

Certain Nonlinear Differential Polynomials Sharing a Nonzero Polynomial with Finite Weight

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ABSTRACT. With the notion of weighted sharing of values we study the uniqueness of meromorphic functions when certain nonlinear differential polynomials share a nonzero polynomial. Our results improve some recent results including that of a present first author [Kyungpook Math. J., 51(2011), 43-58].

1. Introduction

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [6], [15] and [16]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h , we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$ ($r \rightarrow \infty, r \notin E$).

Let f and g be two nonconstant meromorphic functions. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same set of zeros with the same multiplicities and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities.

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Throughout this paper, we need the following definition.

$$\Theta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

In 1959, Hayman [5] proved the following result.

Theorem A. *Let f be a transcendental entire function, and let $n(\geq 1)$ be an integer. Then $f^n f' = 1$ has infinitely many zeros.*

In 2002, Fang and Fang [4] proved the following result.

Theorem B. *Let f and g be two nonconstant entire functions, and let $n(\geq 8)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In the same year Fang [3] investigated the value sharing of more general nonlinear differential polynomial than that was considered in Theorem B and obtained the following result.

Theorem C. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

In 2004, Lin and Yi [12] considered the case of meromorphic function in Theorem B and obtained the following.

Theorem D. *Let f and g be two nonconstant meromorphic functions with $\Theta(\infty, f) > 2/(n+1)$, and let $n(\geq 12)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

Natural inquisition would be to investigate the situation for meromorphic function in Theorem C. In this direction in 2008, Zhang [17] proved the following result.

Theorem E. *Suppose that f is a transcendental meromorphic function with finite number of poles, g is a transcendental entire function, and let n, k be two positive integers with $n \geq 2k + 6$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f \equiv g$.*

To proceed further we require the following definition known as weighted sharing of values introduced by I. Lahiri [7] which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1.1. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

The definition implies that if f, g share a value a with weight k , then z_0 is an a -point of f with multiplicity $m(\leq k)$ if and only if it is an a -point of g with multiplicity $m(\leq k)$ and z_0 is an a -point of f with multiplicity $m(> k)$ if and only

if it is an a -point of g with multiplicity $n(> k)$, where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for any integer $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 2009, using the notion of weighted sharing of values, Xu, Yi and Cao [13] proved the following result.

Theorem F. *Let f and g be two nonconstant meromorphic functions, and $n(\geq 1)$, $k(\geq 1)$ and $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Suppose $[f^n(f - 1)]^{(k)}$ and $[g^n(g - 1)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n > 5k + 11$ or if $l = 1$ and $n > 7k + 23/2$, then $f = g$.*

Note 1.1. The above result is partially correct. According to the method of approach in [13] there is a gap in the proof of Theorem 1.2.

Recently, Li [11] proved the following result which rectify and at the same time improve Theorem F.

Theorem G. *Let f and g be two nonconstant meromorphic functions, and $n(\geq 1)$, $k(\geq 1)$ and $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Suppose $[f^n(f - 1)]^{(k)}$ and $[g^n(g - 1)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n > 3k + 11$ or if $l = 1$ and $n > 5k + 14$, then $f = g$ or $[f^n(f - 1)]^{(k)}[g^n(g - 1)]^{(k)} = 1$.*

Note 1.2. Unfortunately *Theorem G* is also not true for rational functions. One can easily see that in the proof of *Lemma 2.5* in *subcase 3.2*, (2.22) is only true when f and g are transcendental functions. Since in the proof of *Theorem G*, *Lemma 2.5* plays a vital role, the same theorem cease to be hold for rational functions.

In this direction recently the present first author [1] proved the following results first one of which improves Theorem G.

Theorem H. *Let f and g be two transcendental meromorphic functions and $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Suppose for two nonzero constants a and b , $[f^n(af + b)]^{(k)}$ and $[g^n(ag + b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 3k + 9$ or if $l = 1$ and $n \geq 4k + 10$ or if $l = 0$ and $n \geq 9k + 18$, then $f = g$ or $[f^n(af + b)]^{(k)}[g^n(ag + b)]^{(k)} = 1$. The possibility $[f^n(af + b)]^{(k)}[g^n(ag + b)]^{(k)} = 1$ does not occur for $k = 1$.*

Theorem I. *Let f and g be two transcendental entire functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose for two nonzero constants a and b , $[f^n(af + b)]^{(k)}$ and $[g^n(ag + b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 2k + 6$ or if $l = 1$ and $n \geq 5k/2 + 7$ or if $l = 0$ and $n \geq 5k + 12$, then $f = g$.*

Following questions are inevitable:

Question 1.1. Is it possible in any way to remove the second conclusion of Theorem H?

Question 1.2. What can be said if one replace the sharing value 1 by a nonzero polynomial in Theorems H and I?

In the paper, our main concern is to find the affirmative answer of the above questions. We prove following two theorems first one of which improves Theorem H and second one improves Theorem I. We now state the main results of the paper.

Theorem 1.1. *Let f and g be two non-entire transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Suppose for two nonzero constants a and b , $[f^n(af+b)]^{(k)} - P$ and $[g^n(ag+b)]^{(k)} - P$ share $(0, l)$ where $P(\neq 0)$ is a polynomial. If $l \geq 2$ and $n \geq 3k+9$ or if $l = 1$ and $n \geq 4k+10$ or if $l = 0$ and $n \geq 9k+18$, then $f = g$.*

Theorem 1.2. *Let f and g be two transcendental entire functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose for two nonzero constants a and b , $[f^n(af+b)]^{(k)} - P$ and $[g^n(ag+b)]^{(k)} - P$ share $(0, l)$ where $P(\neq 0)$ is a polynomial. If $l \geq 2$ and $n \geq 2k+6$ or if $l = 1$ and $n \geq 5k/2+7$ or if $l = 0$ and $n \geq 5k+12$, then $f = g$.*

However the following question is still open :

Question 1.3. Keeping all other conditions intact, are Theorems 1.1 and 1.2 true for rational functions also ?

Though the standard definitions and notations of the value distribution theory are available in [6], we explain some definitions and notations which are used in the paper.

Definition 1.2.([8]) For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f | = 1)$ the counting function of simple a points of f . For a positive integer p we denote by $N(r, a; f | \leq p)$ the counting function of those a -points of f (counted with multiplicities) whose multiplicities are not greater than p . By $\overline{N}(r, a; f | \leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r, a; f | \geq p)$ and $\overline{N}(r, a; f | \geq p)$.

Definition 1.3.([7]) Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2) + \dots + \overline{N}(r, a; f | \geq k).$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two nonconstant meromorphic functions defined in \mathbb{C} . We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

Lemma 2.1.([14]) *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \dots + a_1 f(z) + a_0,$$

where $a_n (\neq 0), a_{n-1}, \dots, a_1, a_0$ are complex numbers. Then

$$T(r, P_n(f)) = nT(r, f) + O(1).$$

Lemma 2.2.([18]) *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$(2.1) \quad N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f),$$

$$(2.2) \quad N_p(r, 0; f^{(k)}) \leq k\bar{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).$$

Lemma 2.3.([7]) *Let F and G be two nonconstant meromorphic functions sharing $(1, 2)$. Then one of the following cases holds:*

- (i) $T(r) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + S(r)$,
- (ii) $F = G$,
- (iii) $FG = 1$,

where $T(r)$ denotes the maximum of $T(r, F)$ and $T(r, G)$ and $S(r) = o\{T(r)\}$ as $r \rightarrow \infty$, possibly outside a set of finite linear measure.

Lemma 2.4.([2]) *Let F and G be two nonconstant meromorphic functions sharing $(1, 1)$ and $H \neq 0$. Then*

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2}\bar{N}(r, 0; F) + \frac{1}{2}\bar{N}(r, \infty; F) + S(r, F) + S(r, G).$$

Lemma 2.5.([2]) *Let F and G be two nonconstant meromorphic functions sharing $(1, 0)$ and $H \neq 0$. Then*

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\bar{N}(r, 0; F) + \bar{N}(r, 0; G) + 2\bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + S(r, F) + S(r, G).$$

Lemma 2.6.([10]) *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denote the counting function of those zeros of $f^{(k)}$ which are not the zeros of f , where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\bar{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\bar{N}(r, 0; f \mid \geq k) + S(r, f).$$

Lemma 2.7.([6],[15]) *Let f be a transcendental meromorphic function, and let $a_1(z), a_2(z)$ be two distinct meromorphic functions such that $T(r, a_i(z)) = S(r, f)$, $i=1,2$. Then*

$$T(r, f) \leq \bar{N}(r, \infty; f) + \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + S(r, f).$$

Lemma 2.8. *Let f and g be two non-entire transcendental meromorphic functions, let n, k be two positive integers and let P be a nonconstant polynomial. If $n \geq 2k+5$, then*

$$(f^n(af+b))^{(k)}(g^n(ag+b))^{(k)} \neq P^2,$$

where a, b are any two nonzero constants.

Proof. If possible, let

$$(2.3) \quad (f^n(af+b))^{(k)}(g^n(ag+b))^{(k)} = P^2.$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a zero of f with multiplicity $p_1(\geq 1)$. Then it follows from (2.3) that z_1 is a pole of g . Suppose that z_1 is a pole of g of order $q_1(\geq 1)$. Then we have

$$(2.4) \quad np_1 - k = (n+1)q_1 + k.$$

From (2.4) we get $q_1 + 2k = n(p_1 - q_1) \geq n$, i.e., $q_1 \geq n - 2k$. Thus from (2.4) we obtain $np_1 = (n+1)q_1 + 2k$, and so

$$p_1 \geq n - 2k + 1.$$

Let $z_2 \notin \{z : P(z) = 0\}$ be a zero of $af+b$ with multiplicity $p_2(\geq k+1)$. Then from (2.3) it follows that z_2 is a pole of g . Suppose that z_2 is a pole of g of order $q_2(\geq 1)$. Then we have $p_2 - k = (n+1)q_2 + k$, i.e.,

$$p_2 \geq n + 2k + 1.$$

If $z_3 \notin \{z : P(z) = 0\}$ is a zero of $af+b$ with multiplicity $p_3(\leq k)$, then from (2.3) it follows that z_3 may be a zero of $(f^n(af+b))^k$ and if it happens then it will be a pole of g with multiplicity $n+k+1$. Suppose that $z_4 \notin \{z : P(z) = 0\}$ be a pole of f . Then from (2.3) it is clear that z_4 is either a zero of $g^n(ag+b)$ or a zero of $(g^n(ag+b))^{(k)}$. Therefore

$$(2.5) \quad \begin{aligned} \overline{N}(r, \infty; f) &\leq \overline{N}(r, 0; g) + \overline{N}(r, 0; ag+b | \leq k) + \overline{N}(r, 0; ag+b | \geq k+1) \\ &+ \overline{N}(r, 0; h^{(k)} | h \neq 0) + S(r, g), \end{aligned}$$

where $\overline{N}(r, 0; h^{(k)} | h \neq 0)$ denotes the reduced counting function of those zeros of $h^{(k)}$ that are not the zeros of h and $h = g^n(ag+b)$.

By Lemma 2.6 we have

$$\begin{aligned} \overline{N}(r, 0; h^{(k)} \mid h \neq 0) &\leq \frac{1}{n+k+1} [N(r, 0; h^{(k)} \mid h \neq 0)] \\ &\leq \frac{1}{n+k+1} [k\overline{N}(r, \infty; h) + N(r, 0; h \mid < k) + k\overline{N}(r, 0; h \mid \geq k)] \\ &\leq \frac{1}{n+k+1} [k\overline{N}(r, \infty; h) + N_k(r, 0; h)] \\ &\leq \frac{k}{n+k+1} [\overline{N}(r, \infty; g) + \overline{N}(r, 0; g) + \overline{N}(r, 0; ag + b)] \\ &\leq \frac{k}{n+k+1} [\overline{N}(r, \infty; g) + \overline{N}(r, 0; g) + \overline{N}(r, 0; ag + b \mid \leq k) \\ &\quad + \overline{N}(r, 0; ag + b \mid \geq k + 1)]. \end{aligned}$$

So from (2.5) we obtain

$$\begin{aligned} \overline{N}(r, \infty; f) &\leq \left(1 + \frac{k}{n+k+1}\right) \\ &\quad [\overline{N}(r, 0; g) + \overline{N}(r, 0; ag + b \mid \leq k) + \overline{N}(r, 0; ag + b \mid \geq k + 1)] \\ &\quad + \frac{k}{n+k+1} \overline{N}(r, \infty; g) + S(r, g) \\ &\leq \frac{n+2k+1}{n+k+1} \left[\frac{1}{n-2k+1} + \frac{1}{n+k+1} + \frac{1}{n+2k+1} \right] T(r, g) \\ &\quad + \frac{k}{n+k+1} T(r, g) + S(r, g) \\ &\leq \left(\frac{2(n+1)}{(n+k+1)(n-2k+1)} + \frac{k}{(n+k+1)^2} + \frac{k+1}{n+k+1} \right) \\ &\quad T(r, g) + S(r, g). \end{aligned}$$

Using the second fundamental theorem of Nevanlinna we get

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \overline{N}(r, 0; af + b) + S(r, f) \\ &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \overline{N}(r, 0; af + b \mid \leq k) \\ &\quad + \overline{N}(r, 0; af + b \mid \geq k + 1) + S(r, f) \\ &\leq \left[\frac{2(n+1)}{(n+k+1)(n-2k+1)} + \frac{k}{(n+k+1)^2} + \frac{k+1}{n+k+1} \right] T(r, g) \\ (2.6) \quad &+ \left[\frac{2(n+1)}{(n+2k+1)(n-2k+1)} + \frac{1}{n+k+1} \right] T(r, f) + S(r, f) + S(r, g). \end{aligned}$$

Similarly

$$\begin{aligned} T(r, g) &\leq \left[\frac{2(n+1)}{(n+k+1)(n-2k+1)} + \frac{k}{(n+k+1)^2} + \frac{k+1}{n+k+1} \right] T(r, f) \\ (2.7) \quad &+ \left[\frac{2(n+1)}{(n+2k+1)(n-2k+1)} + \frac{1}{n+k+1} \right] T(r, g) + S(r, f) + S(r, g). \end{aligned}$$

Adding (2.6) and (2.7) we obtain

$$\left[\frac{n-1}{n+k+1} - \frac{2(n+1)(2n+3k+2)}{(n-2k+1)(n+2k+1)(n+k+1)} - \frac{k}{(n+k+1)^2} \right] \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

i.e.,

$$[(n-1)(n+k+1)(n-2k+1)(n+2k+1) - 2(n+1)(n+k+1)(2n+3k+2) - k(n-2k+1)(n+2k+1)]\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

contradicting with the fact that $n \geq 2k + 5$. This proves the lemma. □

Lemma 2.9. *Let f and g be two transcendental entire function, let n, k are any two positive integers and let P be a nonconstant polynomial. Then*

$$(f^n(af + b))^{(k)}(g^n(ag + b))^{(k)} \neq P^2,$$

where a, b are any two nonzero constants.

Proof. Suppose that

$$(f^n(af + b))^{(k)}(g^n(ag + b))^{(k)} = P^2.$$

Let z_0 be a zero of f with multiplicity p . Then clearly z_0 is a zero of P . Since P is a polynomial, f has a finite number of zeros. So we put $f(z) = P_1 e^\alpha$, where α is a nonconstant entire function and P_1 is a polynomial. Now

$$(2.8) \quad (af^{n+1})^{(k)} = t_1(\alpha', \alpha'', \dots, \alpha^{(k)}, P_1)e^{(n+1)\alpha},$$

$$(2.9) \quad (bf^n)^{(k)} = t_0(\alpha', \alpha'', \dots, \alpha^{(k)}, P_1)e^{n\alpha},$$

where $t_i(\alpha', \alpha'', \dots, \alpha^{(k)}, P_1)$ ($i = 0, 1$) are differential polynomials in $\alpha', \alpha'', \dots, \alpha^{(k)}$ with coefficients which are rational functions in P_1 or its derivatives. Obviously

$$t_i(\alpha', \alpha'', \dots, \alpha^{(k)}, P_1) \neq 0$$

for $i = 0, 1$, and

$$(f^n(af + b))^{(k)} \neq 0.$$

From (2.8) and (2.9) we have

$$(2.10) \quad t_1(\alpha', \alpha'', \dots, \alpha^{(k)}, P_1)e^{\alpha(z)} + t_0(\alpha', \alpha'', \dots, \alpha^{(k)}, P_1) \neq 0.$$

Since $\alpha(z)$ is an entire function, we obtain $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2, \dots, k$. Thus $T(r, t_i) = S(r, f)$ for $i = 0, 1$. So from (2.10), Lemma 2.1 and Lemma 2.7 we obtain

$$\begin{aligned} T(r, f) &= T(r, t_1 e^\alpha) + S(r, f) \\ &\leq \bar{N}(r, 0; t_1 e^\alpha) + \bar{N}(r, 0; t_1 e^\alpha + t_0) + S(r, f) \\ &\leq S(r, f), \end{aligned}$$

which is a contradiction. This completes the proof of the lemma. \square

Lemma 2.10. *Let f and g be two transcendental meromorphic (entire) functions and let $n(\geq 1)$, $k(\geq 1)$, be two integers. Suppose that $F = \frac{(f^n(af+b))^{(k)}}{P(z)}$ and $G = \frac{(g^n(ag+b))^{(k)}}{P(z)}$. If there exist two nonzero constants c_1 and c_2 such that $\bar{N}(r, c_1; F) = \bar{N}(r, 0; G)$ and $\bar{N}(r, c_2; G) = \bar{N}(r, 0; F)$, then $n \leq 3k + 4$ ($n \leq 2k + 3$).*

Proof. We prove the theorem for two transcendental meromorphic functions. By the second fundamental theorem of Nevanlinna we have

$$(2.11) \quad \begin{aligned} T(r, F) &\leq \bar{N}(r, 0; F) + \bar{N}(r, \infty; F) + \bar{N}(r, c_1; F) + S(r, F) \\ &\leq \bar{N}(r, 0; F) + \bar{N}(r, 0; G) + \bar{N}(r, \infty; F) + S(r, F). \end{aligned}$$

By (2.1), (2.2), (2.11) and Lemma 2.1 we obtain

$$(2.12) \quad \begin{aligned} (n+1)T(r, f) &\leq T(r, F) - \bar{N}(r, 0; F) + N_{k+1}(r, 0; f^n(af+b)) \\ &\quad + O\{\log r\} + S(r, f) \\ &\leq \bar{N}(r, 0; G) + N_{k+1}(r, 0; f^n(af+b)) + \bar{N}(r, \infty; f) \\ &\quad + O\{\log r\} + S(r, f) \\ &\leq N_{k+1}(r, 0; f^n(af+b)) + N_{k+1}(r, 0; g^n(ag+b)) + \bar{N}(r, \infty; f) \\ &\quad + k\bar{N}(r, \infty; g) + O\{\log r\} + S(r, f) + S(r, g) \\ &\leq (k+3)T(r, f) + (2k+2)T(r, g) + O\{\log r\} \\ &\quad + S(r, f) + S(r, g). \end{aligned}$$

Similarly we obtain

$$(2.13) \quad \begin{aligned} (n+1)T(r, g) &\leq (k+3)T(r, g) + (2k+2)T(r, f) + O\{\log r\} \\ &\quad + S(r, f) + S(r, g). \end{aligned}$$

Combining (2.12), (2.13) and noting that $O\{\log r\} = o(T(r, f))$ and $O\{\log r\} = o(T(r, g))$ we get

$$(n - 3k - 4)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which gives $n \leq 3k + 4$. This completes the proof of the lemma. \square

Lemma 2.11. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b) = g^n(ag+b)$$

implies $f = g$, where a, b are two nonzero constants.

Proof. We omit the proof since it can be carried out in the line of Lemma 6 [9]. \square

3. Proof of Theorems

Proof of Theorem 1.1. Let $F(z)$ and $G(z)$ be given as in Lemma 2.10. Then $F(z)$, $G(z)$ are non-entire transcendental meromorphic functions that share $(1, l)$ except the the zeros of the polynomial $P(z)$. So from (2.1) we obtain

$$\begin{aligned}
 N_2(r, 0; F) &\leq N_2\left(r, 0; (f^n(af+b))^{(k)}\right) + S(r, f) \\
 &\leq T\left(r, (f^n(af+b))^{(k)}\right) - (n+1)T(r, f) \\
 &\quad + N_{k+2}(r, 0; f^n(af+b)) + S(r, f) \\
 &\leq T(r, F) - (n+1)T(r, f) + N_{k+2}(r, 0; f^n(af+b)) \\
 (3.1) \quad &\quad + O\{\log r\} + S(r, f).
 \end{aligned}$$

Again by (2.2) we have

$$(3.2) \quad N_2(r, 0; G) \leq k\bar{N}(r, \infty; f) + N_{k+2}(r, 0; g^n(ag+b)) + S(r, g).$$

From (3.1) we get

$$(3.3) \quad \begin{aligned} (n+1)T(r, f) &\leq T(r, F) + N_{k+2}(r, 0; f^n(af+b)) - N_2(r, 0; F) \\ &\quad + O\{\log r\} + S(r, f). \end{aligned}$$

Now we consider the following three cases.

Case 1. Let $l \geq 2$. Let (i) of Lemma 2.3 holds. Then using (3.2) we obtain from (3.3)

$$\begin{aligned}
 (n+1)T(r, f) &\leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + N_{k+2}(r, 0; f^n(af+b)) \\
 &\quad + O\{\log r\} + S(r, f) + S(r, g) \\
 &\leq N_{k+2}(r, 0; f^n(af+b)) + N_{k+2}(r, 0; g^n(ag+b)) + 2\bar{N}(r, \infty; f) \\
 &\quad + (k+2)\bar{N}(r, \infty; g) + O\{\log r\} + S(r, f) + S(r, g) \\
 &\leq (k+3)\{T(r, f) + T(r, g)\} + 2\bar{N}(r, \infty; f) + (k+2)\bar{N}(r, \infty; g) \\
 &\quad + O\{\log r\} + S(r, f) + S(r, g) \\
 &\leq [k+5 - 2\Theta(\infty, f) + \varepsilon]T(r, f) + [2k+5 - (k+2)\Theta(\infty, g) + \varepsilon] \\
 &\quad T(r, g) + S(r, f) + S(r, g) \\
 (3.4) \quad &\leq [3k+10 - 2\Theta(\infty, f) - 2\Theta(\infty, g) - k \min\{\Theta(\infty, f), \Theta(\infty, g)\} \\
 &\quad + 2\varepsilon]T(r) + S(r).
 \end{aligned}$$

In a similar way we can obtain

$$(3.5) \quad \begin{aligned} (n+1)T(r, g) &\leq [3k+10 - 2\Theta(\infty, f) - 2\Theta(\infty, g) - k \min\{\Theta(\infty, f), \Theta(\infty, g)\} \\ &\quad + 2\varepsilon]T(r) + S(r). \end{aligned}$$

From (3.4) and (3.5) we obtain

$$[n - 3k - 9 + 2\Theta(\infty, f) + 2\Theta(\infty, g) + k \min\{\Theta(\infty, f), \Theta(\infty, g)\} - 2\varepsilon]T(r) \leq S(r),$$

contradicting with the facts that $n \geq 3k + 9$, $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$ and $\varepsilon > 0$ be arbitrary. So by Lemma 2.3 either $FG = 1$ or $F = G$. Let $FG = 1$. Then

$$(f^n(af + b))^{(k)}(g^n(ag + b))^{(k)} = P^2,$$

a contradiction by Lemma 2.8. So we have $F = G$. That is

$$[f^n(af + b)]^{(k)} = [g^n(ag + b)]^{(k)}.$$

Integrating we get

$$[f^n(af + b)]^{(k-1)} = [g^n(ag + b)]^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, from Lemma 2.10 we obtain $n \leq 3k + 4$, a contradiction. Hence $c_{k-1} = 0$. Repeating k-times, we obtain

$$(3.6) \quad f^n(af + b) = g^n(ag + b).$$

Now the result follows from Lemma 2.11.

Case 2. Let $l = 1$ and $H \neq 0$. Using Lemma 2.4 and (3.2) we obtain from (3.3)

$$\begin{aligned} (n + 1)T(r, f) &\leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F) \\ &\quad + \frac{1}{2}\overline{N}(r, \infty; F) + N_{k+2}(r, 0; f^n(af + b)) \\ &\quad + O\{\log r\} + S(r, f) + S(r, g) \\ &\leq N_{k+2}(r, 0; f^n(af + b)) + N_{k+2}(r, 0; g^n(ag + b)) \\ &\quad + \frac{1}{2}N_{k+1}(r, 0; f^n(af + b)) + \frac{k+5}{2}\overline{N}(r, \infty; f) \\ &\quad + (k+2)\overline{N}(r, \infty; g) + O\{\log r\} + S(r, f) + S(r, g) \\ &\leq [2k + \frac{13}{2} - (\frac{k}{2} + 2)\Theta(\infty, f) - \frac{1}{2}\Theta(\infty, f) + \varepsilon]T(r, f) \\ &\quad + [2k + 5 - (\frac{k}{2} + 2)\Theta(\infty, g) - \frac{k}{2}\Theta(\infty, g) + \varepsilon]T(r, g) \\ &\quad + O\{\log r\} + S(r, f) + S(r, g) \\ (3.7) \quad &\leq [4k + \frac{23}{2} - \frac{k+5}{2}(\Theta(\infty, f) + \Theta(\infty, g)) + 2\varepsilon]T(r) + S(r). \end{aligned}$$

Similarly

$$(3.8) \quad \begin{aligned} (n + 1)T(r, g) &\leq [4k + \frac{23}{2} - \frac{k+5}{2}(\Theta(\infty, f) + \Theta(\infty, g)) + 2\varepsilon]T(r) \\ &\quad + S(r). \end{aligned}$$

Combining (3.7) and (3.8) we obtain

$$\left[n - 4k - \frac{21}{2} + \frac{k+5}{2}(\Theta(\infty, f) + \Theta(\infty, g)) - 2\varepsilon\right]T(r) \leq S(r),$$

a contradiction since $n \geq 4k + 10$, $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$ and $\varepsilon > 0$ be arbitrary. We now assume that $H = 0$. That is

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$(3.9) \quad \frac{1}{F-1} = \frac{A}{G-1} + B,$$

where $A (\neq 0)$ and B are constants. From (3.9) it is clear that F, G share the value 1 CM and so they share 1 with weight two. Hence we have $n \geq 3k + 9$. Now we discuss the following three subcases.

Subcase 1. Let $B \neq 0$ and $A = B$. Then from (3.9) we get

$$(3.10) \quad \frac{1}{F-1} = \frac{BG}{G-1}.$$

If $B = -1$, then from (3.10) we obtain

$$FG = 1,$$

a contradiction by Lemma 2.8.

If $B \neq -1$, from (3.10), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Now from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r, \infty; G) + S(r, G) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G). \end{aligned}$$

Using (2.1) and (2.2) we obtain from above inequality

$$\begin{aligned} T(r, G) &\leq N_{k+1}(r, 0; f^n(af+b)) + k\overline{N}(r, \infty; f) + T(r, G) \\ &\quad + N_{k+1}(r, 0; g^n(ag+b)) - (n+1)T(r, g) + \overline{N}(r, \infty; g) \\ &\quad + O\{\log r\} + S(r, g). \end{aligned}$$

Hence

$$(n+1)T(r, g) \leq (2k+2)T(r, f) + (k+3)T(r, g) + S(r, g).$$

Thus we obtain

$$(n-3k-4)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

a contradiction as $n \geq 3k + 9$.

Subcase 2. Let $B \neq 0$ and $A \neq B$. Then from (3.9) we get $F = \frac{(B+1)G - (B-A+1)}{BG + (A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Proceeding as in Subcase 1 we obtain a contradiction.

Subcase 3. Let $B = 0$ and $A \neq 0$. Then from (3.9) $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. So by Lemma 2.10 we have $n \leq 3k + 6$, a contradiction. Thus $A = 1$ and hence $F = G$. Now using the same technique as used in Case 1 we can obtain (3.6) which by Lemma 2.11 gives $f = g$.

Case 3. Let $l = 0$ and $H \neq 0$. Using Lemma 2.5 and (3.2) we obtain from (3.3)

$$\begin{aligned}
 (n+1)T(r, f) &\leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) \\
 &\quad + \overline{N}(r, 0; G) + N_{k+2}(r, 0; f^n(af+b)) + 2\overline{N}(r, \infty; F) \\
 &\quad + \overline{N}(r, \infty; G) + O\{\log r\} + S(r, f) + S(r, g) \\
 &\leq N_{k+2}(r, 0; f^n(af+b)) + N_{k+2}(r, 0; g^n(ag+b)) \\
 &\quad + 2N_{k+1}(r, 0; f^n(af+b)) + N_{k+1}(r, 0; g^n(ag+b)) \\
 &\quad + (2k+4)(\overline{N}(r, \infty; f) + (2k+3)\overline{N}(r, \infty; g)) \\
 &\quad + O\{\log r\} + S(r, f) + S(r, g) \\
 &\leq [5k+11 - (2k+4)\Theta(\infty, f) - \varepsilon]T(r, f) + [4k+8 \\
 &\quad - (2k+3)\Theta(\infty, g) - \varepsilon]T(r, g) + O\{\log r\} + S(r, f) + S(r, g) \\
 &\leq [9k+19 - (2k+3)(\Theta(\infty, f) + \Theta(\infty, g)) \\
 &\quad - \min\{\Theta(\infty, f), \Theta(\infty, g)\} + 2\varepsilon]T(r) + S(r).
 \end{aligned}
 \tag{3.11}$$

Similarly

$$\begin{aligned}
 (n+1)T(r, g) &\leq [9k+19 - (2k+3)(\Theta(\infty, f) + \Theta(\infty, g)) \\
 &\quad - \min\{\Theta(\infty, f), \Theta(\infty, g)\} + 2\varepsilon]T(r) + S(r).
 \end{aligned}
 \tag{3.12}$$

From (3.11) and (3.12) we get

$$\begin{aligned}
 &[n - 9k - 18 + (2k+3)(\Theta(\infty, f) + \Theta(\infty, g)) \\
 &\quad + \min\{\Theta(\infty, f), \Theta(\infty, g)\} - 2\varepsilon]T(r) \leq S(r),
 \end{aligned}$$

contradicts with the facts that $n \geq 9k + 18$, $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$ and $\varepsilon > 0$ be arbitrary. We now assume that $H = 0$. Then proceeding in a similar manner as in Case 2 we obtain $f = g$. This completes the proof of the theorem.

Proof of Theorem 1.2. Noting that $\overline{N}(r, \infty; f) = 0$, $\overline{N}(r, \infty; g) = 0$ and using Lemma 2.9 instead of Lemma 2.8 and proceeding in the like manner as the proof of Theorem 1.1 we obtain the result of the theorem.

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