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Polynomial Numerical Index of l_p $(1 < p < \infty)$

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Abstract. We present some estimates for the polynomial numerical index of l_p $(1 < p < \infty)$.

1. Introduction

Given a Banach space E we write B_E for its unit ball and S_E for its unit sphere. The dual space of E is denoted by E^* and let

$$\Pi(E) = \{(x, x^*) \ : \ x \in S_E, \ x^* \in S_{E^*}, x^*(x) = 1\}.$$

A mapping $P : E \to E$ is called a (continuous) k-homogeneous polynomial if there is a (continuous) k-linear mapping $A : E \times \cdots \times E \to E$ such that $P(x) = A(x, \cdots, x)$ for every $x \in E$. Let $\mathcal{P}(^kE : E)$ denote the Banach space of all k-homogeneous polynomials from E to itself, endowed with the polynomial norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$. We refer to a book [6] by Dineen for background on polynomials. It is natural to generalize the concepts of numerical range and numerical radius of linear operators to homogeneous polynomials. The *numerical range* of $P \in \mathcal{P}(^kE : E)$ is defined to be the set of scalars

$$V(P) := \{x^*(Px) : (x, x^*) \in \Pi(E)\}$$

and the *numerical radius* of P is defined by

$$v(P) := \sup \ \{|\lambda| : \lambda \in V(P)\}.$$

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Clearly, $v(\cdot)$ is a semi-norm on $\mathcal{P}(^kE : E)$, and $v(P) \leq \|P\|$ for every $P \in \mathcal{P}(^kE : E)$. As in the linear case, the author *et al.* [3] introduced the concept of the *polynomial numerical index of order* k of E to be the constant

$$\begin{aligned} n^{(k)}(E) &:= \inf\{v(P) : P \in \mathcal{P}(^kE : E), \|P\| = 1\} \\ &= \sup\{M \geq 0 : \|P\| \leq \frac{1}{M} v(P) \text{ for all } P \in \mathcal{P}(^kE : E)\}. \end{aligned}$$

Of course, $n^{(1)}(E)$ coincides with the usual numerical index of the space E. Note that $0 \leq n^{(k)}(E) \leq 1$, and $n^{(k)}(E) > 0$ if and only if $v(\cdot)$ is a norm on $\mathcal{P}(^kE : E)$ equivalent to the usual norm. It is obvious that if E_1, E_2 are isometrically isomorphic Banach spaces, then $n^{(k)}(E_1) = n^{(k)}(E_2)$.

The concept of the numerical index was first suggested by G. Lumer in 1968 (see [14]). The author *et al.* [3] introduced and studied the concept of the polynomial numerical index of order k of a Banach space, generalizing to k-homogeneous polynomials the classical numerical index. Recently, the author *et al.* [11] computed that $\frac{1}{2} = n^{(2)}(c_0) = n^{(2)}(l_1) = n^{(2)}(l_\infty)$ for the real spaces c_0, l_1 and l_∞. Very recently, Garcia *et al.* [9] gave estimates for the polynomial numerical indices of $C(K)$ and $L_1(\mu)$. For general information and background on polynomial numerical index, we refer to ([1]–[4], [7]–[15]).

In this paper, we present some inequality for the norm of 2-homogeneous polynomials from the real space l_p $(1 < p < \infty)$ to itself in terms of their coefficients. Using this we give a lower bound for $n^{(2)}(l_p)$. We also present some upper bounds for $n^{(k)}(l_p)$.

2. Norm and Numerical Radius of 2-Homogeneous Polynomial on l_p

Theorem 2.1. *Let* $1 < p < \infty, 1 = \frac{1}{p} + \frac{1}{q}$ *and* $P(x, y) = (P_1(x, y), P_2(x, y)) \in \mathcal{P}(^2l_p^2 : l_p^2)$, *where* $P_k(x, y) = a_kx^2 + b_ky^2 + c_kxy$ *for* $k = 1, 2$ *and* $(x, y) \in l_p^2$. *Then*

$$(1) \quad \|P\|^p \geq \sup_{0\leq\lambda\leq1} |P_1(\lambda^{\frac{1}{p}}, (1-\lambda)^{\frac{1}{p}})|^p + |P_2(\lambda^{\frac{1}{p}}, (1-\lambda)^{\frac{1}{p}})|^p;$$

$$(2) \quad v(P) \geq \sup_{0\leq\lambda\leq1} |\lambda^{\frac{1}{q}} P_1(\lambda^{\frac{1}{p}}, (1-\lambda)^{\frac{1}{p}}) + (1-\lambda)^{\frac{1}{q}} P_2(\lambda^{\frac{1}{p}}, (1-\lambda)^{\frac{1}{p}})|.$$

Corollary 2.2. *Let* $1 < p < \infty, 1 = \frac{1}{p} + \frac{1}{q}$ *and* $P(x, y) = (P_1(x, y), P_2(x, y)) \in \mathcal{P}(^2l_p^2 : l_p^2)$, *where* $P_k(x, y) = a_kx^2 + b_ky^2 + c_kxy$ *for* $k = 1, 2$ *and* $(x, y) \in l_p^2$. *Then*

$$(1) \quad \|P\|^p \geq \sup_{0\leq\lambda\leq1, k=1,2} \{|P_1([\lambda + \frac{(1-\lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},$$

$$[1 - \lambda - \frac{(1-\lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|^p + |P_2([\lambda + \frac{(1-\lambda)|a_k - b_k|}{(|c_k|^p + |a_k - b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},$$

$$[1-\lambda-\frac{(1-\lambda)|a_k-b_k|}{(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|^p\};$$

$$\begin{aligned}(2)\ v(P) \geq & \sup_{0\leq\lambda\leq 1, k=1,2}\{|[\lambda+\frac{(1-\lambda)|a_k-b_k|}{(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} \\ & P_1([\lambda+\frac{(1-\lambda)|a_k-b_k|}{(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},[1-\lambda-\frac{(1-\lambda)|a_k-b_k|}{(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}) \\ + & [1-\lambda-\frac{(1-\lambda)|a_k-b_k|}{(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}}\times P_2([\lambda+\frac{(1-\lambda)|a_k-b_k|}{(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, \\ & [1-\lambda-\frac{(1-\lambda)|a_k-b_k|}{(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|\}.\end{aligned}$$

Corollary 2.3. *Let $1 < p < \infty, 1 = \frac{1}{p}+\frac{1}{q}$ and $P(x,y) = (P_1(x,y), P_2(x,y)) \in \mathcal{P}(^2l_p^2 : l_p^2)$, where $P_k(x,y) = a_kx^2 + b_ky^2 + c_kxy$ for $k = 1,2$ and $(x,y) \in l_p^2$. Then*

$$\begin{aligned}(1)\ \|P\|^p \geq \sup_{k=1,2}\{ & |P_1([\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},[\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|^p \\ & +|P_2([\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},[\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|^p, \\ & |P_1([\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},[\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|^p \\ & +|P_2([\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},[\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|^p\};\end{aligned}$$

$$\begin{aligned}(2)\ v(P) \geq & \sup_{k=1,2}\{|[\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}}P_1([\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, \\ & [\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})+[\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} \\ & \times P_2([\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},[\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|, \\ & |[\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}}P_1([\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}}, \\ & [\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})+[\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{q}} \\ & \times P_2([\frac{1}{q}-\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}},[\frac{1}{p}+\frac{|a_k-b_k|}{q(|c_k|^p+|a_k-b_k|^p)^{\frac{1}{p}}}]^{\frac{1}{p}})|\}\end{aligned}$$

3. Estimates for the Polynomial Numerical Index of l_p

Theorem 3.1. *Let* $1 < p < \infty, 1 = \frac{1}{p} + \frac{1}{q}$ *and* $P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}(^2 l_p : l_p)$, *where*

$$P_k(x) = \sum_{i \leq j} a_{ij}^{(k)} x_i x_j \quad (x = (x_n)_{n=1}^{\infty} \in l_p).$$

Then

$$\begin{aligned} \|P\| \quad \leq \quad & \| \Big(\sum_{k=1}^{\infty} (|a_{11}^{(k)}| + \frac{1}{2} \sum_{1<j} |a_{1j}^{(k)}|), \ \sum_{k=1}^{\infty} (|a_{22}^{(k)}| + \frac{1}{2}|a_{12}^{(k)}| + \frac{1}{2} \sum_{2<j} |a_{2j}^{(k)}|), \quad \cdots, \\ & \sum_{k=1}^{\infty} (|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m<n} |a_{mn}^{(k)}| + \frac{1}{2} \sum_{n<j} |a_{nj}^{(k)}|), \quad \cdots \Big) \|_q. \end{aligned}$$

Proof. It follows that

$$\begin{aligned} \|P\| &= \sup_{x=(x_n) \in S_{l_p}} (\sum_{k=1}^{\infty} |P_k(x)|^p)^{1/p} \\ &\leq \sup_{x=(x_n) \in S_{l_p}} \Big(\sum_{k=1}^{\infty} \Big(\sum_{i=1}^{\infty} |a_{ii}^{(k)}||x_i|^2 + \sum_{i<j} |a_{ij}^{(k)}||x_i||x_j| \Big)^p \Big)^{1/p} \\ &\leq \sup_{x=(x_n) \in S_{l_p}} \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{\infty} |a_{ii}^{(k)}||x_i|^2 + \sum_{i<j} |a_{ij}^{(k)}||x_i||x_j| \Big) \\ &\leq \sup_{x=(x_n) \in S_{l_p}} \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{\infty} |a_{ii}^{(k)}||x_i|^2 + \sum_{i<j} |a_{ij}^{(k)}| \Big(\frac{|x_i|^2 + |x_j|^2}{2} \Big) \Big) \\ &= \sup_{x=(x_n) \in S_{l_p}} \Big[\sum_{k=1}^{\infty} (|a_{11}^{(k)}| + \frac{1}{2} \sum_{1<j} |a_{1j}^{(k)}|)|x_1|^2 \\ &\quad + \sum_{k=1}^{\infty} (|a_{22}^{(k)}| + \frac{1}{2}|a_{12}^{(k)}| + \frac{1}{2} \sum_{2<j} |a_{2j}^{(k)}|)|x_2|^2 \\ &\quad + \cdots + \sum_{k=1}^{\infty} (|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m<n} |a_{mn}^{(k)}| + \frac{1}{2} \sum_{n<j} |a_{nj}^{(k)}|)|x_n|^2 + \cdots \Big] \\ &\leq \sup_{x=(x_n) \in S_{l_p}} \Big[\sum_{k=1}^{\infty} (|a_{11}^{(k)}| + \frac{1}{2} \sum_{1<j} |a_{1j}^{(k)}|)|x_1| \\ &\quad + \sum_{k=1}^{\infty} (|a_{22}^{(k)}| + \frac{1}{2}|a_{12}^{(k)}| + \frac{1}{2} \sum_{2<j} |a_{2j}^{(k)}|)|x_2| \\ &\quad + \cdots + \sum_{k=1}^{\infty} (|a_{nn}^{(k)}| + \frac{1}{2} \sum_{m<n} |a_{mn}^{(k)}| + \frac{1}{2} \sum_{n<j} |a_{nj}^{(k)}|)|x_n| + \cdots \Big] \end{aligned}$$

$$= \| (\sum_{k=1}^{\infty}(|a_{11}^{(k)}| + \frac{1}{2}\sum_{1<j}|a_{1j}^{(k)}|), \ \sum_{k=1}^{\infty}(|a_{22}^{(k)}| + \frac{1}{2}|a_{12}^{(k)}| + \frac{1}{2}\sum_{2<j}|a_{2j}^{(k)}|), \ \cdots,$$

$$\sum_{k=1}^{\infty}(|a_{nn}^{(k)}| + \frac{1}{2}\sum_{m<n}|a_{mn}^{(k)}| + \frac{1}{2}\sum_{n<j}|a_{nj}^{(k)}|), \ \cdots) \|_q. \qquad \square$$

Theorem 3.2. *Let $1 < p < \infty, 1 = \frac{1}{p} + \frac{1}{q}$. Then*

$$n^{(2)}(l_p) \geq$$

$$\inf\{\frac{\sup_{n\in\mathbb{N}}(\frac{1}{n})^{1+\frac{1}{p}}|\sum_{l=1}^{n}\sum_{i\leq j}a_{ij}^{(l)}|}{\| (\sum_{k=1}^{n}(|a_{11}^{(k)}| + \frac{1}{2}\sum_{j=2}^{n}|a_{1j}^{(k)}|), \ \cdots, \sum_{k=1}^{n}(|a_{nn}^{(k)}| + \frac{1}{2}\sum_{m<n}|a_{mn}^{(k)}|), 0, 0, \ \cdots) \|_q} :$$

$$\textit{nonzero } P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}(^2l_p : l_p), P_k(x) = \sum_{i\leq j} a_{ij}^{(k)}x_ix_j\}.$$

Proof. Suppose that $P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}(^2l_p : l_p)$ with

$$P_k(x) = \sum_{i\leq j} a_{ij}^{(k)}x_ix_j \quad (x = (x_n)_{n=1}^{\infty} \in l_p).$$

Let $n \in \mathbb{N}$, $x_0 = \sum_{k=1}^{n}(\frac{1}{n})^{1/p}e_k$, and $x_0^* = \sum_{k=1}^{n}(\frac{1}{n})^{1/q}e_k$. Then

$$(*) \quad v(P) \geq |x_0^*(P(x_0))| = (\frac{1}{n})^{1+\frac{1}{p}}|\sum_{l=1}^{n}\sum_{i\leq j}a_{ij}^{(l)}|.$$

It follows that by $(*)$ and Theorem 3.1,

$$n^{(2)}(l_p)$$

$$= \inf\{\frac{v(P)}{\|P\|} : P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}(^2l_p : l_p), P \neq 0\}$$

$$\geq \inf\{\frac{\sup_{n\in\mathbb{N}}(\frac{1}{n})^{1+\frac{1}{p}}|\sum_{l=1}^{n}\sum_{i\leq j}a_{ij}^{(l)}|}{\| (\sum_{k=1}^{n}(|a_{11}^{(k)}| + \frac{1}{2}\sum_{j=2}^{n}|a_{1j}^{(k)}|), \ \cdots, \sum_{k=1}^{n}(|a_{nn}^{(k)}| + \frac{1}{2}\sum_{m<n}|a_{mn}^{(k)}|), 0, 0, \ \cdots) \|_q} :$$

nonzero $P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}(^2l_p : l_p)\}$. $\square$

Corollary 3.3. *We have*

$$\inf\{\frac{\sup_{n\in\mathbb{N}}(\frac{1}{n})^{\frac{3}{2}}|\sum_{l=1}^{n}\sum_{i\leq j}a_{ij}^{(l)}|}{\| (\sum_{k=1}^{n}(|a_{11}^{(k)}| + \frac{1}{2}\sum_{j=2}^{n}|a_{1j}^{(k)}|), \ \cdots, \sum_{k=1}^{n}(|a_{nn}^{(k)}| + \frac{1}{2}\sum_{m<n}|a_{mn}^{(k)}|), 0, 0, \ \cdots) \|_2} :$$

nonzero $P(x) = (P_k(x))_{k=1}^{\infty} \in \mathcal{P}(^2l_2 : l_2), P_k(x) = \sum_{i\leq j} a_{ij}^{(k)}x_ix_j\} = 0.$

Theorem 3.4. *Let $k \in \mathbb{N}$ with $k \geq 2$. Then*

$$n^{(2)}(l_k^2) = \inf\{\|Q_P\|_{l_k^2} : P \in \mathcal{P}(^2l_k^2 : l_k^2),$$

$$\|P\| = 1, P(x,y) = (ax^2 + by^2 + cxy, a^{'}x^2 + b^{'}y^2 + c^{'}xy)\},$$

where $Q_P \in \mathcal{P}(^{k+1}l_k^2)$ *with* $Q_P(x,y) = ax^{k+1} + bx^{k-1}y^2 + cx^ky + a^{'}x^2y^{k-1} + b^{'}y^{k+1} + c^{'}xy^k$.

Theorem 3.5. *Let* $1 < p < \infty$ *and* $k \in \mathbb{N}$ *with* $k \geq 2$. *Then*

$$n^{(k)}(l_p) \leq n^{(k)}(l_p^2)$$

$$\leq \min\{(\frac{p-1}{k+p-1})^{1-\frac{1}{p}}\ (\frac{k}{k+p-1})^{\frac{k}{p}},\ (\frac{p+k-1}{2p+k-2})^{\frac{1}{p}}\frac{(\frac{p+k-2}{2p+k-2})^{\frac{p+k-2}{p}}}{(\frac{k-1}{p+k-1})^{\frac{k-1}{p}}}\}.$$

Proof. Let $P_1(x,y) = (x^k, 0)$ and $P_2(x,y) = (x^{k-1}y, 0)$ for $(x,y) \in l_p^2$. Then $P_j \in \mathcal{P}(^kl_p^2 : l_p^2)$ for $j = 1,2$ and $\|P_1\| = 1$. Some computation shows that

$$\|P_2\| = (\frac{k-1}{p+k-1})^{\frac{k-1}{p}}(\frac{p}{p+k-1})^{\frac{1}{p}},$$

$$v(P_2) = (\frac{p+k-2}{2p+k-2})^{\frac{p+k-2}{p}}(\frac{p}{2p+k-2})^{\frac{1}{p}},$$

$$\text{and } v(P_1) = (\frac{p-1}{k+p-1})^{1-\frac{1}{p}}\ (\frac{k}{k+p-1})^{\frac{k}{p}}.$$

It follows that

$$\begin{aligned} n^{(k)}(l_p) &\leq n^{(k)}(l_p^2) \leq \min\{v(P_1), \frac{v(P_2)}{\|P_2\|}\} \\ &= \min\{(\frac{p-1}{k+p-1})^{1-\frac{1}{p}}\ (\frac{k}{k+p-1})^{\frac{k}{p}},\ (\frac{p+k-1}{2p+k-2})^{\frac{1}{p}}\frac{(\frac{p+k-2}{2p+k-2})^{\frac{p+k-2}{p}}}{(\frac{k-1}{p+k-1})^{\frac{k-1}{p}}}\}, \end{aligned}$$

which completes the proof. □

Theorem 3.6. Let $k \in \mathbb{N}$. Then $n^{(2k+1)}(l_{2k}) = 0$ for the real space l_{2k}.

Proof. Let

$$P_0(x) := (-x_1x_2^{2k}, x_1^{2k}x_2, 0, 0, \ldots)\ \ (x = (x_j) \in l_{2k}).$$

Then $\|P_0\| \neq 0$. Note that if $(x_1, x_2, 0, 0, \ldots) \in S_{l_{2k}}$, then $(x_1^{2k-1}, x_2^{2k-1}, 0, 0, \ldots) \in S_{(l_{2k})^*} = S_{l_{\frac{2k}{2k-1}}}$. It follows that

$$\begin{aligned} & v(P_0) \\ = & \sup_{x_1^{2k}+x_2^{2k}=1} |<(sign(x_1^{2k})x_1^{2k-1}, sign(x_2^{2k})x_2^{2k-1}, 0, \ldots),\ P_0(x_1, x_2, 0, \ldots)>| \\ = & \sup\{|<(x_1^{2k-1}, x_2^{2k-1}, 0, 0, \ldots),\ P_0(x_1, x_2, 0, 0, \ldots)>| : x_1^{2k} + x_2^{2k} = 1\} \\ = & \sup\{|-x_1x_2^{2k} + x_1x_2^{2k}| : x_1^{2k} + x_2^{2k} = 1\} = 0. \end{aligned}$$

Thus

$$n^{(2k+1)}(l_{2k}) \leq \frac{v(P_0)}{\|P_0\|} = 0.$$

□

Corollary 3.7. Let $k, s \in \mathbb{N}$ with $s \geq 2k+1$. Then $n^{(s)}(l_{2k}) = 0$ for the real space l_{2k}.

Notice that there exist $1 < p < q < \infty, N \geq 2$ such that

$$n^{(N)}(l_p) = n^{(N)}(l_q).$$

In fact, take $p = 2k, q = 2(k+1), N = 2k+3$. By Corollary 3.7, $n^{(N)}(l_p) = 0 = n^{(N)}(l_q)$.

We are in position to prove our main result.

Theorem 3.8. *Let $1 < p < \infty$. Then for the complex space l_p,*

$$n^{(k)}(l_p) \leq 2^{\frac{1-k}{p}} \quad (\forall k \in \mathbb{N}).$$

Proof. Simple computation shows that

$$\sup_{|x_1|^p+|x_2|^p=1} (|x_1|^{p+k-1} + |x_2|^{p+k-1}) = 2^{\frac{1-k}{p}} = \sup_{|x_1|^p+|x_2|^p=1} (|x_1|^{kp} + |x_2|^{kp})^{\frac{1}{p}}.$$

For every $b \in \mathbb{C}$, define

$$Q_b(x) := (x_1^k + bx_2^k, bx_1^k + x_2^k, 0, 0, \ldots) \in \mathcal{P}(^k l_p) \quad (x = (x_j) \in l_p).$$

Then $\|Q_b\| \geq |Q_b(1,0)| = (1 + |b|^p)^{\frac{1}{p}}$ and

$$\begin{aligned}
& v(Q_b) \\
= & \sup_{|x_1|^p+|x_2|^p=1} |< (sign(x_1^p)x_1^{p-1}, sign(x_2^p)x_2^{p-1}, 0, 0, \ldots),\ Q_b(x_1, x_2, 0, 0, \ldots) > | \\
\leq & \sup_{|x_1|^p+|x_2|^p=1} (|x_1|^{p+k-1} + |x_2|^{p+k-1}) \\
& + |b| \sup_{|x_1|^p+|x_2|^p=1} (|x_1|^{p-1}|x_2|^k + |x_2|^{p-1}|x_1|^k) \\
\leq & 2^{\frac{1-k}{p}} + |b| \sup_{|x_1|^p+|x_2|^p=1} (|x_1|^{(p-1)q} + |x_2|^{(p-1)q})^{\frac{1}{q}} \sup_{|x_1|^p+|x_2|^p=1} (|x_1|^{kp} + |x_2|^{kp})^{\frac{1}{p}} \\
& \text{(by Holder's inequality)} \\
= & 2^{\frac{1-k}{p}} + |b| \sup_{|x_1|^p+|x_2|^p=1} (|x_1|^p + |x_2|^p)^{\frac{1}{q}} \sup_{|x_1|^p+|x_2|^p=1} (|x_1|^{kp} + |x_2|^{kp})^{\frac{1}{p}} \\
= & 2^{\frac{1-k}{p}} + |b| 2^{\frac{1-k}{p}} \\
= & 2^{\frac{1-k}{p}} (1 + |b|).
\end{aligned}$$

It follows that

$$n^{(k)}(l_p) \leq \inf_{b\in\mathbb{C}} \frac{v(Q_b)}{\|Q_b\|} \leq \inf_{b\in\mathbb{C}} \frac{2^{\frac{1-k}{p}}(1+|b|)}{(1+|b|^p)^{\frac{1}{p}}} = 2^{\frac{1-k}{p}}.$$

□

Corollary 3.9. Let $1 < p < \infty$. Then
(1) $n^{(k)}(l_p) \leq 2^{-\frac{1}{p}} < 1 \quad (\forall k \geq 2)$.
(2) $n^{(k)}(l_p) \leq \frac{1}{2} \quad (\forall k \geq p+1)$.
(3) $\frac{2^k}{2+M_k(2^k-2)} \leq n^{(k)}(l_1) \leq (\frac{1}{2})^{k-1}$, where $k \geq 1$ and $M_k = \sum_{j=1}^{k} \frac{j^k}{j!(k-j)!}$.
(4) $\lim_{k\to\infty} n^{(k)}(l_p) = 0$.

Theorem 3.10. Let p be an even number and k an odd number with $k \leq p-1$. Then

$$n^{(k)}(l_p) \leq (\frac{k}{p-1})^{\frac{k}{p-1-k}} - (\frac{k}{p-1})^{\frac{p-1}{p-1-k}}, \text{ when } l_p \text{ is the real space.}$$

Proof. We define

$$Q_0(x) := (-x_2^k, x_1^k, 0, 0, \ldots) \in \mathcal{P}(^k l_p) \quad (x = (x_j) \in l_p).$$

Then $\|Q_0\| = 1$ and

$$\begin{aligned} & v(Q_0) \\ = & \sup_{x_1^p+x_2^p=1} |<(x_1^{p-1}, x_2^{p-1}, 0, 0, \ldots),\ Q_0(x_1, x_2, 0, 0, \ldots)>| \\ = & \sup_{x_1^p+x_2^p=1} |-x_1^{p-1}x_2^k + x_1^k x_2^{p-1}| \\ \leq & \sup_{x_1^p+x_2^p=1} |x_1x_2|^k - |x_1x_2|^{p-1} \\ = & (\frac{k}{p-1})^{\frac{k}{p-1-k}} - (\frac{k}{p-1})^{\frac{p-1}{p-1-k}}, \text{ when } |x_1x_2| = (\frac{k}{p-1})^{\frac{1}{p-1-k}}. \end{aligned}$$

Thus

$$n^{(k)}(l_p) \leq \frac{v(Q_0)}{\|Q_0\|} = (\frac{k}{p-1})^{\frac{k}{p-1-k}} - (\frac{k}{p-1})^{\frac{p-1}{p-1-k}}.$$

□

Theorem 3.11. Let p and s be even numbers with $s < p-1$. Then

$$n^{(s)}(l_p) \leq \min\{(\frac{s-1}{p-1})^{\frac{s-1}{p-s}} - (\frac{s-1}{p-1})^{\frac{p-1}{p-s}}, 2^{-\frac{s}{p}}\}, \text{ when } l_p \text{ is the real space.}$$

Proof. By Theorem 3.10,

$$n^{(s)}(l_p) \leq n^{(s-1)}(l_p) \leq (\frac{s-1}{p-1})^{\frac{s-1}{p-s}} - (\frac{s-1}{p-1})^{\frac{p-1}{p-s}}.$$

We define

$$Q_0(x) := (-x_2^s, x_1^s, 0, 0, \ldots) \in \mathcal{P}(^s l_p) \quad (x = (x_j) \in l_p).$$

Then $\|Q_0\| = 1$ and

$$\begin{aligned} & v(Q_0) \\ = & \sup_{x_1^p + x_2^p = 1} | < (x_1^{p-1}, x_2^{p-1}, 0, 0, \ldots), \ Q_0(x_1, x_2, 0, 0, \ldots) > | \\ = & \sup_{x_1^p + x_2^p = 1} | - x_1^{p-1} x_2^s + x_1^s x_2^{p-1} | \\ = & \sup_{x_1^p + x_2^p = 1} |x_1|^{p-1} |x_2|^s + |x_1|^s |x_2|^{p-1} \\ \leq & \sup_{x_1^p + x_2^p = 1} 2 |x_1 x_2|^s \\ = & 2^{-\frac{s}{p}}. \end{aligned}$$

Thus

$$n^{(k)}(l_p) \leq \frac{v(Q_0)}{\|Q_0\|} = 2^{-\frac{s}{p}}. \qquad \square$$

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