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Fekete-Szegö Inequalities for Quasi-Subordination Functions Classes of Complex Order

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Abstract. In this paper, we obtain Fekete-Szegö inequalities for certain subclasses of analytic univalent functions of complex order associated with quasi-subordination.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and with the following Taylor expansion:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{D}). \tag{1.1}$$

Also, let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions.

For two functions $f, g \in \mathcal{A}$, we say that the function f *is subordinate to* g in $\mathbb{D}$ (f *is majorized by* g, respectively), and write $f \prec g$ (resp. $f << g$) in $\mathbb{D}$, if there exists a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ (resp. if there exists analytic function ϕ with $|\phi(z)| \le 1$ such that $f(z) = \phi(z)g(z)$) for $z \in \mathbb{D}$. In particular, if the function g is univalent in $\mathbb{D}$, the subordination condition is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ (see [18]).

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The notion of a subordination was used very frequently in the literature see, for example [18, 17, 9, 10, 11, 12]. As an example, let us define the concept of the generalized starlikeness and convexity. Let φ be an analytic function with positive real part on $\mathbb{D}$, such that $\varphi(0) = 1, \varphi'(0) > 0$, and which maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Following Ma and Minda [17], let $\mathcal{P}(\varphi)$ denote the family of functions $\psi \prec \varphi$ in $\mathbb{D}$. The class $\mathcal{P}((1+z)/(1-z))$ is a usual class of the Carathéodory functions, denoted generally $\mathcal{P}$. Upholding the notation, let $\mathcal{S}^*(\varphi)$ be the class of functions $f \in \mathcal{S}$, and univalent in $\mathbb{D}$ for which

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (z \in \mathbb{D}), \tag{1.2}$$

and $\mathcal{C}(\varphi)$ be the class of functions $f(z) \in \mathcal{A}$ such that

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \quad (z \in \mathbb{D}). \tag{1.3}$$

The special cases of $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ consist of functions known as the functions starlike (respectively convex) of order α (denoted $\mathcal{S}^*(\alpha)$, and $\mathcal{C}(\alpha)$ resp.) or strongly starlike, and convex of order γ (denoted $\mathcal{S}^*_\gamma$, and $\mathcal{C}_\gamma$ resp.). The last cases correspond to the choice $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$, and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\gamma$ where $\alpha \in [0,1), \gamma \in (0,1]$ (see Robertson [23]). Also, the second author considered the classes related to conic sections [9, 10, 11], where function φ maps the unit disk onto the domain whose boundary is a conic section.

The majorization notion was not exploited so frequently, but several results are well known (see, for example [2, 6, 7, 16]).

Joining the notion of subordination and majorization Robertson [22] (see also [2]) introduced the concept of quasi-subordination. For two analytic functions f and g we say that the function f is *quasi-subordinated to* g, and write $f \prec_q g$, if there exists analytic functions ϕ and w with $|\phi(z)| \leq 1, w(0) = 0$, $|w(z)| < 1$, and such that $f(z) = \phi(z)g(w(z))$. Observe that, when $\phi(z) = 1$, then $f(z) = g(w(z))$, so that $f \prec g$ in $\mathbb{D}$. Also notice that if $w(z) = z$, then $f(z) = \phi(z)g(z)$ and it is said that f is majorized by g in $\mathbb{D}$.

Applying the notion of the quasi-subordination we define the class of generalized starlikeness and convexity with complex parameter, below.

Definition 1.1. Let $f \in \mathcal{A}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Also, let $\varphi \in \mathcal{P}$ be univalent and starlike with respect to 1. We say that f is in the class $\mathcal{S}^*_q(\lambda, \varphi)$, if

$$1 + \frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec_q \varphi(z) \quad (z \in \mathbb{D}). \tag{1.4}$$

Example 1.2. Let $\lambda \in \mathbb{C}\setminus\{0\}$, and let $\varphi \in \mathcal{P}$ be defined as in Definition 1. Defining the function f as follows

$$\frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right) = z\left(\varphi(z) - 1\right) \prec_q \varphi(z) - 1$$

or, equivalently

$$\left(\frac{f(z)}{z}\right)^{\frac{1}{\lambda}} = \exp\left(-z + \int\limits_0^z \varphi(\xi)d\xi\right),$$

we obtain that $f \in \mathcal{S}_q^*(\lambda, \varphi)$.

Definition 1.3. For $f \in \mathcal{A}$, $\lambda \in \mathbb{C} \setminus \{0\}$, and φ as in the Definition 1.1, we say that f is in the class $\mathcal{C}_q(\lambda, \varphi)$, if

$$1 + \frac{1}{\lambda}\frac{zf''(z)}{f'(z)} \prec_q \varphi(z) \quad (z \in \mathbb{D}). \tag{1.5}$$

By the well known Alexander equivalence the following relation holds

$$f \in \mathcal{C}_q^*(\lambda, \varphi) \Longleftrightarrow zf' \in \mathcal{S}_q^*(\lambda, \varphi). \tag{1.6}$$

Example 1.4. Since

$$\frac{1}{\lambda}\frac{zf''(z)}{f'(z)} = z\left(\varphi(z) - 1\right) \prec_q \varphi(z) - 1, \tag{1.7}$$

the function f defined by

$$\left(f'(z)\right)^{\frac{1}{\lambda}} = \exp\left(-z + \int\limits_0^z \varphi(\xi)d\xi\right)$$

is in $\mathcal{C}_q^*(\lambda, \varphi)$.

The classes $\mathcal{S}_q^*(\lambda, \varphi)$ and $\mathcal{C}_q(\lambda, \varphi)$ become the classes $\mathcal{S}_\lambda^*(\varphi)$ and $\mathcal{C}_\lambda(\varphi)$, respectively, for the case, when $\phi(z) = 1$ (see Ravichandran et al. [21]).

Upon setting $\lambda = 1$, the classes $\mathcal{S}_q^*(\lambda, \varphi)$ and $\mathcal{C}_q(\lambda, \varphi)$ reduce to the classes $\mathcal{S}_q^*(\varphi)$ and $\mathcal{C}_q(\varphi)$, respectively (see Mohd and M. Darus [19]), which are a quasi-subordination analog of the Ma-Minda starlike and convex classes.

Also, we note that if $|\theta| < \frac{\pi}{2}$ and $0 \le \rho < 1$, then $\mathcal{S}_q^*((1-\rho)e^{-i\theta}\cos\theta, \varphi) = \mathcal{S}_q^\theta(\rho, \varphi)$, which is a generalization of θ-spirallike functions of order ρ with $\phi(z) = 1$ and $w(z) = z$ (see [15]). Precisely

$$\mathcal{S}_q^\theta(\rho, \varphi) = \left\{ f \in \mathcal{A} : \frac{e^{i\theta}\,\frac{zf'(z)}{f(z)} - \rho\cos\theta - i\sin\theta}{(1-\rho)\cos\theta} \prec_q \varphi(z) \right\}.$$

Similarly, for the same range of parameters, we obtain $\mathcal{C}_q((1-\rho)e^{-i\theta}\cos\theta, \varphi) = \mathcal{C}_q^\theta(\rho, \varphi)$. The last class is defined as follows

$$\mathcal{C}_q^\theta(\rho, \varphi) = \left\{ f \in \mathcal{A} : \frac{e^{i\theta}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \rho\cos\theta - i\sin\theta}{(1-\rho)\cos\theta} \prec_q \varphi(z) \right\},$$

and it generalizes a class introduced by [3] (it is enough to set $\phi(z) = 1$, $w(z) = z$) which is related to the class $\mathcal{S}_q^{\theta}(\rho, \varphi)$ by the Alexander relation.

The Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for normalized univalent functions of the form (1.1) is well known from its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1932 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in many subclasses of the family of univalent functions (see e.g. [5, 8, 13, 20, 24, 1]). For a brief history of the Fekete- Szegö problem for class of starlike, convex, and close-to convex functions, see the recent paper by Srivastava et al. [25]. In the present paper, we obtain the Fekete-Szegö inequalities for functions in the classes $\mathcal{S}_q^*(\lambda, \varphi)$ and $\mathcal{C}_q(\lambda, \varphi)$.

To prove our results, we need the following lemmas.

Lemma 1.5.([[4], p.198]) *If w be the function analytic in the open unit disc $\mathbb{D}$, satisfying $w(0) = 0$ and $|w(z)| < 1$, and let $w(z) = d_1 z + d_2 z^2 + \cdots$. Then $|d_1| \le 1$, and for any natural number $n \ge 2$,*

$$|d_n| \le 1 - |d_1|^2.$$

The result is sharp for the functions given by

$$w(z) = z \text{ or } w(z) = z^n.$$

Lemma 1.6.([14]) *If $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is a function with positive real part, then $|b_1| \le 2$, and for any complex number μ,*

$$|b_2 - \mu b_1^2| \le 2 \max\{1, |2\mu - 1|\}, \tag{1.8}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2} \quad p(z) = \frac{1 + z}{1 - z}.$$

Lemma 1.7.([14]) *If ϕ be the function analytic in the open unit disc $\mathbb{D}$, satisfying $|\phi(z)| < 1$, and let $\phi(z) = c_0 + c_1 z + c_2 z^2 + \ldots$. Then $|c_0| \le 1$, and*

$$|c_1| \le 1 - |c_0|^2.$$

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that the function f is of the form (1.1), $w(z) = d_1 z + d_2 z^2 + \cdots$, $\phi(z) = c_0 + c_1 z + c_2 z^2 + \ldots$, and $\varphi(z) = 1 + b_1 z + b_2 z^2 + \ldots$.

Theorem 2.1. *Let $\lambda \in \mathbb{C} \setminus \{0\}$, and let $\varphi \in \mathcal{P}$ be univalent and starlike with respect to 1. If $f \in \mathcal{S}_q^*(\lambda, \varphi)$, then*

$$|a_2| \le |\lambda||b_1|, \tag{2.1}$$

$$|a_3| \leq \frac{|\lambda|}{2}\left[|b_1|(|c_0|+1)+2|c_0|\max\left\{1,|2\lambda c_0-1|\right\}\right], \tag{2.2}$$

and for any complex number μ*,*

$$|a_3-\mu a_2^2| \leq \frac{|\lambda|}{2}\left[|b_1|(|c_0|+1)+2|c_0|\max\left\{1,|2\lambda(1-2\mu)c_0-1|\right\}\right]. \tag{2.3}$$

The result is sharp.

Proof. Let $f \in \mathcal{S}_q^*(\lambda,\varphi)$. Then there exist an analytic functions ϕ and w in $\mathbb{D}$, with $|\phi| \leq 1, w(0)=0$ and $|w(z)|<1$ in $\mathbb{D}$, and such that

$$\frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)=\phi(z)\left[(\varphi(w(z))-1)\right]. \tag{2.4}$$

Since

$$\frac{zf'(z)}{f(z)}-1=a_2z+(-a_2^2+2a_3)z^2+...,$$

and

$$\phi(z)\left(\varphi(w(z))-1\right)=b_1c_0d_1z+(b_1c_1d_1+c_0(b_1d_2+b_2d_1^2)z^2+...,$$

then, comparing both sides of (2.4) we see that

$$a_2=\lambda b_1c_0d_1,$$

from which, by the inequalities $|c_0| \leq 1, |d_1| \leq 1$, we immediately obtain (2.1). Moreover, we have

$$-a_2^2+2a_3=\lambda b_1c_1d_1+\lambda c_0(b_1d_2+b_2d_1^2),$$

or equivalently

$$a_3=\frac{\lambda}{2}\left(b_1c_1d_1+c_0b_1d_2+c_0b_2d_1^2+\lambda c_0^2b_1^2d_1^2\right).$$

Further,

$$a_3-\mu a_2^2=\frac{\lambda}{2}\left(b_1c_1d_1+c_0b_1d_2+c_0b_2d_1^2+\lambda(1-2\mu)c_0^2b_1^2d_1^2\right), \tag{2.5}$$

then applying $|d_n| \leq 1$ and $|c_1| \leq 1$, we have

$$|a_3-\mu a_2^2| \leq \frac{|\lambda|}{2}\left[|b_1|(|c_0|+1)+|c_0||d_1|^2|b_2+\lambda(1-2\mu)c_0b_1^2|\right].$$

Using the estimate $|d_1| \leq 1$, and Lemma 1.6 we obtain

$$|a_3-\mu a_2^2| \leq \frac{|\lambda|}{2}\left[|b_1|(|c_0|+1)+2|c_0|\max\left\{1,|2\lambda(1-2\mu)c_0-1|\right\}\right].$$

The case $\mu = 0$ gives the estimate (2.2). The result is sharp for the functions

$$\frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right) = \phi(z)\left(\varphi(z^2) - 1\right), \tag{2.6}$$

and

$$\frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right) = \phi(z)\left(\varphi(z) - 1\right). \tag{2.7}$$

This completes the proof of Theorem 2.1. □

Remark 2.2. The case $\lambda = 1$ in Theorem 2.1 reduces to the result obtained by Mohd and Darus [19, Theorem 2.1], and when $\phi(z) \equiv 1$ in Theorem 2.1, then the obtained result improves the result Ravichandran et al. [21, Theorem 4.1].
Setting $\phi(z) \equiv 1$, and $\lambda = 1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.3. *If f given by*(1.1) *belongs to the class $\mathcal{S}^*(\varphi)$, then*

$$|a_2| \le |b_1|, \quad |a_3| \le \frac{|b_1|}{2}.$$

The result is sharp.
Putting $\lambda = (1-\rho)e^{-i\theta}\cos\theta$ $\left(|\theta| < \frac{\pi}{2},\ 0 \le \rho < 1\right)$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.4. *If f given by* (1.1) *belongs to the class $\mathcal{S}_q^\theta(\rho, \varphi)$, then*

$$|a_2| \le |b_1|(1-\rho)\cos\theta$$

$$|a_3| \le \frac{(1-\rho)\cos\theta}{2}\left[|b_1| + 2|c_0|\max\{1, |2c_0(1-\rho)e^{-i\theta}\cos\theta - 1|\}\right]$$

and, for any complex number μ,

$$|a_3 - \mu a_2^2| \le \frac{(1-\rho)\cos\theta}{2}\left[|b_1| + 2|c_0|\max\left\{1, |2c_0(1-2\mu)(1-\rho)e^{-i\theta}\cos\theta - 1|\right\}\right].$$

The result is sharp.

Theorem 2.5. *If f given by* (1.1)*, and satisfies*

$$\frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right) \ll \varphi(z) - 1,$$

then

$$|a_2| \le |\lambda||b_1|, \quad |a_3| \le \frac{|\lambda||b_1|}{2}.$$

The result is sharp.

Proof. The result follows by taking $w(z) \equiv z$, and $\phi(z) \equiv 1$ in the proof of Theorem 2.1. □

Setting $\lambda = (1-\rho)e^{-i\theta}\cos\theta$ ($|\theta| < \dfrac{\pi}{2}$, $0 \le \rho < 1$) in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *If $f(z)$ given by* (1.1) *and satisfies*

$$\frac{e^{i\theta}}{(1-\rho)\cos\theta}\left(\frac{zf'(z)}{f(z)} - 1\right) \ll \varphi(z) - 1,$$

then

$$|a_2| \le |b_1|(1-\rho)\cos\theta, \quad |a_3| \le \frac{(1-\rho)|b_1|\cos\theta}{2}.$$

The result is sharp.

Theorem 2.7. *If f given by* (1.1) *belongs to the class $\mathcal{C}_q(\lambda, \varphi)$, then*

$$|a_2| \le \frac{|\lambda||b_1|}{2}$$

$$|a_3| \le \frac{|\lambda|}{6}\left[|b_1| + 2|c_0| \max\left\{1, |2\lambda c_0 - 1|\right\}\right]$$

and, for any complex number μ,

$$|a_3 - \mu a_2^2| \le \frac{|\lambda|}{6}\left[|b_1| + 2|c_0| \max\left\{1, |\lambda(2-3\mu)c_0 - 1|\right\}\right].$$

The result is sharp.

Proof. From (1.6) $zf' \in \mathcal{S}_q^*(\lambda, \varphi)$, then (2.4) becomes

$$\frac{1}{\lambda}\left(\frac{z\left(zf'(z)\right)'}{zf'(z)} - 1\right) = \phi(z)\left[(\varphi(w(z)) - 1)\right]$$

or equivalently

$$\frac{1}{\lambda}\left(\frac{zf''(z)}{f'(z)}\right) \prec \phi(z)\left[\varphi(w(z)) - 1\right].$$

Using arguments similar to those in the proof of Theorem 2.1, we can obtain the required estimates. □

Remark 2.8. Putting $\lambda = 1$ in Theorem 2.7, we obtain the result obtained by Mohd and Darus [19, Theorem 2.3].

For the case $\phi(z) \equiv 1$ Theorem 2.7 reduces to the following corollary.

Corollary 2.9. *If f given by* (1.1) *belongs to the class $\mathcal{C}_\lambda(\varphi)$, then*

$$|a_2| \le \frac{|\lambda||b_1|}{2}, \quad |a_3| \le \frac{|\lambda||b_1|}{6}.$$

The result is sharp.

Setting $\phi(z) = \lambda = 1$ in Theorem 2.7, we obtain the following corollary.

Corollary 2.10. *If f given by* (1.1) *belongs to the class $\mathcal{C}(\varphi)$, then*

$$|a_2| \leq \frac{|b_1|}{2}, \quad |a_3| \leq \frac{|b_1|}{6}.$$

The result is sharp.

Putting $\lambda = (1-\rho)e^{-i\theta}\cos\theta$ ($|\theta| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Theorem 2.7, we obtain the following corollary.

Corollary 2.11. *If $f(z)$ given by* (1.1) *belongs to the class $\mathcal{C}_q^\theta(\rho, \varphi)$, then*

$$|a_2| \leq \frac{|b_1|(1-\rho)\cos\theta}{2}$$

$$|a_3| \leq \frac{(1-\rho)\cos\theta}{6}\left[|b_1| + 2|c_0| \max\left\{1, |2(1-\rho)c_0 e^{-i\theta}\cos\theta - 1|\right\}\right]$$

and μ is a complex number

$$|a_3 - \mu a_2^2| \leq \frac{(1-\rho)\cos\theta}{6}\left[|b_1| + 2|c_0| \max\left\{1, |(1-\rho)e^{-i\theta}\cos\theta(2-3\mu)c_0 - 1|\right\}\right].$$

The result is sharp.

Theorem 2.12. *If f given by* (1.1)*, and satisfies*

$$\frac{1}{\lambda}\left(\frac{zf''(z)}{f'(z)}\right) \ll \varphi(z) - 1,$$

then

$$|a_2| \leq \frac{|\lambda||b_1|}{2}, \quad |a_3| \leq \frac{|\lambda||b_1|}{6}.$$

The result is sharp.

Set $\lambda = (1-\rho)e^{-i\theta}\cos\theta$ ($|\theta| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Theorem 2.12. Then we obtain the following corollary.

Corollary 2.13. *If f given by* (1.1) *and satisfies*

$$\frac{e^{i\theta}}{(1-\rho)\cos\theta}\left(\frac{zf''(z)}{f'(z)}\right) \ll \varphi(z) - 1,$$

then

$$|a_2| \leq \frac{|b_1|(1-\rho)\cos\theta}{2}, \quad |a_3| \leq \frac{|b_1|(1-\rho)\cos\theta}{6}.$$

The result is sharp.

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