KYUNGPOOK Math. J. 55(2015), 689-694
http://dx.doi.org/10.5666/KMJ.2015.55.3.689
pISSN 1225-6951 eISSN 0454-8124

Inequalities for a Polynomial and its Derivative

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Abstract. In this paper we consider the class of polynomials of the type $p(z) = z^s\left(a_0+\sum_{j=\mu}^{n-s}a_jz^j\right)$, $1\le\mu\le n-s$, $0\le s\le n-1$ having some zeros at origin and rest of zeros on or outside the boundary of a prescribed disk, and obtain the generalization of well known results.

1. Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{1.1}$$

Inequality (1.1) is a well known result of S-Bernstein [4]. Equality holds in (1.1) if and only if $p(z)$ has all its zeros at the origin. If we restrict ourselves to the class of polynomial not vanishing in $|z|<1$, then

$$\max_{|z|=1}|p'(z)|\le \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.2}$$

Equality holds in (1.2) for $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$. Inequality (1.2) was conjectured by Erdös and later verified lax [6].

As an extension of (1.2) Malik [7] proved that if $p(z)\ne 0$ in $|z|<k$, $k\ge 1$, then

$$\max_{|z|=1}|p'(z)|\le \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.3}$$

The result is sharp and equality holds for $p(z)=(z+k)^n$. under the same hypothesis, Govil [10]. proved that

$$\max_{|z|=1}|p'(z)|\le \frac{n}{1+k}\Big\{\max_{|z|=1}|p(z)|-\min_{|z|=k}|p(z)|\Big\}. \tag{1.4}$$

Received January 25, 2014; accepted August 21, 2014.
2010 Mathematics Subject Classification: 30A10, 30C10, 30D15, 41A17.
Key words and phrases: Derivative, polynomial, inequality, Zeros.

Chan and Malik [2] generalized inequality (1.4) and proved the following.

Theorem A. *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu < n$ *be a polynomial of degree* n *having all its zeros in* $|z| \ge k$, $k \ge 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)| . \tag{1.5}$$

The result is best possible and equality holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

Pukhta [8] improved the inequality (1.5) and proved the following result.

Theorem B. *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu < n$ *be a polynomial of degree* n *having all its zeros in* $|z| \ge k$, $k \ge 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^\mu} \Big\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \Big\}. \tag{1.6}$$

The result is best possible and equality holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

Theorem C. *Let* $p(z)$ *be a polynomial of degree* n *having all its zeros in* $|z| = k$, $k \le 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)| . \tag{1.7}$$

Theorem C is due to Govil [5].

For the polynomial of type $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$, Theorem C was generalized by Dewan and Hans [3] and proved the following.

Theorem D. *Let* $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \le 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)| . \tag{1.8}$$

In this paper we consider the class of polynomials of type $p(z) = z^s \left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right)$, $1 \le \mu < n-s$, $0 \le s \le n-1$ and proved the following generalizations of Theorem B and Theorem D.

Theorem 1. *Let* $p(z) = z^s \left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right)$, $1 \le \mu < n-s$, $0 \le s \le n-1$, *be a polynomial of degree* n *having s-fold zeros at the origin and remaining n-s zeros in* $|z| > k$, $k \ge 1$, *then*

$$\begin{aligned}\max_{|z|=1} |p'(z)| \le{}& \frac{\begin{pmatrix} (n-s)^2|a_0| + (n-s)\mu|a_\mu|k^{\mu+1} \\ +s(n-s)|a_0|(1+k^{\mu+1}) \\ +s\mu|a_\mu|(k^{\mu+1}+k^{2\mu}) \end{pmatrix}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})} \max_{|z|=1} |p(z)| \\ &- \frac{(n-s)^2|a_0| + (n-s)\mu|a_\mu|k^{\mu+1}}{k^s(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})} \min_{|z|=k} |p(z)|.\end{aligned}$$

If we take $\mu = 1$ in Theorem 1, we get the following result.

Corollary 1. *Let* $p(z) = z^s \left(a_0 + \sum_{j=0}^{n-s} a_j z^j\right)$, $0 \le s \le n-1$, *be a polynomial of degree* n *having* s*-fold zeros at the origin and remaining* $n-s$ *zeros in* $|z| > k$, $k \ge 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{\begin{pmatrix} (n-s)^2|a_0| + (n-s)|a_1|k^2 \\ +s(n-s)|a_0|(1+k^2) + 2s|a_1|k^2 \end{pmatrix}}{(n-s)|a_0|(1+k^2) + 2|a_1|k^2} \max_{|z|=1} |p(z)| - \frac{(n-s)^2|a_0| + (n-s)|a_1|k^2}{k^s(n-s)|a_0|(1+k^2) + 2|a_1|k^2} \min_{|z|=k} |p(z)| .$$

Next we prove the following result which is an extension of Theorem D for the polynomial of the type $p(z) = z^s \left(a_{n-s}z^{n-s} + \sum_{j=\mu}^{n-s} a_{n-s-j} z^{n-s-j}\right)$, $1 \le \mu < n-s$, $0 \le s \le n-1$.

Theorem 2. Let $p(z) = z^s \left(a_{n-s}z^{n-s} + \sum_{j=\mu}^{n-s} a_{n-s-j} z^{n-s-j}\right)$, $1 \le \mu < n-s$, $0 \le s \le n-1$ be a polynomial of degree n having s-fold zeros at the origin and remaining $n-s$ zeros in $|z| > k$, $k \ge 1$, then

$$\max_{|z|=1} |p'(z)| \le \frac{(n-s)}{k^{n-s-\mu+1}} \left\{ \frac{(n-s)|a_{n-s}|k^{2\mu} + \mu|a_{n-s-\mu}|k^{\mu-1}}{\begin{pmatrix} (\mu|a_{n-s-\mu}|(1+k^{\mu-1}) \\ +(n-s)|a_{n-s}|k^{\mu-1}(1+k^{\mu+1}) \end{pmatrix}} + s \right\} \max_{|z|=1} |p(z)| .$$

By choosing $\mu = 1$ in Theorem 2, we get the following result.

Corollary 2. *Let* $(z) = z^s \left(a_{n-s}z^{n-s} + \sum_{j=1}^{n-s} a_{n-s-j} z^{n-s-j}\right)$, $0 \le s \le n-1$ *be a polynomial of degree* n *having* s*-fold zeros at the origin and remaining* $n-s$ *zeros in* $|z| = k$, $k \ge 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{(n-s)}{k^{n-s}} \left\{ \frac{(n-s)|a_{n-s}|k^2 + |a_{n-s-1}|}{2|a_{n-s-1}| + (n-s)|a_{n-s}|(1+k^2)} + s \right\} \max_{|z|=1} |p(z)| .$$

For the proof of theorems, we need the following lemmas.

2. Lemmas

Lemma 1. *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu < n$ *is a polynomial of degree* n *having all its zeros on* $|z| \ge k$, $k \ge 1$, *then*

$$\max_{|z|=1} |p'(z)| \le n \left[\frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{n|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \right] \max_{|z|=1} |p(z)| .$$

The above lemma is due to Qazi [9, lemma 1].

Lemma 2. *If* $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-\nu}$, $1 \le \mu < n$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \le 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-\mu+1}} \left[\frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|(k^{2\mu} + k^{\mu-1}) + \mu|a_{n-1}|(k^{2\mu} - k^{\mu-1})} \right] \max_{|z|=1} |p(z)| .$$

The above lemma is due to M.S. Pukhta, Abdullah Mir and T.A. Raja [1].

3. Proof of the Theorems

Proof of Theorem 1. Let

$$p(z) = z^s H(z) \quad \text{where } H(z) = a_0 + \sum_{j=\mu}^{n-s} a_j z^j \tag{1}$$

is a polynomial of degree $n - s$ having all its zeros in $|z| > k$, $k \ge 1$. From (1) we have

$$\begin{aligned} zp'(z) &= sz^s H(z) + z^{s+1} H'(z) \\ &= sp(z) + z^{s+1} H'(z) . \end{aligned}$$

This gives for $|z| = 1$,

$$|p'(z)| \le s|p(z)| + |H'(z)| . \tag{2}$$

Inequality (2) holds for all points on $|z| = 1$ and hence

$$|p'(z)| \le s|p(z)| + \max_{|z|=1} |H'(z)| . \tag{3}$$

Let $m = \min_{|z|=k} |H(z)|$, then $m \le |H(z)|$ for $|z| = k$. As all $n - s$ zeros of $|H(z)|$ lie in $|z| > k$, $k \ge 1$, therefore for every complex number λ such $|\lambda| < 1$, it follows by Rouche's Theorem that all the zeros of the polynomial $H(z) - \lambda m$ of degree $n - s$ lie in $|z| > k$, $k \ge 1$. Therefore, By using Lemma 1 to the polynomial $H(z) - \lambda m$ of degree $n - s$, we get

$$\begin{aligned} \max_{|z|=1} |H'(z)| \le (n-s) &\left\{ \frac{(n-s)|a_0| + \mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1 + k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \right\} \\ &\cdot \max_{|z|=1} |H(z) - \lambda m| . \end{aligned} \tag{4}$$

Choosing the argument of λ such that

$$|H(z) - \lambda m| = |H(z)| - |\lambda| m \quad \text{for } |z| = 1 . \tag{5}$$

And letting $|\lambda| \to 1$, we get from (4) and (5)

$$\max_{|z|=1}|H'(z)| \le (n-s)\left\{\frac{(n-s)|a_0| + \mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right\} \cdot \max_{|z|=1}(|H(z)| - m) \tag{6}$$

Combining inequalities (3) and (6), we obtain

$$\begin{aligned}|H'(z)| &\le (n-s)\left\{\frac{(n-s)|a_0| + \mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right\} \\ &\quad \cdot \max_{|z|=1}|H(z)| - |H'(z)| \\ &\le (n-s)\left\{\frac{(n-s)|a_0| + \mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right\} m + s|p(z)| \end{aligned} \tag{7}$$

From (1) we have $|p(z)| = |H(z)|$ on $|z| = 1$ and that one can easily obtain

$$m = \min_{|z|=k}|H(z)| = \frac{1}{k^s}\min_{|z|=k}|p(z)|.$$

This gives from (7)

$$\begin{aligned}\max_{|z|=1}|p'(z)| \le{}& \frac{\left(\begin{array}{l}(n-s)^2|a_0| + (n-s)\mu|a_\mu|k^{\mu+1} \\ +s(n-s)|a_0|(1+k^{\mu+1}) + s\mu|a_\mu|(k^{\mu+1}+k^{2\mu})\end{array}\right)}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\max_{|z|=1}|p(z)| \\ &- \frac{(n-s)^2|a_0| + (n-s)\mu|a_\mu|k^{\mu+1}}{k^s(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\min_{|z|=k}|p(z)|\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2 Let

$$p(z) = z^s H(z) \quad \text{where } H(z) = a_{n-s}z^{n-s} + \sum_{j=\mu}^{n-s} a_{n-s-j}z^{n-s-j} \tag{8}$$

is a polynomial of degree $n-s$ having all its zeros on $|z| = k$, $k \le 1$. By using Lemma 2 to the polynomial $H(z)$ of degree $n-s$, we get

$$\max_{|z|=1}|H'(z)| \le \frac{(n-s)}{k^{n-s-\mu+1}}\left\{\frac{(n-s)|a_{n-s}|k^{2\mu} + \mu|a_{n-s-\mu}|k^{\mu-1}}{\left(\begin{array}{l}\mu|a_{n-s-\mu}|(1+k^{\mu-1}) \\ +(n-s)|a_{n-s}|k^{\mu-1}(1+k^{\mu+1})\end{array}\right)} + s\right\}\max_{|z|=1}|H(z)| \tag{9}$$

From (7) one can easily obtain for $|z| = 1$

$$|p'(z)| \le s|p(z)| + \max_{|z|=1}|H'(z)| \tag{10}$$

On combining the inequalities (9) and (10) and using the fact that for $|z| = 1$, $|p(z)| = |H(z)|$, we get

$$\max_{|z|=1}|p'(z)| \leq \frac{(n-s)}{k^{n-s-\mu+1}}\left\{\frac{(n-s)|a_{n-s}|k^{2\mu}+\mu|a_{n-s-\mu}|k^{\mu-1}}{\left(\begin{array}{l}\mu|a_{n-s-\mu}|(1+k^{\mu-1})\\ +(n-s)|a_{n-s}|k^{\mu-1}(1+k^{\mu+1})\end{array}\right)}+s\right\}\max_{|z|=1}|p(z)|$$

This completes the proof of Theorem 2. □

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