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Certain Subclasses of Bi-Starlike and Bi-Convex Functions of Complex Order

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Abstract. In this paper, we introduce and investigate an interesting subclass $\mathcal{M}_\Sigma(\gamma,\lambda,\delta,\varphi)$ of analytic and bi-univalent functions of complex order in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{M}_\Sigma(\gamma,\lambda,\delta,\varphi)$ we investigate the coefficient estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The results presented in this paper would generalize and improve some recent works of [1],[5],[9].

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$.

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For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U})$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions which are in (or which are not in) the class Σ, together with various other properties of the bi-univalent function class Σ one can refer the work of Srivastava et al. [20] and references therein. In fact, the study of the coefficient problems involving bi-univalent functions was reviewed recently by Srivastava et al. [20]. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1] - [9], [11] - [13], [16] - [19] and [21] - [24]). The aforecited all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [20]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \dots)$ for functions $f \in \Sigma$ is still an open problem.

Let φ be an analytic and univalent function with positive real part in $\mathbb{U}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, and symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \text{ with } B_1 > 0. \tag{1.3}$$

Throughout this paper we assume that the function φ satisfies the above conditions one or otherwise stated.

We now introduce the function class $\mathcal{S}^*(\gamma, \delta, \varphi)$ of Mocanu-convex functions of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$) of Ma-Minda type as follows :

$$\mathcal{S}^*(\gamma, \delta, \varphi) := \Big\{ f : \; f \in \mathcal{A} \text{ and } 1 + \frac{1}{\gamma}\left((1-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right) \prec \varphi(z) \; (\delta \geq 0) \Big\}.$$

A function f is bi-Mocanu-convex function of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$) in $\mathbb{U}$ of Ma-Minda type if both f and f^{-1} are Mocanu-convex functions of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$) in $\mathbb{U}$ of Ma-Minda type. The class is denoted by $\mathcal{S}^*_\Sigma(\gamma, \delta, \varphi)$. For $\gamma = 1$, the class $\mathcal{S}^*(\gamma, \delta, \varphi)$ leads to the class $\mathcal{M}(\delta, \varphi)$ of Mocanu-convex functions in $\mathbb{U}$ of Ma-Minda type. A function f is bi-Mocanu-convex in $\mathbb{U}$ of Ma-Minda type if both f and f^{-1} are Mocanu-convex in $\mathbb{U}$ of Ma-Minda type (see [1]). The class is denoted by $\mathcal{M}^*_\Sigma(\delta, \varphi)$. For $\delta = 0$ and $\delta = 1$, the class $\mathcal{S}^*(\gamma, \delta, \varphi)$ reduces respectively, to the familiar classes $\mathcal{S}^*(\gamma, \varphi)$ and $\mathcal{K}(\gamma, \varphi)$ of Ma-Minda starlike and convex functions of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$) in $\mathbb{U}$ (see [15]). Also, a function f is bi-starlike and bi-convex of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$) of Ma-Minda type in $\mathbb{U}$ if both f and f^{-1} are, respectively, Ma-Minda starlike and Ma-Minda convex of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$) in $\mathbb{U}$. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\gamma, \varphi)$ and $\mathcal{K}_\Sigma(\gamma, \varphi)$ (see for more details [5]). Furthermore the classes $\mathcal{S}^*_\Sigma(1, \varphi)$ $:= \mathcal{S}^*_\Sigma(\varphi)$ and $\mathcal{K}_\Sigma(1, \varphi) := \mathcal{K}_\Sigma(\varphi)$ are, respectively bi-starlike of Ma-Minda type in $\mathbb{U}$ and bi-convex of Ma-Minda type in $\mathbb{U}$ (see [1]) and for its other subclasses one can refer the reference therein.

Recently Srivastava et al. [18] introduced a general class of bi-univalent functions for investigating the extensions, generalizations and improvements of the various subclasses of bi-univalent functions which were considered by a number of earlier researchers (see, [1, 3, 6, 20, 24, 23] and others). With this motivation in this paper we define the following unified subclass of bi-univalent function class Σ:

A function $f \in \Sigma$ is said to be in the class $\mathcal{M}_\Sigma(\gamma, \lambda, \delta, \varphi)$, $0 \neq \gamma \in \mathbb{C}$, $\delta \geq 0$, if the following subordinations hold:

$$1 + \frac{1}{\gamma}\left((1-\delta)\frac{z\mathcal{F}'_\lambda(z)}{\mathcal{F}_\lambda(z)} + \delta\left(1 + \frac{z\mathcal{F}''_\lambda(z)}{\mathcal{F}'_\lambda(z)}\right) - 1\right) \prec \varphi(z) \tag{1.4}$$

and for $g(w) = f^{-1}(w)$,

$$(1.5) \qquad 1+\frac{1}{\gamma}\left((1-\delta)\frac{w\mathcal{G}'_\lambda(w)}{\mathcal{G}_\lambda(w)}+\delta\left(1+\frac{w\mathcal{G}''_\lambda(w)}{\mathcal{G}'_\lambda(w)}\right)-1\right)\prec\varphi(w),$$

where

$$\mathcal{F}_\lambda(z)=(1-\lambda)f(z)+\lambda zf'(z), \quad \mathcal{G}_\lambda(w)=(1-\lambda)g(w)+\lambda wg'(w) \qquad (0\le\lambda\le 1).$$

It is interesting to note that the special values of δ, γ, λ and φ, the class $\mathcal{M}_\Sigma(\gamma,\lambda,\delta,\varphi)$ unifies the following known and new classes:

1. $\mathcal{M}_\Sigma(\gamma,0,\delta,\frac{1+(1-2\alpha)z}{1-z})=\mathcal{S}^*_\Sigma(\gamma,\delta,\alpha) \quad (0\le\alpha<1)$
2. $\mathcal{M}_\Sigma(\gamma,0,\delta,\left(\frac{1+z}{1-z}\right)^\beta)=\mathcal{S}^*_{\Sigma,\beta}(\gamma,\delta) \quad (0<\beta\le 1)$
3. $\mathcal{M}_\Sigma(1,0,\delta,\frac{1+(1-2\alpha)z}{1-z})=\mathcal{B}_\Sigma(\alpha,\delta) \quad (0\le\alpha<1)$ [9, Definition 3.1., p.1500]
4. $\mathcal{M}_\Sigma(1,0,\delta,\left(\frac{1+z}{1-z}\right)^\beta)=\mathcal{M}^{\beta,\delta}_\Sigma$ $(0<\beta\le 1)$ [9, Definition 2.1., p.1497]
5. $\mathcal{M}_\Sigma(1,0,\delta,\varphi)=\mathcal{M}_\Sigma(\delta,\varphi)$ [1, p.348]
6. $\mathcal{M}_\Sigma(\gamma,0,0,\varphi)=\mathcal{S}^*_\Sigma(\gamma,\varphi)$ [5, p.50]
7. $\mathcal{M}_\Sigma(1,0,0,\varphi)=\mathcal{S}^*_\Sigma(\varphi)$ [1, p.345]
8. $\mathcal{M}_\Sigma(1,0,0,\frac{1+(1-2\alpha)z}{1-z})=\mathcal{S}^*_\Sigma(\alpha) \quad (0\le\alpha<1)$
9. $\mathcal{M}_\Sigma(1,0,0,\left(\frac{1+z}{1-z}\right)^\beta)=\mathcal{S}^*_\Sigma(\beta)$ $(0<\beta\le 1)$
10. $\mathcal{M}_\Sigma(\gamma,0,1,\varphi)=\mathcal{K}_\Sigma(\gamma,\varphi)$ [5, p.50]
11. $\mathcal{M}_\Sigma(1,0,1,\varphi)=\mathcal{K}_\Sigma(\varphi)$ [1, p.345]
12. $\mathcal{M}_\Sigma(1,0,1,\frac{1+(1-2\alpha)z}{1-z})=\mathcal{K}_\Sigma(\alpha) \quad (0\le\alpha<1)$.

In this paper we introduce the unified bi-univalent function class $\mathcal{M}_\Sigma(\gamma,\lambda,\delta,\varphi)$ as defined above and obtain the coefficient estimates for Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging $\mathcal{M}_\Sigma(\gamma,\lambda,\delta,\varphi)$. Some interesting applications of the results presented here are also discussed.

In order to derive our results, we shall need the following lemma:

Lemma 2.1.(see [14]) *If $p\in\mathcal{P}$, then $|p_i|\le 2$ for each i, where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{U}$, for which*

$$\Re\{p(z)\}>0 \quad (z\in\mathbb{U}),$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}).$$

2. Coefficient Estimates for the Function Class $\mathcal{M}_\Sigma(\gamma, \lambda, \delta, \varphi)$

In this section we find the estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the unified bi-univalent function class $\mathcal{M}_\Sigma(\gamma, \lambda, \delta, \varphi)$.

Theorem 2.2. *If $f \in \mathcal{M}_\Sigma(\gamma, \lambda, \delta, \varphi)$, then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|\gamma(2(1+2\delta)(1+2\lambda)-(1+3\delta)(1+\lambda)^2)B_1^2+(1+\delta)^2(1+\lambda)^2(B_1-B_2)|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{|\gamma|[B_1+|B_2-B1|]}{2(1+2\delta)(1+2\lambda)-(1+3\delta)(1+\lambda)^2}. \tag{2.2}$$

Proof. Since $f \in \mathcal{M}_\Sigma(\gamma, \lambda, \delta, \varphi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{1}{\gamma}\left((1-\delta)\frac{z\mathcal{F}'_\lambda(z)}{\mathcal{F}_\lambda(z)} + \delta\left(1 + \frac{z\mathcal{F}''_\lambda(z)}{\mathcal{F}'_\lambda(z)}\right) - 1\right) = \varphi(r(z)) \tag{2.3}$$

and

$$1 + \frac{1}{\gamma}\left((1-\delta)\frac{w\mathcal{G}'_\lambda(w)}{\mathcal{G}_\lambda(w)} + \delta\left(1 + \frac{w\mathcal{G}''_\lambda(w)}{\mathcal{G}'_\lambda(w)}\right) - 1\right) = \varphi(s(z)). \tag{2.4}$$

Define the functions p and q by

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$$

and

$$q(z) = \frac{1+s(z)}{1-s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \ldots$$

or equivalently,

$$\begin{aligned} (2.5)\; r(z) &= \frac{p(z)-1}{p(z)+1} \\ &= \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 + \frac{p_1}{2}\left(\frac{p_1^2}{2} - p_2\right) - \frac{p_1 p_2}{2}\right) z^3 + \ldots\right) \end{aligned}$$

and

$$\begin{aligned} (2.6)\; s(z) &= \frac{q(z)-1}{q(z)+1} \\ &= \frac{1}{2}\left(q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \left(q_3 + \frac{q_1}{2}\left(\frac{q_1^2}{2} - q_2\right) - \frac{q_1 q_2}{2}\right) z^3 + \ldots\right). \end{aligned}$$

It is clear that p and q are analytic in $\mathbb{U}$ and $p(0) = 1 = q(0)$. Also p and q have positive real part in $\mathbb{U}$, and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. In the view of (2.3), (2.4), (2.5) and (2.6), clearly

$$1 + \frac{1}{\gamma}\left((1-\delta)\frac{z\mathcal{F}'_\lambda(z)}{\mathcal{F}_\lambda(z)} + \delta\left(1 + \frac{z\mathcal{F}''_\lambda(z)}{\mathcal{F}'_\lambda(z)}\right) - 1\right) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{2.7}$$

and

$$1 + \frac{1}{\gamma}\left((1-\delta)\frac{w\mathcal{G}'_\lambda(w)}{\mathcal{G}_\lambda(w)} + \delta\left(1 + \frac{w\mathcal{G}''_\lambda(w)}{\mathcal{G}'_\lambda(w)}\right) - 1\right) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{2.8}$$

Using (2.5) and (2.6) together with (1.3), it is evident that

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2}B_1p_1z + \left(\frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2\right)z^2 + \ldots \tag{2.9}$$

and

$$\varphi\left(\frac{q(w)-1}{q(w)+1}\right) = 1 + \frac{1}{2}B_1q_1w + \left(\frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2\right)w^2 + \ldots. \tag{2.10}$$

Since $f \in \Sigma$ is of the form (1.1), a computation shows that its inverse $g = f^{-1}$ has the expression given by (1.2). It follows from (2.7), (2.8), (2.9) and (2.10) that

$$\frac{1}{\gamma}(1+\delta)(\lambda+1)a_2 = \frac{1}{2}B_1p_1 \tag{2.11}$$

$$\frac{1}{\gamma}[2(1+2\delta)(1+2\lambda)a_3 - (1+3\delta)(1+\lambda)^2a_2^2] = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2 \tag{2.12}$$

$$-\frac{1}{\gamma}(1+\delta)(\lambda+1)a_2 = \frac{1}{2}B_1q_1 \tag{2.13}$$

and

$$\frac{1}{\gamma}[4((1+2\delta)(1+2\lambda) - (1+3\delta)(1+\lambda)^2)a_2^2 - 2(1+2\delta)(1+2\lambda)a_3] = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2. \tag{2.14}$$

From (2.11) and (2.13), it follows that

$$p_1 = -q_1 \tag{2.15}$$

and

$$\frac{8}{\gamma^2}(1+\delta)^2(\lambda+1)^2 a_2^2 = B_1^2(p_1^2+q_1^2). \tag{2.16}$$

Now (2.12), (2.14) and (2.16) yield

$$a_2^2 = \frac{\gamma^2 B_1^3(p_2+q_2)}{4[\gamma(2(1+2\delta)(1+2\lambda)-(1+3\delta)(1+\lambda)^2)B_1^2+(1+\delta)^2(1+\lambda)^2(B_1-B_2)]}.$$

Thus the desired estimate on $|a_2|$ as asserted in (2.1), follows using the Lemma 2.1 that $|p_2| \leq 2$ and $|q_2| \leq 2$. By subtracting (2.12) from (2.14) and a computation using (2.11) finally lead to

$$a_3 = \frac{\gamma B_1(p_2+q_2)+\gamma(B_2-B_1)p_1^2}{8(1+2\delta)(1+2\lambda)-4(1+3\delta)(1+\lambda)^2} + \frac{B_1\gamma(p_2-q_2)}{8(1+2\delta)(1+2\lambda)}.$$

Applying Lemma 2.1 once again, we readily get the estimate given in (2.2). □

3. Consequences and Corollaries

Taking $\delta = 0$ and $\lambda = 0$ in Theorem 2.2, we have the following coefficient estimates for bi-starlike functions of complex order.

Corollary 3.1.([5]) *If* $f \in \mathcal{S}^*_\Sigma(\gamma,\varphi)$, *then*

$$|a_2| \leq \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{|\gamma B_1^2+(B_1-B_2)|}} \quad \text{and} \quad |a_3| \leq |\gamma|[B_1+|B_2-B1|].$$

Taking $\delta = 1$ and $\lambda = 0$ in Theorem 2.2, we have the following coefficient estimates for bi-convex functions of complex order.

Corollary 3.2.([5]) *If* $f \in \mathcal{K}_\Sigma(\gamma,\varphi)$, *then*

$$|a_2| \leq \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{|2[\gamma B_1^2+2(B_1-B_2)]|}} \qquad \text{and} \qquad |a_3| \leq \frac{|\gamma|[B_1+|B_2-B1|]}{2}.$$

Remark 3.3. For $\gamma = 1$, putting $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\beta$ $(0 < \beta \leq 1)$ and $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ in Corollary 3.1 we have results as in [1, Remark 2.2] and taking $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$ in Corollary 3.2 the estimates coincide with [1, Remark 2.3].

Taking $\lambda = 0$ in Theorem 2.2, we have the following coefficient estimates for bi-Mocanu-convex functions of complex order γ of Ma-Minda type.

Corollary 3.4. *If* $f \in \mathcal{M}_\Sigma(\gamma,\delta,\varphi)$, *then*

$$|a_2| \leq \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{|(\delta+1)[\gamma B_1^2+(\delta+1)(B_1-B_2)]|}}$$

and

$$|a_3| \leq \frac{|\gamma|[B_1 + |B_2 - B1|]}{\delta + 1}.$$

Remark 3.5. For $\gamma = 1$, Corollary 3.4 reduces to estimates in [1, Theorem 2.3, p.348]. If we set $\gamma = 1$ in Corollary 3.4, then for $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\beta}$ $(0 < \beta \leq 1)$, it respectively reduces to [9, Theorem 3.2, p.1500] and [9, Theorem 2.2, 1498].

Remark 3.6. Taking $\delta = 0$, we have the class $\mathcal{M}_\Sigma(\gamma, \lambda, 0, \varphi) \equiv \mathcal{P}_\Sigma(\gamma, \lambda, \varphi)$ as defined below:

A function $f \in \Sigma$ is said to be in the class $\mathcal{P}_\Sigma(\gamma, \lambda, \varphi)$, $0 \neq \gamma \in \mathbb{C}$, $0 \leq \lambda \leq 1$, if the following subordinations hold:

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda wg'(w)} - 1\right) \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$. A function in the class $\mathcal{P}_\Sigma(\gamma, \lambda, \varphi)$ is called both bi-$\lambda-$convex functions and bi-λ-starlike functions of complex order γ of Ma-Minda type.

For functions in the class $\mathcal{P}_\Sigma(\gamma, \lambda, \varphi)$, the following coefficient estimation holds.

Corollary 3.7.([5]) *If* $f \in \mathcal{P}_\Sigma(\gamma, \lambda, \varphi)$, *then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|\gamma(1 + 2\lambda - \lambda^2)B_1^2 + (1+\lambda)^2(B_1 - B_2)|}}$$

and

$$|a_3| \leq \frac{|\gamma|[B_1 + |B_2 - B1|]}{1 + 2\lambda - \lambda^2}.$$

Remark 3.8. Taking $\delta = 1$, we have the class $\mathcal{M}_\Sigma(\gamma, \lambda, 1, \varphi) \equiv \mathcal{K}_\Sigma(\gamma, \lambda, \varphi)$ as defined below:

A function $f \in \Sigma$ is said to be in the class $\mathcal{K}_\Sigma(\gamma, \lambda, \varphi)$, $0 \neq \gamma \in \mathbb{C}$, $0 \leq \lambda \leq 1$, if the following subordinations hold:

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + (1+2\lambda)z^2 f''(z) + \lambda z^3 f'''(z)}{zf'(z) + \lambda z^2 f''(z)} - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + (1+2\lambda)w^2 g''(w) + \lambda w^3 g'''(w)}{wg'(w) + \lambda w^2 g''(w)} - 1\right) \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$.

For functions in the class $\mathcal{K}_\Sigma(\gamma, \lambda, \varphi)$, the following coefficient estimation holds.

Corollary 3.9. *If $f \in \mathcal{K}_\Sigma(\gamma, \lambda, \varphi)$, then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|\gamma(2 + 4\lambda - 4\lambda^2) B_1^2 + 4(1+\lambda)^2 (B_1 - B_2)|}}$$

and

$$|a_3| \leq \frac{|\gamma| [B_1 + |B_2 - B1|]}{2 + 4\lambda - 4\lambda^2}.$$

Remark 3.10. Furthermore, various other interesting corollaries and consequences of our results could be derived similarly by specializing φ.

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