapsto L(x, y, r), \tag{2.1}$$

be a smooth positive function, for any compact domain D of M the L-energy functional of φ is defined by

$$E_L(\varphi; D) = \int_D L\big(x, \varphi(x), e(\varphi)(x)\big)\, v_g, \tag{2.2}$$

where $e(\varphi)$ is the energy density of φ defined by

$$e(\varphi) = \frac{1}{2}\, h\big(d\varphi(e_i), d\varphi(e_i)\big), \tag{2.3}$$

v_g is the volume element, here $\{e_i\}$ is a orthonormal frame on (M, g).

Definition 2.1. A map is called L-harmonic if it is a critical point of the L-energy functional over any compact subset D of M.

3. The First Variation of the L-Energy Functional

Let $L : M \times N \times \mathbb{R} \to (0, \infty)$, $(x, y, r) \mapsto f(x, y, r)$, we denote by

$$\partial_r = \partial/\partial r, \quad L' = \partial_r(L), \quad L'' = \partial_r\big(\partial_r(L)\big)$$

and let L'_φ, $L''_\varphi \in C^\infty(M)$ defined by

$$L'_\varphi(x) = L'\big(x, \varphi(x), e(\varphi)(x)\big), \quad L''_\varphi(x) = L''\big(x, \varphi(x), e(\varphi)(x)\big). \tag{3.1}$$

Theorem 3.1. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt} E_L(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_L(\varphi), v)\, v_g, \tag{3.2}$$

where $v = \dfrac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ,

$$\tau_L(\varphi) = L'_\varphi\, \tau(\varphi) + d\varphi\big(\operatorname{grad}^M L'_\varphi\big) - (\operatorname{grad}^N L) \circ \varphi, \tag{3.3}$$

and $\tau(\varphi)$ *is the tension field of* φ *given by*

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi. \tag{3.4}$$

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \to N$ by

$$\phi(x,t) = \varphi_t(x), \quad (x,t) \in M \times (-\epsilon, \epsilon), \tag{3.5}$$

let ∇^{ϕ} denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon)$, we have

$$[\partial_t, X] = 0. \tag{3.6}$$

Using (2.2) we obtain

$$\frac{d}{dt} E_L(\varphi_t; D)\Big|_{t=0} = \int_D \partial_t \Big(L\big(x, \varphi_t(x), e(\varphi_t)(x)\big)\Big)\Big|_{t=0} v_g, \tag{3.7}$$

first, note that

$$\partial_t \Big(L\big(x, \varphi_t(x), e(\varphi_t)(x)\big)\Big)\Big|_{t=0} = dL\big(d\phi(\partial_t)\big)\Big|_{t=0} + dL\big(\partial_t(e(\varphi_t))\big)\Big|_{t=0}, \tag{3.8}$$

the first term on the left-hand side of (3.8) is

$$dL\big(d\phi(\partial_t)\big)\Big|_{t=0} = h\big((\operatorname{grad}^N L) \circ \varphi, v\big). \tag{3.9}$$

Calculating in a normal frame at $x \in M$, we have

$$\begin{aligned} \partial_t(e(\varphi_t)) &= h\big(\nabla^{\phi}_{\partial_t} d\varphi_t(e_i), d\varphi_t(e_i)\big) \\ &= h\big(\nabla^{\phi}_{e_i} d\phi(\partial_t), d\varphi_t(e_i)\big), \end{aligned} \tag{3.10}$$

the second term on the left-hand side of (3.8) is

$$\begin{aligned} dL\big(\partial_t(e(\varphi_t))\big)\Big|_{t=0} &= L'_{\varphi}\, h\big(\nabla^{\varphi}_{e_i} v, d\varphi(e_i)\big) \\ &= e_i\big(h(v, L'_{\varphi}\, d\varphi(e_i))\big) - h\big(v, \nabla^{\varphi}_{e_i} L'_{\varphi}\, d\varphi(e_i)\big), \end{aligned} \tag{3.11}$$

where the last equality holds since $d\phi(\partial_t)\Big|_{t=0} = v$, define a 1-form on M by

$$\omega(X) = h(v, L'_{\varphi}\, d\varphi(X)), \quad X \in \Gamma(TM), \tag{3.12}$$

by (3.11) and (3.12) we get

$$\begin{aligned} dL\big(\partial_t(e(\varphi_t))\big)\Big|_{t=0} &= \operatorname{div} \omega - h\big(v, d\varphi(\operatorname{grad}^M L'_{\varphi})\big) \\ &\quad - h\big(v, L'_{\varphi}\, \tau(\varphi)\big). \end{aligned} \tag{3.13}$$

Substituting (3.8), (3.9) and (3.13) in (3.7), and consider the divergence theorem, the Theorem 3.1 follows. □

From Theorem 3.1 and Definition 2.1, we obtain

Corollary 3.1. *A smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds, is L-harmonic if and only if*

$$\tau_L(\varphi) = L'_\varphi \, \tau(\varphi) + d\varphi\big(\operatorname{grad}^M L'_\varphi\big) - (\operatorname{grad}^N L) \circ \varphi = 0. \tag{3.14}$$

Remark 3.1. If $L(x, y, t) = F(t)$, then

$$\tau_F(\varphi) = \tau_F(\varphi) = F'\big(e(\varphi)\big)\, \tau(\varphi) + d\varphi\big(\operatorname{grad}^M F'\big(e(\varphi)\big)\big).$$

Theorem 3.2. *Let $\varphi : M \to N$ be a smooth map of two Riemannian manifolds and let $i : N \hookrightarrow P$ be the inclusion map of a submanifold, then φ is f-harmonic if and only if $\tau_f(i \circ \varphi)$ is normal to N, where $f \in C^\infty(M \times P \times \mathbb{R})$ be a smooth positive function.*

Proof. The L-tension field of the composition $i \circ \varphi : M \to P$ is given by

$$\tau_L(i \circ \varphi) \quad = \quad L'_{i\circ\varphi} \tau(i \circ \varphi) + di(d\varphi(\operatorname{grad}^M L'_{i\circ\varphi})) - (\operatorname{grad}^P L) \circ i \circ \varphi,$$

where $L'_{i\circ\varphi} : M \to (0, \infty)$ defined by

$$L'_{i\circ\varphi}(x) = L'(x, i(\varphi(x)), e(i \circ \varphi)(x)) = L'(x, \varphi(x), e(\varphi)(x)) = L'_\varphi(x),$$

for all $x \in M$, because the energy density of $i \circ \varphi$ is $e(\varphi)$, thus

$$\tau_L(i \circ \varphi) \quad = \quad L'_\varphi \tau(i \circ \varphi) + di(d\varphi(\operatorname{grad}^M L'_\varphi)) - (\operatorname{grad}^P L) \circ i \circ \varphi,$$

since the tension field of the composition $i \circ \varphi$ is given by

$$\tau(i \circ \varphi) = di(\tau(\varphi)) + \operatorname{trace} \nabla di(d\varphi, d\varphi),$$

and $(\operatorname{grad}^P L) \circ i \circ \varphi = di(\operatorname{grad}^N L) \circ \varphi + (\operatorname{grad}^P L)^\perp \circ i \circ \varphi$, we obtain

$$\begin{aligned}
\tau_L(i \circ \varphi) \quad &= \quad L'_\varphi di(\tau(\varphi)) + L'_\varphi \operatorname{trace} \nabla di(d\varphi, d\varphi) \\
&\qquad + di(d\varphi(\operatorname{grad}^M L'_\varphi)) - di(\operatorname{grad}^N L) \circ \varphi \\
&\qquad - (\operatorname{grad}^P L)^\perp \circ i \circ \varphi \\
&= \quad di(\tau_L(\varphi)) + L'_\varphi \operatorname{trace} \nabla di(d\varphi, d\varphi) \\
&\qquad - (\operatorname{grad}^P L)^\perp \circ i \circ \varphi.
\end{aligned}$$

So $\tau_L(i \circ \varphi) - di(\tau_L(\varphi))$ is normal to N. □

4. The Second Variation of the L-Energy Functional

Theorem 4.1. *Let $\varphi : (M, g) \to (N, h)$ be an f-harmonic map between Riemannian manifolds and $\{\varphi_{t,s}\}_{t,s\in(-\epsilon,\epsilon)}$ be a two-parameter variation with compact support in D. Set*

$$v = \frac{\partial\varphi_{t,s}}{\partial t}\Big|_{t=s=0}, \quad w = \frac{\partial\varphi_{t,s}}{\partial s}\Big|_{t=s=0}. \tag{4.1}$$

Under the notation above we have the following

$$\frac{\partial^2}{\partial t\partial s}E(\varphi_{t,s}; D)\Big|_{t=s=0} = \int_D h(J_{\varphi,L}(v), w)\, v_g, \tag{4.2}$$

where $J_{\varphi,L}(v) \in \Gamma(\varphi^{-1}TN)$ given by

$$\begin{aligned} J_{\varphi,L}(v) \;\; = \;\; & -L'_\varphi \text{ trace } R^N(v, d\varphi)d\varphi - \text{trace } \nabla^\varphi L'_\varphi \nabla^\varphi v \\ & +\big(\nabla^N_v \text{ grad}^N f\big) \circ \varphi + < \nabla^\varphi v, d\varphi > (\text{grad}^N L') \circ \varphi \\ & - \text{trace } \nabla^\varphi < \nabla^\varphi v, d\varphi > L''_\varphi\, d\varphi. \end{aligned} \tag{4.3}$$

Here $<,>$ denote the inner product on $T^*M \otimes \varphi^{-1}TN$ and R^N is the curvature tensor on (N, h).

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to N$ by

$$\phi(x, t, s) = \varphi_{t,s}(x), \quad (x, t, s) \in M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon), \tag{4.4}$$

let ∇^ϕ denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$, we have

$$[\partial_t, X] = 0, \quad [\partial_s, X] = 0, \quad [\partial_t, \partial_s] = 0, \tag{4.5}$$

Then, by (2.2) we obtain

$$\frac{\partial^2}{\partial t\partial s}E_L(\varphi_{t,s}; D)\Big|_{t=s=0} = \int_D \frac{\partial^2}{\partial t\partial s} f\big(x, \varphi_{t,s}(x), e(\varphi_{t,s})(x)\big)\Big|_{t=s=0} v_g, \tag{4.6}$$

first, note that

$$\frac{\partial}{\partial t} f\big(x, \varphi_{t,s}(x), e(\varphi_{t,s})(x)\big) = dL(d\phi(\partial_t)) + dL(\partial_t(e(\varphi_{t,s}))), \tag{4.7}$$

$$dL(d\phi(\partial_t)) = h(d\phi(\partial_t), (\text{grad}^N f) \circ \varphi), \tag{4.8}$$

$$dL(\partial_t(e(\varphi_{t,s}))) = h(\nabla^\phi_{\partial_t} d\phi(e_i), d\phi(e_i))\, L'_{\varphi_{t,s}}, \tag{4.9}$$

when we pass to the seconde derivative, we get

$$\begin{aligned}
\frac{\partial^2}{\partial t\partial s}L\big(x,\varphi_{t,s}(x),e(\varphi_{t,s})(x)\big) &= h(\nabla^\phi_{\partial_s}d\phi(\partial_t),(\operatorname{grad}^N L)\circ\varphi)\\
&\quad +h(d\phi(\partial_t),\nabla^\phi_{\partial_s}(\operatorname{grad}^N L)\circ\varphi)\\
&\quad +h(\nabla^\phi_{\partial_s}\nabla^\phi_{\partial_t}d\phi(e_i),d\phi(e_i))L'_{\varphi_{t,s}}\\
&\quad +h(\nabla^\phi_{\partial_t}d\phi(e_i),\nabla^\phi_{\partial_s}d\phi(e_i))L'_{\varphi_{t,s}}\\
&\quad +h(\nabla^\phi_{\partial_t}d\phi(e_i),d\phi(e_i))\,\partial_s(L'_{\varphi_{t,s}}).
\end{aligned} \tag{4.10}$$

by (4.1) and the property of the gradient operator we have

$$h(d\phi(\partial_t),\nabla^\phi_{\partial_s}(\operatorname{grad}^N L)\circ\varphi)\Big|_{t=s=0} = h(w,(\nabla^N_v \operatorname{grad}^N L)\circ\varphi), \tag{4.11}$$

by (6.5) and the definition of the curvature tensor of (N,h) we have

$$\begin{aligned}
h(\nabla^\phi_{\partial_s}\nabla^\phi_{\partial_t}d\phi(e_i),d\phi(e_i))L'_{\varphi_{t,s}}\Big|_{t=s=0} &= L'_\varphi\, h(R^N(w,d\varphi(e_i))v,d\varphi(e_i))\\
&\quad +L'_\varphi\, h(\nabla^\phi_{e_i}\nabla^\phi_{\partial_s}d\phi(\partial_t),d\varphi(e_i))\Big|_{t=s=0},
\end{aligned} \tag{4.12}$$

by (4.13), the property of the curvature tensor of (N,h) and the compatibility of ∇^ϕ with the metric h we have

$$\begin{aligned}
h(\nabla^\phi_{\partial_s}\nabla^\phi_{\partial_t}d\phi(e_i),d\phi(e_i))L'_{\varphi_{t,s}}\Big|_{t=s=0} &= -L'_\varphi\, h(R^N(v,d\varphi(e_i))d\varphi(e_i),w)\\
&\quad +e_i(h(\nabla^\phi_{\partial_s}d\phi(\partial_t),L'_\varphi\, d\varphi(e_i)))\Big|_{t=s=0},\\
&\quad -h(\nabla^\phi_{\partial_s}d\phi(\partial_t),\nabla^\varphi_{e_i}L'_\varphi\, d\varphi(e_i))\Big|_{t=s=0},
\end{aligned} \tag{4.13}$$

$$h(\nabla^\phi_{\partial_t}d\phi(e_i),\nabla^\phi_{\partial_s}d\phi(e_i))L'_{\varphi_{t,s}}\Big|_{t=s=0} = e_i(h(L'_\varphi\nabla^\varphi_{e_i}v,w))-h(\nabla^\varphi_{e_i}L'_\varphi\nabla^\varphi_{e_i}v,w). \tag{4.14}$$

Note that

$$\begin{aligned}
\partial_s(L'_{\varphi_{t,s}}) &= \partial_s\big(L'(x,\varphi_{t,s}(x),e(\varphi_{t,s})(x))\big)\\
&= dL'\big(d\phi(\partial_s)\big)+dL'\big(\partial_s(e(\varphi_{t,s}))\big),
\end{aligned} \tag{4.15}$$

by a simple calculation we have

$$dL'\big(d\phi(\partial_s)\big)\Big|_{t=s=0} = h(w,(\operatorname{grad}^N L')\circ\varphi), \tag{4.16}$$

$$dL'\big(\partial_s(e(\varphi_{t,s}))\big)\Big|_{t=s=0} = L''_\varphi\, h(\nabla^\varphi_{e_i} w, d\varphi(e_i)), \tag{4.17}$$

then we get

$$\begin{aligned} h(\nabla^\phi_{\partial_t} d\phi(e_i), d\phi(e_i))\partial_s(L'_{\varphi_{t,s}})\Big|_{t=s=0} &= \ < \nabla^\varphi v, d\varphi > \ h(w, (\mathrm{grad}^N L')\circ\varphi) \\ &\quad + < \nabla^\varphi v, d\varphi > \ L''_\varphi\, h(\nabla^\varphi_{e_i} w, d\varphi(e_i)) \\ &= h(w, < \nabla^\varphi v, d\varphi > \ (\mathrm{grad}^N L')\circ\varphi) \\ &\quad + e_i(h(w, < \nabla^\varphi v, d\varphi > \ L''_\varphi\, d\varphi(e_i))) \\ &\quad - h(w, \nabla^\varphi_{e_i} < \nabla^\varphi v, d\varphi > \ L''_\varphi\, d\varphi(e_i)). \end{aligned} \tag{4.18}$$

From the formulas (6.5), (4.10), (4.11), (4.13), (4.14), (4.18), the divergence theorem and the L-harmonicity of φ, the Theorem 4.1 follows. □

Corollary 4.1. *If $L(x,y,r) = F(r)$, then we obtain:*

$$\begin{aligned} \frac{\partial^2}{\partial t\partial s} E(\varphi_{t,s}; D)\Big|_{t=s=0} &= \int_D F''\Big(\frac{|d\varphi|^2}{2}\Big) < \nabla^\varphi v, d\varphi >< \nabla^\varphi w, d\varphi > \ v_g \\ &\quad - \int_D F'\Big(\frac{|d\varphi|^2}{2}\Big) h(\mathrm{trace}\ R^N(v, d\varphi)d\varphi, w)\, v_g \\ &\quad + \int_D F'\Big(\frac{|d\varphi|^2}{2}\Big) < \nabla^\varphi v, \nabla^\varphi w > \ v_g \end{aligned} \tag{4.19}$$

(We recover the result obtained by M. Ara in [1].)

Proof. we have:

$$\begin{aligned} J_{\varphi,L}(v) &= -L'_\varphi\ \mathrm{trace}\ R^N(v, d\varphi)d\varphi - \mathrm{trace}\ \nabla^\varphi L'_\varphi \nabla^\varphi v \\ &\quad + \big(\nabla^N_v\ \mathrm{grad}^N L\big)\circ\varphi + < \nabla^\varphi v, d\varphi > \ (\mathrm{grad}^N L')\circ\varphi \\ &\quad - \mathrm{trace}\ \nabla^\varphi < \nabla^\varphi v, d\varphi > \ L''_\varphi\, d\varphi \\ &= -L'_\varphi\ \mathrm{trace}\ R^N(v, d\varphi)d\varphi - \mathrm{trace}\ \nabla^\varphi L'_\varphi \nabla^\varphi v \\ &\quad - \mathrm{trace}\ \nabla^\varphi < \nabla^\varphi v, d\varphi > \ L''_\varphi\, d\varphi \end{aligned} \tag{4.20}$$

remark that :

$$L'_\varphi = F'\Big(\frac{|d\varphi|^2}{2}\Big) \tag{4.21}$$

$$L''_\varphi = F''\Big(\frac{|d\varphi|^2}{2}\Big) \tag{4.22}$$

$$\begin{aligned} -h(\mathrm{trace}\ \nabla^\varphi L'_\varphi \nabla^\varphi v, w) &= -h(\nabla^\varphi_{e_i} L'_\varphi \nabla^\varphi_{e_i} v, w) \\ &= -e_i\big(h(L'_\varphi \nabla^\varphi_{e_i} v, w)\big) + h(L'_\varphi \nabla^\varphi_{e_i} v, \nabla^\varphi_{e_i} w) \\ &= -\,\mathrm{div}\ \omega + L'_\varphi < \nabla^\varphi v, \nabla^\varphi w > \end{aligned} \tag{4.23}$$

where: $\omega(X) = L'_\varphi\, h(\nabla^\varphi_X v, w)$.

$$\begin{array}{rcl} - & h(\text{trace}\ \nabla^\varphi & < \nabla^\varphi v, d\varphi > L''_\varphi\, d\varphi, w) \\ & = & -h(\nabla^\varphi_{e_i} < \nabla^\varphi v, d\varphi > L''_\varphi\, d\varphi(e_i), w) \\ & = & -e_i\big(h(< \nabla^\varphi v, d\varphi > L''_\varphi\, d\varphi(e_i), w)\big) \\ & & +h(< \nabla^\varphi v, d\varphi > L''_\varphi\, d\varphi(e_i), \nabla^\varphi_{e_i} w) \\ & = & -\operatorname{div} \eta + L''_\varphi < \nabla^\varphi v, d\varphi >< \nabla^\varphi w, d\varphi > \end{array} \tag{4.24}$$

where: $\eta(X) = L''_\varphi < \nabla^\varphi v, d\varphi > h(d\varphi(X), w)$.

Substituting (4.20),(4.21),(4.22),(4.23),(4.24) in (4.2) we obtain (4.19). □

5. L-Biharmonic Maps.

Definition 5.1. A natural generalization of L-harmonic maps is given by integrating the square of the norm of the L-tension field. More precisely, the L-bienergy functional of a smooth map $\varphi : (M, g) \to (N, h)$ is defined by

$$E_{2,L}(\varphi; D) = \frac{1}{2} \int_D |\tau_L(\varphi)|^2\, v_g. \tag{5.1}$$

Definition 5.2. A map is called L-biharmonic if it is a critical point of the L-energy functional over any compact subset D of M.

Theorem 5.1. (First variation of the L-bienergy functional) *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, D a compact subset of M and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation with compact support in D. Then*

$$\frac{d}{dt} E(\varphi_t; D)\Big|_{t=0} = \int_D h(\tau_{2,L}(\varphi), v)\, v_g, \tag{5.2}$$

where in normal frame at $x \in M$, we have

$$\begin{array}{rcl} \tau_{2,L}(\varphi) & = & -L'_\varphi\ \text{trace}\ R^N(\tau_L(\varphi), d\varphi)d\varphi - \text{trace}\ \nabla^\varphi L'_\varphi \nabla^\varphi \tau_L(\varphi) \\ & & +\big(\nabla^N_{\tau_L(\varphi)}\ \text{grad}^N L\big) \circ \varphi + < \nabla^\varphi \tau_L(\varphi), d\varphi > (\text{grad}^N L') \circ \varphi \\ & & -\,\text{trace}\ \nabla^\varphi < \nabla^\varphi \tau_L(\varphi), d\varphi > L''_\varphi\, d\varphi. \end{array} \tag{5.3}$$

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \to N$ by $\phi(x, t) = \varphi_t(x)$. First note that

$$\frac{d}{dt} E_{2,L}(\varphi_t; D) = \int_D h(\nabla^\phi_{\partial_t} \tau_L(\varphi_t), \tau_L(\varphi_t))\, v_g. \tag{5.4}$$

Calculating in a normal frame at $x \in M$ we have

$$\nabla^\phi_{\partial_t} \tau_L(\varphi_t) = \nabla^\phi_{\partial_t} \nabla^\phi_{e_i} L'_{\varphi_t}\, d\varphi_t(e_i) - \nabla^\phi_{\partial_t} (\text{grad}^N L) \circ \varphi_t, \tag{5.5}$$

by the definition of the curvature tensor of (N,h) we have

$$\nabla^{\phi}_{\partial_t}\nabla^{\phi}_{e_i}L'_{\varphi_t}\,d\varphi_t(e_i)=L'_{\varphi_t}\,R^N(d\phi(\partial_t),d\varphi_t(e_i))d\varphi_t(e_i)+\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_t}L'_{\varphi_t}\,d\varphi_t(e_i), \tag{5.6}$$

by the compatibility of ∇^ϕ with h we have

$$\begin{aligned} h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_t}L'_{\varphi_t}\,d\varphi_t(e_i),\tau_L(\varphi_t)) &= e_i\big(h(\nabla^{\phi}_{\partial_t}L'_{\varphi_t}\,d\varphi_t(e_i),\tau_L(\varphi_t))\big)\\ &\quad -h(\nabla^{\phi}_{\partial_t}L'_{\varphi_t}\,d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t)), \end{aligned} \tag{5.7}$$

the second term on the left-hand side of (6.16) is

$$\begin{aligned} -h(\nabla^{\phi}_{\partial_t}L'_{\varphi_t}\,d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t)) &= -\partial_t(L'_{\varphi_t})\,h(d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t))\\ &\quad -L'_{\varphi_t}\,h(\nabla^{\phi}_{\partial_t}d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t)), \end{aligned} \tag{5.8}$$

be a simple calculation we have

$$\partial_t(L'_{\varphi_t})=d\phi(\partial_t)(L')+L''_{\varphi_t}\,h(\nabla^{\phi}_{e_j}d\phi(\partial_t),d\varphi_t(e_j)), \tag{5.9}$$

then the first term on the left-hand side of (6.17) is

$$\begin{aligned} &-\partial_t(L'_{\varphi_t})\,h(d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t))\\ &= -h(d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t))\,h((\mathrm{grad}^N L')\circ\varphi_t,d\phi(\partial_t))\\ &\quad -e_j\big(h(d\phi(\partial_t),L''_{\varphi_t}\,h(d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t))\,d\varphi_t(e_j))\big)\\ &\quad +h\big(d\phi(\partial_t),\nabla^{\phi}_{e_j}L''_{\varphi_t}h(d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t))\,d\varphi_t(e_j)\big), \end{aligned} \tag{5.10}$$

(5.11)

the second term on the left-hand side of (6.17) is

$$\begin{aligned} -L'_{\varphi_t}\,h(\nabla^{\phi}_{\partial_t}d\varphi_t(e_i),\nabla^{\phi}_{e_i}\tau_L(\varphi_t)) &= -e_i\big(h(d\phi(\partial_t),L'_{\varphi_t}\nabla^{\phi}_{e_i}\tau_L(\varphi_t))\big),\\ &\quad +h(d\phi(\partial_t),\nabla^{\phi}_{e_i}L'_{\varphi_t}\nabla^{\phi}_{e_i}\tau_L(\varphi_t))\big), \end{aligned} \tag{5.12}$$

and notice that

$$-h\big(\nabla^{\phi}_{\partial_t}(\mathrm{grad}^N L)\circ\varphi_t,\tau_L(\varphi_t)\big) = -h\big((\nabla^N_{\tau_L(\varphi_t)}\mathrm{grad}^N L)\circ\varphi_t,d\phi(\partial_t)\big). \tag{5.13}$$

From (6.13), (6.14), (6.15), (6.16), (6.17), (6.19), (6.20), (6.21), $v=d\phi(\partial_t)$ when $t=0$ and the divergence theorem, we deduce the Theorem 5.1. □

Remark 5.1. If $L(x,y,r)=\tilde{L}(x,y)\,r$ for all $(x,y,r)\in M\times N\times\mathbb{R}$, by a simple calculation the term

$$\big(\nabla^N_{\tau_L(\varphi)}\ \mathrm{grad}^N L\big)\circ\varphi$$

is replaced by

$$e(\varphi)\big(\nabla^N_{\tau_{\tilde{L}}(\varphi)}\ \mathrm{grad}^N \tilde{f}\big)\circ\varphi-\mathrm{trace}\ \nabla^{\varphi}v(\tilde{L})\,d\varphi.$$

6. Stress L-Energy and L-Bienergy Tensors

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds and $f \in C^\infty(M\times N\times\mathbb{R})$, $(x, y, r) \mapsto f(x, y, r)$ be a smooth positif function, consider a smooth one-parameter variation of the metric g, i.e. a smooth family of metrics (g_t) $(-\epsilon < t < \epsilon)$ such that $g_0 = g$, write $\delta = \frac{\partial}{\partial t}\Big|_{t=0}$, then $\delta g \in \Gamma(\odot^2 T^*M)$ is a symmetric 2-covariant tensor field on M ([2]). Take local coordinates (x^i) on M, and write the metric on M in the usual way as $g_t = g_{ij}(t, x)\, dx^i\, dx^j$, we now compute

$$
\begin{aligned}
\frac{d}{dt}E_L(\varphi; D)\Big|_{t=0} &= \int_D \delta\big(L(x, \varphi(x), e(\varphi)(x))\big)\, v_g \\
&\quad + \int_D L(x, \varphi(x), e(\varphi)(x))\, \delta(v_{g_t}),
\end{aligned}
\tag{6.1}
$$

using the previous notation, we obtain

$$
\frac{d}{dt}E_L(\varphi; D)\Big|_{t=0} = \int_D \delta\big(e(\varphi)\big)\, L'_\varphi\, v_g + \int_D L_\varphi\, \delta(v_{g_t}),
\tag{6.2}
$$

let $\langle\, ,\, \rangle$ the induced Riemannian metric on $\otimes^2 T^*M$, we have

$$
\delta\big(e(\varphi)\big) = -\frac{1}{2}\langle\varphi^*h, \delta g\rangle, \quad \delta(v_{g_t}) = \frac{1}{2}\langle g, \delta g\rangle v_g,
\tag{6.3}
$$

where φ^*h is the pull-back of the metric h, by (6.2) and (6.3), we get

Theorem 6.1.

$$
\frac{d}{dt}E_L(\varphi; D)\Big|_{t=0} = \frac{1}{2}\int_D < L_\varphi\, g - L'_\varphi\, \varphi^*h, \delta g > \ v_g,
\tag{6.4}
$$

Definition 6.1. A stress L-energy tensor of the smooth map φ is defined by

$$
S_L(\varphi) = L_\varphi\, g - L'_\varphi\, \varphi^*h.
\tag{6.5}
$$

Special cases of stress L-energy tensors

1. If $L(x, y, r) = r$ for all $(x, y, r) \in M \times N \times \mathbb{R}$, the stress L-energy tensor of the smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds given by ([14])

$$
S_L(\varphi) = S(\varphi) = e(\varphi)\, g - \varphi^*h.
\tag{6.6}
$$

2. Let L_1 be a smooth positif function in M, if $L(x, y, r) = L_1(x)\, r$ for all $(x, y, r) \in M \times N \times \mathbb{R}$, the stress L-energy tensor of the smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds given by ([15])

$$
S_L(\varphi) = S_{L_1}(\varphi) = L_1\, e(\varphi)\, g - L_1\, \varphi^*h.
\tag{6.7}
$$

3. Let L_1 be a smooth positif function in $M \times N$, if $L(x,y,r) = L_1(x,y)\,r$ for all $(x,y,r) \in M \times N \times \mathbb{R}$, the stress L-energy tensor of the smooth map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds given by ([7])

$$S_L(\varphi) = S_{L_1}(\varphi) = (L_1)_\varphi\, e(\varphi)\, g - (L_1)_\varphi\, \varphi^* h, \tag{6.8}$$

4. Let F be a smooth positif function in $\mathbb{R}$, if $L(x,y,r) = F(r)$ for all $(x,y,r) \in M \times N \times \mathbb{R}$, the stress L-energy tensor of a smooth map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds is given by

$$S_L(\varphi) = S_F(\varphi) = F\big(e(\varphi)\big)\, g - F'\big(e(\varphi)\big)\, \varphi^* h. \tag{6.9}$$

Stress L-bi-energy tensors.

Let $\varphi : (M,g) \to (N,h)$ be a smooth map between two Riemannian manifolds and $f \in C^\infty(M \times N \times \mathbb{R})$, $(x,y,r) \mapsto f(x,y,r)$ be a smooth positif function. Consider the f-bi-energy functional

$$E_{2,f}(\varphi; D) = \frac{1}{2}\int_D |\tau_f(\varphi)|^2 v_g, \tag{6.10}$$

where $\tau_f(\varphi) = f'_\varphi\, \tau(\varphi) + d\varphi\big(\operatorname{grad}^M f'_\varphi\big) - (\operatorname{grad}^N f) \circ \varphi$ is the f-tension field of φ. Take local coordinates (x^i) on M, and write the metric on M in the usual way as $g_t = g_{ij}(t,x)\, dx^i\, dx^j$, we have

$$\frac{d}{dt} E_{2,L}(\varphi; D)\Big|_{t=0} = \frac{1}{2}\int_D \delta(|\tau_L(\varphi)|^2) v_g + \frac{1}{2}\int_D |\tau_L(\varphi)|^2 \delta(v_{g_t}), \tag{6.11}$$

the calculation of the first term breaks down in three lemmas.

Lemma 6.1. *The vector field* $\xi = (\operatorname{div}^M \delta g)^\sharp - \frac{1}{2}\operatorname{grad}^M(\operatorname{trace}\delta g)$ *satisfies*

$$\begin{aligned}
\delta(|\tau_L(\varphi)|^2) \;=\;& -L''_\varphi h(\tau(\varphi), \tau_L(\varphi)) < \varphi^* h, \delta g > -2L'_\varphi < h(\nabla d\varphi, \tau_L(\varphi)), \delta g > \\
& -2L'_\varphi h(d\varphi(\xi), \tau_L(\varphi)) - h(d\varphi(\operatorname{grad}^M L''_\varphi), \tau_L(\varphi)) < \varphi^* h, \delta g > \\
& + h(\operatorname{grad}^N L', \tau_L(\varphi)) < \varphi^* h, \delta g > -2 < dL'_\varphi \odot h(d\varphi, \tau_L(\varphi)), \delta g > \\
& - L''_\varphi h(d\varphi(\operatorname{grad}^M < \varphi^* h, \delta g >), \tau_L(\varphi)).
\end{aligned}$$

Proof. In local coordinates (x^i) on M and (y^α) on N, we have

$$\delta(|\tau_L(\varphi)|^2) = \delta\big(\tau_L(\varphi)^\alpha \tau_L(\varphi)^\beta h_{\alpha\beta}\big) = 2\,\delta(\tau_L(\varphi)^\alpha)\tau_L(\varphi)^\beta h_{\alpha\beta}, \tag{6.12}$$

first, note that

$$\begin{aligned}
\delta(\tau_L(\varphi)^\alpha) \;&=\; \delta\big(L'_\varphi \tau(\varphi)^\alpha + \theta^\alpha - \eta^\alpha\big) \\
&=\; \delta(L'_\varphi)\, \tau(\varphi)^\alpha + L'_\varphi\, \delta(\tau(\varphi)^\alpha) + \delta(\theta^\alpha) - \delta(\eta^\alpha)
\end{aligned} \tag{6.13}$$

where $\tau(\varphi)^\alpha = g^{ij}\big(\varphi^\alpha_{i,j} +^N \Gamma^\alpha_{\mu\sigma}\varphi^\mu_i\varphi^\sigma_j -^M \Gamma^k_{ij}\varphi^\alpha_k\big)$ is the component of $\tau(\varphi)$,

$$\theta^\alpha = g^{ij}(L'_\varphi)_i\varphi^\alpha_j, \quad \eta^\alpha = h^{\alpha\mu}L_\mu$$

by (6.2), the first term in the right-hand side of (6.13) is

$$\delta(L'_\varphi)\,\tau(\varphi)^\alpha = \delta(e(\varphi))L''_\varphi\,\tau(\varphi)^\alpha = -\frac{1}{2} < \varphi^* h, \delta g > L''_\varphi\,\tau(\varphi)^\alpha, \tag{6.14}$$

by ([14]), the second term on the right-hand side of (6.13) is

$$L'_\varphi\,\delta(\tau(\varphi)^\alpha) = -L'_\varphi g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^\alpha_{ij} - L'_\varphi\xi^k\varphi^\alpha_k, \tag{6.15}$$

the third term on the right-hand side of (6.13) is

$$\begin{aligned}
\delta(\theta^\alpha) &= \delta(g^{ij})(L'_\varphi)_i\varphi^\alpha_j + g^{ij}\delta\big((L'_\varphi)_i\big)\varphi^\alpha_j \\
&= \delta(g^{ij})(L'_\varphi)_i\varphi^\alpha_j + g^{ij}\big(\delta(L'_\varphi)\big)_i\varphi^\alpha_j \\
&= \delta(g^{ij})(L'_\varphi)_i\varphi^\alpha_j + g^{ij}\big(\delta(e(\varphi))L''_\varphi\big)_i\varphi^\alpha_j \\
&= \delta(g^{ij})(L'_\varphi)_i\varphi^\alpha_j - \frac{1}{2}g^{ij}\big(< \varphi^* h, \delta g > L''_\varphi\big)_i\varphi^\alpha_j \\
&= \delta(g^{ij})(L'_\varphi)_i\varphi^\alpha_j - \frac{1}{2}g^{ij} < \varphi^* h, \delta g >_i L''_\varphi\varphi^\alpha_j \\
&\quad -\frac{1}{2}g^{ij} < \varphi^* h, \delta g > (L''_\varphi)_i\varphi^\alpha_j,
\end{aligned} \tag{6.16}$$

the fourth term on the right-hand side of (6.13) is

$$-\delta(\eta^\alpha) = -h^{\alpha\mu}\delta(L_\mu) = -h^{\alpha\mu}\delta(e(\varphi))L'_\mu = \frac{1}{2}h^{\alpha\mu} < \varphi^* h, \delta g > L'_\mu, \tag{6.17}$$

and note that

$$\begin{aligned}
2\,\delta(L'_\varphi)\,\tau(\varphi)^\alpha\tau_L(\varphi)^\beta h_{\alpha\beta} &= - < \varphi^* h, \delta g > L''_\varphi\,\tau(\varphi)^\alpha\tau_L(\varphi)^\beta h_{\alpha\beta} \\
&= -L''_\varphi h(\tau(\varphi), \tau_L(\varphi)) < \varphi^* h, \delta g >,
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
2\,L'_\varphi\,\delta(\tau(\varphi)^\alpha)\tau_L(\varphi)^\beta h_{\alpha\beta} &= -2L'_\varphi g^{ai}g^{bj}\delta(g_{ab})(\nabla d\varphi)^\alpha_{ij}\tau_L(\varphi)^\beta h_{\alpha\beta} \\
&\quad -2L'_\varphi\xi^k\varphi^\alpha_k\tau_L(\varphi)^\beta h_{\alpha\beta} \\
&= -2L'_\varphi < h(\nabla d\varphi, \tau_L(\varphi)), \delta g > \\
&\quad -2L'_\varphi h(d\varphi(\xi), \tau_L(\varphi)),
\end{aligned} \tag{6.19}$$

$$\begin{aligned}
2\,\delta(\theta^\alpha\tau_L(\varphi)^\beta h_{\alpha\beta} &= 2\delta(g^{ij})(L'_\varphi)_i\varphi^\alpha_j\tau_L(\varphi)^\beta h_{\alpha\beta} \\
&\quad -g^{ij} < \varphi^* h, \delta g >_i L''_\varphi\varphi^\alpha_j\tau_L(\varphi)^\beta h_{\alpha\beta} \\
&\quad -g^{ij} < \varphi^* h, \delta g > (L''_\varphi)_i\varphi^\alpha_j\tau_L(\varphi)^\beta h_{\alpha\beta} \\
&= -2 < dL'_\varphi \odot h(d\varphi, \tau_L(\varphi)), \delta g > \\
&\quad -L''_\varphi h(d\varphi(\mathrm{grad}^M < \varphi^* h, \delta g >), \tau_L(\varphi)) \\
&\quad -h(d\varphi(\mathrm{grad}^M L''_\varphi), \tau_L(\varphi)) < \varphi^* h, \delta g >,
\end{aligned} \tag{6.20}$$

$$\begin{aligned} -2\,\delta(\eta^\alpha)\tau_L(\varphi)^\beta h_{\alpha\beta} &= h^{\alpha\mu} < \varphi^*h, \delta g > L'_\mu \tau_L(\varphi)^\beta h_{\alpha\beta} \\ &= h(\operatorname{grad}^N L', \tau_L(\varphi)) < \varphi^*h, \delta g > . \end{aligned} \tag{6.21}$$

Substituting (6.13), (6.18), (6.19), (6.20) and (6.21) in (6.12), the Lemma 6.1 follows. □

Lemma 6.2.([7])

$$\begin{aligned} \int_D L'_\varphi h(d\varphi(\xi), \tau_L(\varphi)) v_g &= \int_D \langle -\operatorname{sym}\left(\nabla L'_\varphi h(d\varphi, \tau_L(\varphi))\right) \\ &\quad + \frac{1}{2}\operatorname{div}^M\left(L'_\varphi h(d\varphi, \tau_L(\varphi))^\sharp\right) g, \delta g\rangle\, v_g. \end{aligned}$$

Lemma 6.3. *Let* $\omega = L''_\varphi\, h(d\varphi, \tau_L(\varphi))$*, then*

$$-\int_D L''_\varphi h(d\varphi(\operatorname{grad}^M < \varphi^*h, \delta g >), \tau_L(\varphi))\, v_g = \int_D < \varphi^*h, \delta g > \operatorname{div}\omega\, v_g.$$

Proof. Note that

$$\operatorname{div}(< \varphi^*h, \delta g > \omega) =< \varphi^*h, \delta g > \operatorname{div}\omega + \omega(\operatorname{grad}^M < \varphi^*h, \delta g >),$$

and consider the divergence theorem, the Lemma 3.1 follows. □

Theorem 6.2. *Let* $\varphi : (M, g) \to (N, h)$ *be a smooth map and let* $\{g_t\}$ *a one parameter variation of* g*. Then*

$$\frac{d}{dt} E_{2,L}(\varphi; D)\Big|_{t=0} = \frac{1}{2}\int_D < S_{2,L}(\varphi), \delta g > \ v_g,$$

where $S_{2,L}(\varphi) \in \Gamma(\odot^2 T^*M)$ *is given by*

$$\begin{aligned} S_{2,L}(\varphi)(X, Y) &= -\frac{1}{2}|\tau_f(\varphi)|^2 g(X, Y) - L'_\varphi < d\varphi, \nabla^\varphi \tau_L(\varphi) > g(X, Y) \\ &\quad + L'_\varphi h(d\varphi(X), \nabla^\varphi_Y \tau_L(\varphi)) + L'_\varphi h(d\varphi(Y), \nabla^\varphi_X \tau_L(\varphi)) \\ &\quad - \tau_L(\varphi)(L) g(X, Y) - \tau_L(\varphi)(L') h(d\varphi(X), d\varphi(Y)) \\ &\quad + L''_\varphi < d\varphi, \nabla^\varphi \tau_L(\varphi) > h(d\varphi(X), d\varphi(Y)). \end{aligned}$$

$S_{2,L}(\varphi)$ *is called the stress* f*-bi-energy tensor of* φ*.*

Proof. By (6.2), (6.11), lemma 6.1, lemma 6.2 and lemma 6.3, we obtain

$$\begin{aligned} S_{2,L}(\varphi) &= -L''_\varphi h(\tau(\varphi), \tau_L(\varphi))\varphi^*h - 2L'_\varphi h(\nabla d\varphi, \tau_L(\varphi)) \\ &\quad + 2\operatorname{sym}\left(\nabla L'_\varphi h(d\varphi, \tau_L(\varphi))\right) - \operatorname{div}^M\left(L'_\varphi h(d\varphi, \tau_L(\varphi))^\sharp\right) g \\ &\quad - h(d\varphi(\operatorname{grad}^M L''_\varphi), \tau_L(\varphi))\varphi^*h + h(\operatorname{grad}^N L', \tau_L(\varphi))\varphi^*h \\ &\quad - 2dL'_\varphi \odot h(d\varphi, \tau_L(\varphi)) + \operatorname{div}\left(L''_\varphi\, h(d\varphi, \tau_L(\varphi))\right)\varphi^*h \\ &\quad + \frac{1}{2}\,|\tau_L(\varphi)|^2 g, \end{aligned} \tag{6.22}$$

note that, for all $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} 2\,\mathrm{sym}\left(\nabla L'_\varphi h(d\varphi, \tau_L(\varphi))\right)(X,Y) &= 2L'_\varphi h(\nabla d\varphi(X,Y), \tau_L(\varphi)) \\ &\quad + L'_\varphi h(d\varphi(X), \nabla^\varphi_Y \tau_L(\varphi)) \\ &\quad + L'_\varphi h(d\varphi(Y), \nabla^\varphi_X \tau_L(\varphi)) \\ &\quad + X(L'_\varphi) h(d\varphi(Y), \tau_L(\varphi)) \\ &\quad + Y(L'_\varphi) h(d\varphi(X), \tau_L(\varphi)), \end{aligned} \tag{6.23}$$

$$\begin{aligned} -2dL'_\varphi \odot h(d\varphi, \tau_L(\varphi))(X,Y) &= -X(L'_\varphi) h(d\varphi(Y), \tau_L(\varphi)) \\ &\quad - Y(L'_\varphi) h(d\varphi(X), \tau_L(\varphi)), \end{aligned} \tag{6.24}$$

calculating in a normal frame at x, we have

$$\begin{aligned} \mathrm{div}\left(L'_\varphi h(d\varphi, \tau_L(\varphi))^\sharp\right) &= e_i(g(L'_\varphi h(d\varphi, \tau_L(\varphi))^\sharp, e_i)) \\ &= e_i(L'_\varphi h(d\varphi(e_i), \tau_L(\varphi))) \\ &= e_i(L'_\varphi) h(d\varphi(e_i), \tau_L(\varphi)) + L'_\varphi h(\nabla^\varphi_{e_i} d\varphi(e_i), \tau_L(\varphi)) \\ &\quad + L'_\varphi h(d\varphi(e_i), \nabla^\varphi_{e_i} \tau_L(\varphi)) \\ &= h(d\varphi(\mathrm{grad}^M L'_\varphi), \tau_L(\varphi)) + L'_\varphi h(\tau(\varphi), \tau_L(\varphi)) \\ &\quad + L'_\varphi < d\varphi, \nabla^\varphi \tau_L(\varphi) >, \end{aligned} \tag{6.25}$$

let $\tau_L(\varphi) = L'_\varphi \tau(\varphi) + d\varphi(\mathrm{grad}^M L'_\varphi) - (\mathrm{grad}^N L) \circ \varphi$, so

$$\mathrm{div}\left(L'_\varphi h(d\varphi, \tau_L(\varphi))^\sharp\right) = |\tau_L(\varphi)|^2 + \tau_L(\varphi)(L) + L'_\varphi < d\varphi, \nabla^\varphi \tau_L(\varphi) >, \tag{6.26}$$

with the same method of (6.25), we find that

$$\begin{aligned} \mathrm{div}\left(L''_\varphi h(d\varphi, \tau_L(\varphi))\right) &= h(d\varphi(\mathrm{grad}^M L''_\varphi), \tau_L(\varphi)) + L''_\varphi h(\tau(\varphi), \tau_L(\varphi)) \\ &\quad + L''_\varphi < d\varphi, \nabla^\varphi \tau_L(\varphi) > . \end{aligned} \tag{6.27}$$

Substituting (6.23), (6.24), (6.26) and (6.27) in (6.22), the Theorem 6.2 follows. □

Special cases of stress L-bi-energy tensors

1. If $L(x,y,r) = r$ for all $(x,y,r) \in M \times N \times \mathbb{R}$, the stress L-bi-energy tensor of the smooth map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds given by ([14])

$$\begin{aligned} S_{2,L}(\varphi)(X,Y) &= S_2(\varphi)(X,Y) \\ &= -\frac{1}{2}|\tau(\varphi)|^2 g(X,Y) - < d\varphi, \nabla^\varphi \tau(\varphi) > g(X,Y) \\ &\quad + h(d\varphi(X), \nabla^\varphi_Y \tau(\varphi)) + h(d\varphi(Y), \nabla^\varphi_X \tau(\varphi)). \end{aligned}$$

2. Let L_1 be a smooth positif function in M, if $L(x,y,r) = L_1(x)\,r$ for all $(x,y,r) \in M \times N \times \mathbb{R}$, the stress L-bi-energy tensor of the smooth map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds given by ([15])

$$\begin{aligned} S_{2,L}(\varphi)(X,Y) &= S_{2,L_1}(\varphi)(X,Y) \\ &= -\frac{1}{2}|\tau_{L_1}(\varphi)|^2 g(X,Y) - L_1 < d\varphi, \nabla^\varphi \tau_{L_1}(\varphi) > g(X,Y) \\ &\quad + L_1 h(d\varphi(X), \nabla^\varphi_Y \tau_{L_1}(\varphi)) + L_1 h(d\varphi(Y), \nabla^\varphi_X \tau_{L_1}(\varphi)). \end{aligned}$$

3. Let L_1 be a smooth positif function in $M \times N$, if $L(x,y,r) = L_1(x,y)\,r$ for all $(x,y,r) \in M \times N \times \mathbb{R}$, the stress L-bi-energy tensor of the smooth map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds given by ([7])

$$\begin{aligned} S_{2,L}(\varphi)(X,Y) &= S_{2,L_1}(\varphi)(X,Y) \\ &= -\frac{1}{2}|\tau_{L_1}(\varphi)|^2 g(X,Y) - (L_1)_\varphi < d\varphi, \nabla^\varphi \tau_{L_1}(\varphi) > g(X,Y) \\ &\quad + (L_1)_\varphi h(d\varphi(X), \nabla^\varphi_Y \tau_{L_1}(\varphi)) + (L_1)_\varphi h(d\varphi(Y), \nabla^\varphi_X \tau_{L_1}(\varphi)) \\ &\quad - \tau_{L_1}(\varphi)(L_1) e(\varphi) g(X,Y) + \tau_{L_1}(\varphi)(L_1)\, h(d\varphi(X), d\varphi(Y)). \end{aligned}$$

4. Let F be a smooth positif function in $\mathbb{R}$, if $L(x,y,r) = F(r)$ for all $(x,y,r) \in M \times N \times \mathbb{R}$, the stress L-bi-energy tensor of the smooth map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds given by ([1])

$$\begin{aligned} & S_{2,L}(\varphi)(X,Y) \\ = \ & S_{2,F}(\varphi)(X,Y) \\ = \ & -\frac{1}{2}|\tau_F(\varphi)|^2 g(X,Y) - F'\big(e(\varphi)\big) < d\varphi, \nabla^\varphi \tau_F(\varphi) > g(X,Y) \\ & + F'\big(e(\varphi)\big) h(d\varphi(X), \nabla^\varphi_Y \tau_F(\varphi)) + F'\big(e(\varphi)\big) h(d\varphi(Y), \nabla^\varphi_X \tau_F(\varphi)) \\ & + F''\big(e(\varphi)\big) < d\varphi, \nabla^\varphi \tau_F(\varphi) > h(d\varphi(X), d\varphi(Y)). \end{aligned}$$

References

[1] M. Ara, *Geometry of F-harmonic maps*, Kodai Math. J., **22**(1999), 243-263.

[2] P. Baird, J. C. Wood, *Harmonic morphisms between Riemannain manifolds*, Clarendon Press Oxford 2003.

[3] Y. J. Chiang, *f-biharmonic Maps between Riemannian Manifolds*, Department of Mathematics, University of Mary Washington Fredericksburg, VA 22401, USA 2012.

[4] Y.-J. Chiang and H. Sun,*Biharmonic maps on V-manifolds*, Int. J. Math. Math. Sci., **27(8)**(2001), 477-484.

[5] N. Course, *f-harmonic maps which map the boundary of the domain to one point in the target*, New York Journal of Mathematics, **13**(2007), 423-435.

[6] A.M. Cherif and M. Djaa, *On generalized f-harmonic morphisms*, Comment. Math. Univ. Carolin., **55(1)**(2014), 1727.

[7] M. Djaa and A. M. Cherif, *On Generalized f-biharmonic Maps and Stress f-bienergy Tensor*, Journal of Geometry and Symmetry in Physics JGSP, **29**(2013), 65-81.

[8] M. Djaa, A. M. Cherif, K. Zegga S. Ouakkas, *On the Generalized of f-harmonic and f-bi-harmonic Maps*, International Electronic Journal of Geometry, **1**(2012), 1-11.

[9] J. Eells, *p-Harmonic and Exponentially Harmonic Maps,* lecture given at Leeds University, June 1993.

[10] J. Eells and J.H. Sampson, *Harmonic mappings of riemannian manifolds*, Amer. J. Math., **86**(1964), 109-160.

[11] S. Feng and Y. Han, *Liouville Type Theorems of f-Harmonic Maps with Potential*, Results in Mathematics, **66**(2014), 43-64.

[12] G. Y. Jiang, *2-Harmonic Maps Between Riemannian Manifolds*, Annals of Math., China, **7A(4)**(1986), 389-402.

[13] E. Loubeau and C. Oniciuc. *On the biharmonic and harmonic indices of the hopf map*, Transactions of the american mathematical society., **359**(2007), 5239-5256.

[14] E. Loubeau, S. Montaldo, And C. Oniciuc, *the stress-energy tensor for biharmonic maps*, arXiv:math/0602021v1 [math.DG] 1 Feb 2006.

[15] S. Ouakkas, R. Nasri and M. Djaa. *On the f-harmonic and f-biharmonic maps*, J. P. Journal. of Geom. and Top., **10(1)**(2010), 11-27.

[16] Y. L. Ou, On f-harmonic morphisms between Riemannian manifolds, arxiv:1103.5687 (2011), Chinese Ann Math, series B (2014).