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Pointless Form of Rough Sets

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Abstract. In this paper we introduce the pointfree version of rough sets. For this we consider a lattice L instead of the power set $P(X)$ of a set X. We study the properties of lower and upper pointfree approximation, precise elements, and their relation with prime elements. Also, we study lower and upper pointfree approximation as a Galois connection, and discuss the relations between partitions and Galois connections.

1. Introduction

Rough set theory [10, 11], a new mathematical approach to deal with inexact, uncertain or vague knowledge, has recently received wide attention on the research areas in both of the real-life applications and the theory itself. Rough set theory is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations.

There are at least two methods for the development of this theory, constructive and axiomatic. In constructive method, the lower and upper approximations are constructed from the basic notions, such as equivalence relations on a universe and neighborhood systems. In rough sets, the equivalence classes are the building blocks for the construction of the lower and upper approximations.

Rough sets are a suitable mathematical model of vague concepts, i.e., concepts

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without sharp boundaries. Rough set theory is emerging as a powerful theory dealing with imperfect data. It is an expanding research area which stimulates explorations on both real-world applications and on the theory itself. It has found practical applications in many areas such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on, see [2, 9, 12, 13].

Pointless or (as it is also known) pointfree topology has been the focus of mathematicians since the early 1910's. Initial interest was sparked by the German mathematician Felix Hausdorff [6] who is believed to be the first to consider, instead of points in the space, the "notion of (open) set (or neighbourhood) as primitive..." [8]. Consequently, after 1914, it was common knowledge that a topological space gives rise to a lattice of open sets. A detailed outline of the history and development of pointfree topology can be found in Johnstone ([7],[8]).

In this paper we try to construct the pointfree form of Rough set theory. For this we consider a lattice L rather than the power set $P(X)$ of a set X.

The necessary background on lattices, frames and boolean algebra is given in section 1.

In section 2 we introduce a partition in a lattice and then pointfree approximation is introduced by taking a lattice L with a partition (L, θ). So we introduce pointfree versions of lower and upper rough approximation maps. We prove that the three fundamental relations of rough approximation maps hold in pointfree form when L is a frame (Lemma 3.5). Example 3.6 shows that a frame is required in this case. We introduce the concepts of rough and exact for elements of a frame for providing the concepts of roughness and preciseness in the pointfree form. Then we study the behavior of precise element related with lattice concepts such as complement, pseudocomplement, and Heyting operation.

In section 3, we find some good relations between partitions, lower and upper rough approximation maps of L, with Galois connections of L. We prove that $\big(\underline{\theta}, \overline{\theta}\big)$ is a Galois connection (Proposition 4.1). The set of all partitions with the partial order refinement is a $\vee$-semi lattice (Theorem 4.3), also the set of all Galois connections with a partial order is a $\vee$-semi lattice (Theorem 4.5). The map given by $\theta \rightsquigarrow (\overline{\theta}, \underline{\theta})$ from partitions to Galois connections is monotone but does not preserve $\vee$ (Proposition 4.7 and Example 4.8).

In section 4, we explain when a prime element $p \in L$ is a precise element (Theorem 5.2).

2. Preliminaries

In this section we give the necessary background on Rough set theory, lattices, frames and boolean algebra.

Rough set Theory: For an equivalence relation θ on A, the equivalence class of a is denoted by $[a]_\theta$. A pair (A, θ), where θ is an equivalence relation on A, is called an *approximation space* [10]. For an approximation space (A, θ), by an *upper rough approximation* in (A, θ) we mean a mapping $\overline{Apr} : \mathcal{P}(A) \to \mathcal{P}(A)$ defined for every $X \in \mathcal{P}(A)$ by

$$\overline{Apr}(X) = \{a \in A : [a]_\theta \cap X \neq \emptyset\}.$$

Also, by a *lower rough approximation* in (A, θ) we mean a mapping $\underline{Apr} : \mathcal{P}(A) \to \mathcal{P}(A)$ defined for every $X \in \mathcal{P}(A)$ by

$$\underline{Apr}(X) = \{a \in A : [a]_\theta \subseteq X\}.$$

The following proposition is well known and easily seen.

Proposition 2.1. *Let (A, θ) be an approximation space. For every subsets $X, Y \subseteq A$, we have*

1. $\underline{Apr}(X) \subseteq X \subseteq \overline{Apr}(X)$.
2. *If* $X \subseteq Y$, *then* $\underline{Apr}(X) \subseteq \underline{Apr}(Y)$ *and* $\overline{Apr}(X) \subseteq \overline{Apr}(Y)$.
3. $\overline{Apr}(X \cup Y) = \overline{Apr}(X) \cup \overline{Apr}(Y)$ *and* $\overline{Apr}(X \cap Y) \subseteq \overline{Apr}(X) \cap \overline{Apr}(Y)$.
4. $\underline{Apr}(X \cap Y) = \underline{Apr}(X) \cap \underline{Apr}(Y)$ *and* $\underline{Apr}(X \cup Y) \supseteq \underline{Apr}(X) \cup \underline{Apr}(Y)$.
5. $\underline{Apr}\left(\underline{Apr}(X)\right) = \underline{Apr}(X)$ *and* $\overline{Apr}\left(\overline{Apr}(X)\right) = \overline{Apr}(X)$.

Lattices, Frames and boolean algebras:

Recall that a poset $(L, \leq)$ is called a *lattice* if for every $a, b \in L$, both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist. We denote $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$. The top and the bottom elements are denoted by 1 and 0, respectively. A lattice that has top and bottom element is called *bounded*. In this paper, all lattices are bounded. We denote the two elements lattice $\{0, 1\}$ by $\mathbf{2}$.

Let L be a lattice. For $a, b \in L$, we say that c (usually denoted by $a \to b$) is a relative pseudocomplement of a with respect to b, if c is the largest element with $a \wedge c \leq b$. A pseudocomplement of an element a in a lattice L with 0 is the largest element b such that $b \wedge a = 0$. If it exists, it is usually denoted by $a^\star$. An element $a \in L$ is said to be *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$, we denoted b by a'. A *boolean algebra* is a distributive lattice every element of which is complemented.

A *prime element* of L is an element $p \in L$ such that $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$.

A poset L is called a *complete lattice* if for every subset S of L, both $\sup S = \bigvee S$ and $\inf S = \bigwedge S$ exist. A complete lattice L is called a *frame* if for every subset S and element a of L, $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$.

Proposition 2.2. *Let L be a pseudocomplemented distributive lattice. Then the set $R_G(L) = \{a \in L : a = a^{**}\}$ is a Boolean algebra with the operations given by*

$$a \vee^{R_G(L)} b = (a \vee^L b)^{**}, a \wedge^{R_G(L)} b = a \wedge^L b, a' = a^*$$

and 0, 1 *the same as in* L.

Galois connections [5]**:** Let S and T be two posets. A pair (g, d) of monotone functions $g : S \to T$ and $d : T \to S$ is called a Galois connection between S and T if for all $(s, t) \in S \times T$,

$$g(s) \geq t \Leftrightarrow s \geq d(t).$$

where g, d are called the upper and the lower adjunctions, respectively.

Galois connections are efficient tools in dealing with ordered sets. They appeared in the literature in two equivalent versions. The version we adopt here uses order-preserving maps, which is more popular in computer science, and the other version uses order-reversing maps, which occurs in FCA [4], etc.

Definition 2.3. Let L be a poset.

1. A *projection operator* (shortly projection) is an idempotent, monotone self-map $p : L \to L$.
2. A *closure operator* is a projection operator c on L with $1_L \leq c$.
3. A *kernel operator* is a projection operator k on L with $k \leq 1_L$.

3. Partitions and Precise Elements in a Lattice

It is well known that every complete lattice has a top and a bottom element, which we denote by 1 and 0, respectively.

Definition 3.1. Let L be a complete lattice. A subset θ of L is called a *partition* of L, if

1. $0 \notin \theta$.
2. For every two distinct $a, b \in \theta$, $a \wedge b = 0$.
3. $\bigvee \theta = 1$.

Definition 3.2. Let L be a complete lattice and θ be a partition of L. For every $x \in L$, define

$$\overline{\theta}(x) = \bigvee \{a \in \theta : a \wedge x \neq 0\}$$

and

$$\underline{\theta}(x) = \bigvee \{a \in \theta : a \leq x\}.$$

Lemma 3.3. *Let L be a complete lattice. Then $\overline{\theta}, \underline{\theta} : L \to L$ are ordered preserving.*

Proof. It is clear. □

Remark 3.4. If L is a complete lattice and θ is a partition of L, then, by the fixed point lemma for complete lattices, there are $x_0, x_1 \in L$ such that $\overline{\theta}(x_0) = x_0$, and $\underline{\theta}(x_1) = x_1$.

Lemma 3.5. *Let L be a complete lattice and θ a partition of L. For all $x \in L$*

1. $\underline{\theta}(x) \leq \overline{\theta}(x)$.
2. $\underline{\theta}(x) \leq x$.
3. *If* L *is a frame,* $x \leq \overline{\theta}(x)$.

Proof. (1) Since $0 \notin \theta$, we conclude that $\{a \in \theta : a \leq x\} \subseteq \{a \in \theta : a \wedge x \neq 0\}$, which proves (1).

(2) It is clear.

(3) Let L be a frame and $x \in L$. Let $\theta_1 = \{y \in \theta : x \wedge y = 0\}$. Since L is a frame, we conclude that

$$x = (x \wedge \overline{\theta}(x)) \vee (x \wedge \bigvee \theta_1) = (x \wedge \overline{\theta}(x)) \vee 0 = x \wedge \overline{\theta}(x).$$

Therefore $x \leq \overline{\theta}(x)$. □

Example 3.6. (1) Let $L = \{0, a, b, c, 1\}$ be the non distributive lattice M_5 (see [1]). Consider the partition $\theta = \{b, c\}$. We have $a \not\leq \overline{\theta}(a) = 0$.

(2) Let $L = \{0, a, b, c, 1\}$ be the non distributive lattice N_5 with $a \leq b$ and $b \wedge c = 0$ (see [1]). Consider the partition $\theta = \{a, c\}$. We have $b \not\leq \overline{\theta}(b) = a$.

Remark 3.7. It is well known that L is a non distributive lattice if and only if M_5 or N_5 can be embedded into L (see Theorem 3.6, [1]). If L is a non distributive lattice, we can find a partition θ that the inequality $x \leq \overline{\theta}(x)$ does not hold.

Throughout this paper L is a frame with the least element 0 and the greatest element 1 and θ is a partition of L.

Definition 3.8. An element $a \in L$ is called a *rough element* if $\underline{\theta}(a) < \overline{\theta}(a)$, otherwise it is called a *precise element*, i.e., $\underline{\theta}(a) = \overline{\theta}(a)$.

Theorem 3.9. *Let* θ *be a partition of* L *and* $x \in L$*.*

1. $\overline{\theta}(x)$ *is a precise element.*
2. $\underline{\theta}(x)$ *is a precise element.*

Proof. (1) Let $x \in L$, and $a \in \theta$. We have

$$a \wedge \overline{\theta}(x) = a \wedge \bigvee \{b \in \theta : b \wedge x \neq 0\} = \bigvee \{a \wedge b : b \in \theta, b \wedge x \neq 0\}$$

Since θ is a partition, we conclude that for every $b \in \theta$, $b \neq a$ implies that $a \wedge b = 0$. It follows that

$$a \wedge \overline{\theta}(x) = 0 \Leftrightarrow a \wedge x = 0$$

$$a \wedge \overline{\theta}(x) = a \Leftrightarrow a \wedge x \neq 0.$$

Hence,

$$a \wedge \overline{\theta}(x) \neq 0 \Leftrightarrow a \wedge x \neq 0 \Leftrightarrow a \leq \overline{\theta}(x)$$

and thus

$$\bigvee\{a \in \theta : a \wedge \overline{\theta}(x) \neq 0\} = \bigvee\{a \in \theta : a \wedge x \neq 0\} = \bigvee\{a \in \theta : a \leq \overline{\theta}(x)\}.$$

Therefore,

$$\overline{\theta}(\overline{\theta}(x)) = \overline{\theta}(x) = \underline{\theta}(\overline{\theta}(x)).$$

It completes the proof.

(2) First note that for every $a \in \theta$, $a \wedge \underline{\theta}(x) = a \wedge \bigvee\{b \in \theta : b \leq x\} = \bigvee\{a \wedge b : b \in \theta, b \leq x\}$. Hence

$$a \leq x \Leftrightarrow a \wedge \underline{\theta}(x) = a \Leftrightarrow a \leq \underline{\theta}(x) \Leftrightarrow a \wedge \underline{\theta}(x) \neq 0$$

Therefore, $\bigvee\{a \in \theta : a \leq \underline{\theta}(x)\} = \bigvee\{a \in \theta : a \leq x\} = \bigvee\{a \in \theta : a \wedge \underline{\theta}(x) \neq 0\}$, which follows that $\underline{\theta}(\underline{\theta}(x)) = \underline{\theta}(x) = \overline{\theta}(\underline{\theta}(x))$. □

Theorem 3.10. *Let θ be a partition of L and $x \in L$. Then*

1. $\underline{\theta}(x^*)$ *is the complement of* $\overline{\theta}(x)$ *and* $\underline{\theta}(x^*) = (\overline{\theta}(x))^*$.
2. $\overline{\theta}(x^*)$ *is the complement of* $\underline{\theta}(x)$ *and* $\overline{\theta}(x^*) = (\underline{\theta}(x))^*$.

Proof. (1) For every $a \in \theta$, if $a \leq x^*$, then $a \wedge x = 0$ and

$$a \wedge \overline{\theta}(x) = a \wedge \bigvee\{b \in \theta : b \wedge x \neq 0\} = \bigvee\{a \wedge b : b \in \theta, b \wedge x \neq 0\} = 0.$$

So, we have

$$\underline{\theta}(x^*) \wedge \overline{\theta}(x) = \bigvee\{a \wedge \overline{\theta}(x) : a \in \theta, a \leq x^*\} = 0$$

and

$$\underline{\theta}(x^*) \vee \overline{\theta}(x) = \bigvee\{a \in \theta : a \wedge x = 0 \text{ or } a \wedge x \neq 0\} = \bigvee \theta = 1.$$

Therefore, $\underline{\theta}(x^*)$ is the complement of $\overline{\theta}(x)$.

(2) Let $a \in \theta$. If $a \not\leq x$, then $a \wedge \underline{\theta}(x) = \bigvee\{a \wedge b : b \in \theta, b \leq x\} = 0$. So,

$$\begin{array}{rcl} \underline{\theta}(x) \wedge \overline{\theta}(x^*) & = & \bigvee\{a \wedge \underline{\theta}(x) : a \in \theta, a \wedge x^* \neq 0\} \\ & \leq & \bigvee\{a \wedge \underline{\theta}(x) : a \in \theta, a \not\leq x\} \\ & = & 0 \end{array}$$

and

$$\begin{array}{rcl} \underline{\theta}(x) \vee \overline{\theta}(x^*) & = & \bigvee\{a \in \theta : a \wedge x^* \neq 0\} \vee \bigvee\{a \in \theta : a \leq x\} \\ & \geq & \bigvee\{a \in \theta : a \not\leq x \text{ or } a \leq x\} \\ & = & \bigvee \theta \\ & = & 1. \end{array}$$

Therefore, $\overline{\theta}(x^*)$ is the complement of $\underline{\theta}(x)$. □

Theorem 3.11. *Let θ be a partition of L. For $x \in L$, if x is a precise element, then*

1. *x^* is a precise element.*
2. *x^* is the complement of x.*

Proof. By hypothesis, $\underline{\theta}(x) = x = \overline{\theta}(x)$ and by Theorem 3.10, we have $\overline{\theta}(x^*) = (\underline{\theta}(x))^* = x^* = (\overline{\theta}(x))^* = \underline{\theta}(x^*)$, that is, x^* is a precise element. Also, $x^* \vee x = \overline{\theta}(x^*) \vee \underline{\theta}(x) = 1$ and $x^* \wedge x = \underline{\theta}(x^*) \wedge \overline{\theta}(x) = 0$, by Theorem 3.10. Therefore x^* is the complement of x. □

Proposition 3.12. *Let θ be a partition of frame L such that $\overline{\theta}$ preserves binary meets. Then*

1. *For every $a \in L$ and $b \in Fix(\overline{\theta})$, $a \longrightarrow b \in Fix(\overline{\theta})$.*
2. *For every $a \in L$ and $b \in Fix(\overline{\theta})$, $\overline{\theta}(a) \longrightarrow b = a \longrightarrow b$.*

Proof. (1) Let $a \in L$, and let $b \in Fix(\overline{\theta})$. Then

$$\begin{aligned} a \wedge \overline{\theta}(a \longrightarrow b) &\leq \overline{\theta}(a) \wedge \overline{\theta}(a \longrightarrow b) \\ &= \overline{\theta}(a \wedge (a \longrightarrow b)) \\ &= \overline{\theta}(a \wedge b) \\ &\leq \overline{\theta}(b) \\ &= b. \end{aligned}$$

Hence $\overline{\theta}(a \longrightarrow b) \leq a \longrightarrow b$ and since $a \longrightarrow b \leq \overline{\theta}(a \longrightarrow b)$, we conclude that $\overline{\theta}(a \longrightarrow b) = a \longrightarrow b \in Fix(\overline{\theta})$.

(2) Let $a \in L$, and let $b \in Fix(\overline{\theta})$. So,

$$\begin{aligned} a \wedge (a \longrightarrow b) \leq b &\Rightarrow \overline{\theta}(a) \wedge \overline{\theta}(a \longrightarrow b) = \overline{\theta}(a \wedge (a \longrightarrow b)) \leq \overline{\theta}(b) = b \\ &\Rightarrow a \longrightarrow b \leq \overline{\theta}(a \longrightarrow b) \leq \overline{\theta}(a) \longrightarrow b. \end{aligned}$$

Also, since $\overline{\theta}(a) \longrightarrow b \leq a \longrightarrow b$, we conclude that $\overline{\theta}(a) \longrightarrow b = a \longrightarrow b$. □

Corollary 3.13. *Let θ be a partition of a frame L such that $\overline{\theta}$ preserves binary meets. Then for every $a, b \in L$,*

1. *$\overline{\theta}(a \longrightarrow \overline{\theta}(b)) = a \longrightarrow \overline{\theta}(b)$.*
2. *$\overline{\theta}(a) \longrightarrow \overline{\theta}(b) = a \longrightarrow \overline{\theta}(b)$.*
3. *$\overline{\theta}(a \longrightarrow \underline{\theta}(b)) = a \longrightarrow \underline{\theta}(b)$.*
4. *$\overline{\theta}(a) \longrightarrow \underline{\theta}(b) = a \longrightarrow \underline{\theta}(b)$.*

Proof. By Proposition 3.12, it is clear. □

Proposition 3.14. *Let θ be a partition of a frame L. Then the following statements are equivalent.*

1. *$\overline{\theta}$ preserves binary meets.*
2. *For every $a, b \in L$, $\underline{\theta}(\overline{\theta}(a) \longrightarrow b) = a \longrightarrow \underline{\theta}(b)$.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in L$. Then for every $x \in L$,

$$\begin{aligned}
\underline{\theta}(\overline{\theta}(a) \longrightarrow b) \geq x &\Leftrightarrow \overline{\theta}(a) \longrightarrow b \geq \overline{\theta}(x) \\
&\Leftrightarrow \overline{\theta}(a \wedge x) = \overline{\theta}(a) \wedge \overline{\theta}(x) \leq b \\
&\Leftrightarrow a \wedge x \leq \underline{\theta}(b) \\
&\Leftrightarrow x \leq a \longrightarrow \underline{\theta}(b).
\end{aligned}$$

Therefore, $\underline{\theta}(\overline{\theta}(a) \longrightarrow b) = a \longrightarrow \underline{\theta}(b)$.

(2) $\Rightarrow$ (1) Let $a, b \in L$. Then for every $x \in L$,

$$\begin{aligned}
\overline{\theta}(a \wedge b) \leq x &\Leftrightarrow a \wedge b \leq \underline{\theta}(x) \\
&\Leftrightarrow a \leq (b \longrightarrow \underline{\theta}(x)) = \underline{\theta}(\overline{\theta}(b) \longrightarrow x) \\
&\Leftrightarrow \overline{\theta}(a) \leq \overline{\theta}(b) \longrightarrow x \\
&\Leftrightarrow \overline{\theta}(a) \wedge \overline{\theta}(b) \leq x.
\end{aligned}$$

Therefore, $\overline{\theta}(a \wedge b) = \overline{\theta}(a) \wedge \overline{\theta}(b)$. □

Corollary 3.15. *Let θ be a partition of L. Then, $\overline{\theta} : L \longrightarrow L$ is a frame map if and only if for every $a, b \in L$, $\underline{\theta}(\overline{\theta}(a) \longrightarrow b) = a \longrightarrow \underline{\theta}(b)$.*

Proposition 3.16. *Let θ be a partition of L. Then the following statements are equivalent.*

1. *$\overline{\theta}$ is a one-one map.*
2. *For every $x \in L$ and $a \in \theta$, $a \wedge x = 0$ or $a \leq x$.*
3. *For every $x \in L$, $\overline{\theta}(x) = x$.*
4. *For every $x \in L$, there exists a unique $\theta_x \subseteq \theta$ such that $x = \bigvee \theta_x$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in L$ and $a \in \theta$. If $a \wedge x \neq 0$, then $\overline{\theta}(a) = a = \overline{\theta}(a \wedge x)$, which follows that $a = a \wedge x$, that is, $a \leq x$.

(2) $\Rightarrow$ (3) Let $x \in L$. Then $x \leq \overline{\theta}(x) = \bigvee\{a \in \theta : a \wedge x \neq 0\} = \bigvee\{a \in \theta : a \leq x\} \leq x$. Therefore, $\overline{\theta}(x) = x$.

(3) $\Rightarrow$ (4) Let $x \in L$ and $\theta_x = \{a \in \theta | a \wedge x \neq 0\}$. Then $x = \overline{\theta}(x) = \bigvee \theta_x$. Now suppose that there exists $\theta_1 \subseteq \theta$ such that $x = \bigvee \theta_1$. If $b \in \theta_x \setminus \theta_1$, then $b = b \wedge x = b \wedge \bigvee \theta_1 = 0 \in \theta$, which is a contradiction. If $c \in \theta_1 \setminus \theta_x$, then $c = c \wedge x = c \wedge \bigvee \theta_x = 0 \in \theta$, which is again a contradiction. Therefore, $\theta_1 = \theta_x$.

(4) $\Rightarrow$ (1) Let $x \in L$. Then there exists a unique $\theta_x \subseteq \theta$ such that $x = \bigvee \theta_x$. Hence $\overline{\theta}(x) = \overline{\theta}(\bigvee \theta_x) = \bigvee_{a \in \theta_x} \overline{\theta}(a) = \bigvee_{a \in \theta_x} a = x$. Also, since $\theta_x \subseteq \{a \in \theta | a \leq x\}$, we conclude that $x = \bigvee \theta_x \leq \underline{\theta}(x)$. By Lemma 3.5, $x = \underline{\theta}(x)$. □

Theorem 3.17. *Let L be a frame. Then, $\overline{\theta}$ is a one-one map if and only if θ is the set of all atoms of L and L is an atomic complete boolean algebra.*

Proof. Suppose that $\overline{\theta}$ is a one-one map. By Proposition 3.16(4), every element of L is precise, and so by Theorem 3.11 every element of L is complemented, therefore

L is a complete boolean algebra. Assume $a \in \theta$ and $0 \neq x \leq a$. By Proposition 3.16(4) there exists a unique $\theta_x \subseteq \theta$ such that $x = \bigvee \theta_x$. Let $b \in \theta_x$. So, $a \geq x \geq b$, hence $a = b$, and thus $x = a$. Therefore a is an atom. Now, let a be an atom. Again, by Proposition 3.16(4), there exists a unique $\theta_a \subseteq \theta$ such that $a = \bigvee \theta_a$. Let $b \in \theta_a$. Hence $b \leq a$, since a is an atom, $b = a$, and so $a \in \theta$. It gives that θ is equal to the set of all atoms of L. Therefore L is atomic.

Conversely, suppose L is an atomic complete boolean algebra and θ is the set of all atoms of L. Hence every element $x \in L$ is a join of atoms of L. So, by Proposition 3.16, $\overline{\theta}$ is one-one. □

Proposition 3.18. *Let θ be a partition of a complete Boolean algebra L. Then the following statements are equivalent.*

1. *$\overline{\theta}$ preserves binary meets.*
2. *$\overline{\theta} = id_L$.*
3. *$\overline{\theta}$ is one-one.*
4. *L is atomic and θ is the set of all atoms of L.*

Proof. (1) ⇒ (2) Let $x \in L \setminus \{0,1\}$. It is clear that

$$0 = \overline{\theta}(x \wedge x') = \overline{\theta}(x) \wedge \overline{\theta}(x') = \bigvee \{a \wedge b : a, b \in \theta \& a \wedge x \neq 0 \& a \wedge x' \neq 0\}$$

Hence for every $a \in \theta$, $a \wedge x = 0$ if and only if $a \wedge x' \neq 0$ and also,

$$\begin{array}{rcl} \overline{\theta}(x) & = & (x \wedge \overline{\theta}(x)) \vee (x' \wedge \overline{\theta}(x)) \\ & = & (x \wedge \overline{\theta}(x)) \vee \bigvee \{x' \wedge a : a \in \theta \& x \wedge a \neq 0\} \\ & = & x \wedge \overline{\theta}(x). \end{array}$$

Hence $x \leq \overline{\theta}(x) \leq x$, that is, $\overline{\theta}(x) = x$. (2) ⇒ (1) is clear, (2)⇔(3) by Proposition 3.16, and (3)⇔(4) by Theorem 3.17. □

Corollary 3.19. *Let θ be a partition of a set X. Then the following statements are equivalent.*

1. *$\overline{Apr}_\theta$ preserves binary meets.*
2. *$\theta = \{\{x\} : x \in X\}$.*

Proof. Consider $L = P(X)$, which is an atomic complete boolean algebra whose set of all atoms is $\{\{x\} : x \in X\}$. So, by Theorem 3.17, Propositions 3.16, and 3.18, the proof is complete. □

4. Partitions and Galois Connections

Let L be a frame. For a partition θ of L, by Lemmas 3.3 and 3.5, we have $\overline{\theta}$ is a closure operator on L, and also, $\underline{\theta}$ is a kernel operator on L (Definition 2.3).

Proposition 4.1. *Let θ be a partition of a frame L. Then $\left(\underline{\theta}, \overline{\theta}\right)$ is a Galois* *connection.*

Proof. Let $x, y \in L$ and $\underline{\theta}(x) \geq y$. For every $a \in \theta$, we have

$$\begin{array}{rcl} a \wedge y \neq 0 & \Rightarrow & 0 \neq a \wedge \underline{\theta}(x) = \bigvee\{a \wedge b : b \in \theta \& b \leq x\} \\ & \Rightarrow & a \leq x \\ & \Rightarrow & a \wedge x = a. \end{array}$$

Hence

$$\begin{array}{rcl} x \wedge \overline{\theta}(y) & = & \bigvee\{a \wedge x : a \in \theta \& a \wedge y \neq 0\} \\ & = & \bigvee\{a \in \theta : a \wedge y \neq 0\} \\ & = & \overline{\theta}(y). \end{array}$$

Therefore $x \geq \overline{\theta}(y)$.

Let $x, y \in L$ and $x \geq \overline{\theta}(y) = \bigvee\{a \in \theta : a \wedge y \neq 0\}$. For every $a \in \theta$, if $a \wedge y \neq 0$, then $a \leq x$, which follows that if $a \nleq x$, then $a \wedge y = 0$. So that

$$\begin{array}{rcl} y \wedge \underline{\theta}(x) & = & \bigvee\{a \wedge y : a \in \theta \& a \leq x\} \\ & = & \bigvee\{a \wedge y : a \in \theta \& a \leq x\} \vee \bigvee\{a \wedge y : a \in \theta \& a \nleq x\} \\ & = & y \wedge \bigvee \theta \\ & = & y. \end{array}$$

Therefore $\underline{\theta}(x) \geq y$. □

Definition 4.2. Let θ_1 and θ_2 be two partitions. We say that θ_2 is a *refinement* of θ_1, and denote it by $\theta_1 \preceq \theta_2$, if for every $a \in \theta_1$, there exists $S \subseteq \theta_2$ such that $a = \bigvee S$. The set of all the partitions of L is denoted by $Part(L)$.

Theorem 4.3. *$(Part(L), \preceq)$ is a $\vee$-semi lattice.*

Proof. Reflexivity and transitivity of $\preceq$ is clear, thus we show that $\preceq$ is anti-symmetric. Assume $\theta_1 \preceq \theta_2$ and $\theta_2 \preceq \theta_1$. Let $a \in \theta_1$. By the definition of a refinement, there exists $S \subseteq \theta_2$ such that $a = \bigvee S$. Let $b \in S$. So, $0 \neq b \leq a$. On the other hand there exists $T \subseteq \theta_1$ such that $b = \bigvee T$. If $a \notin T$, $b = a \wedge b = a \wedge \bigvee T = \bigvee\{a \wedge t : t \in T\} = 0$, which contradicts $b \in \theta_2$. So, $a \in T$, thus $a \leq b$, and therefore $a = b \in \theta_2$. Hence $\theta_1 \subseteq \theta_2$. Similarly, $\theta_2 \subseteq \theta_1$. So $\theta_1 = \theta_2$ and we conclude that $(Part(L), \preceq)$ is a partial ordered set.

Let $\theta_1, \theta_2 \in Part(L)$. We put $\theta = \{a_1 \wedge a_2 | (a_1, a_2) \in \theta_1 \times \theta_2\} \setminus \{0\}$. Then $1 = \bigvee \theta_1 \wedge \bigvee \theta_2 = \bigvee_{a_1 \in \theta_1} \bigvee_{a_2 \in \theta_2} a_1 \wedge a_2 = \bigvee \theta$. If $a \in \theta_1$, then $a = a \wedge \bigvee \theta_2 = \bigvee\{a \wedge b | b \in \theta_2\}$. Hence $\theta_1 \preceq \theta$ and similarly $\theta_2 \preceq \theta$. Now, suppose that there exists $\theta_3 \in Part(L)$ such that $\theta_1 \preceq \theta_3$ and $\theta_2 \preceq \theta_3$. If $c \in \theta$, then there exists $(a, b) \in \theta_1 \times \theta_2$ such that $c = a \wedge b$. Since there exist $S_a, S_b \subseteq \theta_3$ such that $a = \bigvee S_a$ and $b = \bigvee S_b$, we conclude that $c = \bigvee\{x \wedge y | (x, y) \in S_a \times S_b\} = \bigvee\{x \wedge x | (x, x) \in S_a \times S_b\} = \bigvee\{x | x \in S_a \cap S_b\}$. Therefore $\theta \preceq \theta_3$ and so $\theta_1 \vee \theta_2 = \theta$. Hence $Part(L)$ is a $\vee$-semi lattice. □

Definition 4.4. Let (f_1, g_1) and (f_2, g_2) be two Galois connections of L, that is $f_i \dashv g_i$ for $i = 1, 2$. Define $(f_1, g_1) \leq (f_2, g_2)$, if $f_1 \geq f_2$ and $g_1 \leq g_2$. The set of all Galois connections is denoted by $Gal(L)$.

Theorem 4.5. *$(Gal(L), \leq)$ is a $\vee$-semi lattice.*

Proof. It is clear that $Gal(L)$ is a partially ordered set. Let $(f_1, g_1), (f_2, g_2) \in Gal(L)$. Then, for every $s, t \in L$,

$$\begin{aligned} f_1(t) \wedge f_2(t) \geq s \quad &\Leftrightarrow \quad f_1(t) \geq s \& f_2(t) \geq s \\ &\Leftrightarrow \quad t \geq g_1(s) \& t \geq g_2(s) \\ &\Leftrightarrow \quad t \geq g_1(s) \vee g_2(s) \end{aligned}$$

Therefore $(f_1 \wedge f_2, g_1 \vee g_2) \in Gal(L)$. It is clear that $(f_1 \wedge f_2, g_1 \vee g_2)$ is an upper bound for $\{(f_1, g_1), (f_2, g_2)\}$. Let $(f, g) \in Gal(L)$ be an upper bound for $\{(f_1, g_1), (f_2, g_2)\}$. Then

$$\begin{aligned} \forall 1 \leq i \leq 2 \ (f_i \geq f \& g_i \leq g) \quad &\Rightarrow \quad f_1 \wedge f_2 \geq f \& g_1 \vee g_2 \leq g \\ &\Rightarrow \quad (f_1 \wedge f_2, g_1 \vee g_2) \leq (f, g). \end{aligned}$$

Hence $(f_1, g_1) \vee (f_2, g_2) = (f_1 \wedge f_2, g_1 \vee g_2)$ and we conclude that $Gal(L)$ is a $\vee$-semi lattice. □

Lemma 4.6. *If θ is a partition of L and $\theta' \subset \theta \subset \theta''$, then θ', θ'' are not partitions.*

Proof. If θ' is a partition, and $x \in \theta \setminus \theta'$, then, $1 = \bigvee \theta' = \bigvee \theta$, and hence $x = x \wedge \bigvee \theta' = \bigvee \{x \wedge a : a \in \theta'\} = 0$, which contradicts $x \in \theta$.

Now, if θ'' is a partition, then, by the above proof, θ is not a partition. Hence θ'' is not a partition, too. □

Proposition 4.7. *The map $\phi : Part(L) \to Gal(L)$ given by $\theta \rightsquigarrow (\overline{\theta}, \underline{\theta})$ is an order preserving map and a monomorphism.*

Proof. Suppose $\phi(\theta_1) = \phi(\theta_2)$. So, for $a \in \theta_1$, $a = \overline{\theta}_1(a) = \overline{\theta}_2(a) = \bigvee\{b \in \theta_2 : a \wedge b \neq 0\} = \bigvee S$, where $S \subseteq \theta_2$. So, by the definition of a refinement, $\theta_1 \preceq \theta_2$. Similarly $\theta_2 \preceq \theta_1$. Therefore $\theta_1 = \theta_2$. To show the order preserving, let $\theta_1 \preceq \theta_2$. We show that for every $x \in L$, $\overline{\theta}_1(x) \geq \overline{\theta}_2(x)$, and $\underline{\theta}_1(x) \leq \underline{\theta}_2(x)$. For every $a \in \theta_1$, there exists $S_a \subseteq \theta_2$ such that $a = \bigvee S_a$. Hence $1 = \bigvee \theta_1 = \bigvee_{a \in \theta_1} \bigvee S_a$, which follows that $\bigcup_{a \in \theta_1} S_a$ is a partition of L. By Lemma 4.6, $\theta_2 = \bigcup_{a \in \theta_1} S_a$. If $(x, b) \in L \times \theta_2$ and $b \wedge x \neq 0$, then there exists $a \in \theta_1$ such that $b \in S_a$ and $a = \bigvee S_a$, which follows that $a \wedge x \neq 0$ and $b \leq a$. Therefore

$$\overline{\theta_2}(x) = \bigvee\{b \in \theta_2 | b \wedge x \neq 0\} \leq \bigvee\{a \in \theta_1 | a \wedge x \neq 0\} = \overline{\theta_1}(x).$$

Also,

$$a \leq x \Leftrightarrow \forall b \in S_a (b \leq x).$$

Hence $\underline{\theta_1}(x) \leq \underline{\theta_2}(x)$. □

Example 4.8. Let $X = \{1, 2, \ldots, 12\}$ and $L = \mathcal{P}(X)$. If

$$\theta_1 = \{\{1\}, \{2, 3\}, \{4\}, \{5, 9\}, \{6, 7, 10, 11\}, \{8, 12\}\},$$

$$\theta_2 = \{\{1, 2, 5, 6\}, \{3, 4, 7, 8\}, \{9, 10\}, \{11, 12\}\}$$

and $A = \{4, 6, 7\}$, then $\theta_1 \vee \theta_2 = \{\{x\} | x \in X\}$, $\overline{\theta_1}(A) = \{4, 6, 7, 10, 11\}$, $\overline{\theta_2}(A) = \{4\}$ and $\overline{\theta_1 \vee \theta_2}(A) = A \neq \{4\} = \overline{\theta_1} \wedge \overline{\theta_2}(A)$. Hence $\phi : Part(L) \to Gal(L)$ does not preserve binary joins.

Proposition 4.9. *If L is a compact frame and $\theta \in Part(L)$, then θ is a finite set.*

Proof. Since L is a compact frame and $1 = \bigvee \theta$, we conclude that there exists $\theta' \subseteq \theta$ such that θ' is a finite set and $1 = \bigvee \theta'$. It is clear that $\theta' \in Part(L)$ and by Lemma 4.6, $\theta' = \theta$ □

Proposition 4.10. *Let L be a complete Boolean algebra.*

1. *For every $x \in L$, $\bigwedge_\theta \overline{\theta}(x) = \bigvee_\theta \underline{\theta}(x) = x$, where the suprimum and infimum are taken over all partitions θ of L.*
2. *In the poset $Gal(L)$, $\bigvee_\theta (\overline{\theta}, \underline{\theta}) = (id_L, id_L) = 1_{Gal(L)}$.*

Proof. (1) Let $x \in L \setminus \{0, 1\}$. Since L is a Boolean algebra, x has a complement x'. Hence $\theta_x = \{x, x'\}$ is a partition of L and $\overline{\theta_x}(x) = x = \underline{\theta_x}(x)$. Since $x \leq \bigwedge_\theta \overline{\theta}(x) \leq \overline{\theta_x}(x) = x$ and $x = \underline{\theta_x}(x) \leq \bigvee_\theta \underline{\theta}(x) \leq x$, we conclude that $\bigwedge_\theta \overline{\theta}(x) = x = \bigvee_\theta \underline{\theta}(x)$.

(2) By (1) and noting that (id_L, id_L) is a Galois connection, which is the top element of $Gal(L)$. □

Proposition 4.11. *Let L be a frame. If θ is a partition of $R_G(L)$, then θ is a partition of L.*

Proof. Let θ be a partition of $R_G(L)$. Using Proposition 2.2, we have

$$1 = \bigvee^{R_G(L)} \theta = (\bigvee^{L} \theta)^{**} \Rightarrow 0 = (\bigvee^{L} \theta)^{*} \Rightarrow 1 = \bigvee^{L} \theta$$

since for $x \in L$ we have $x^* = 0$ if and only if $x = 1$. So that θ is a partition of L.□

5. Prime Elements and Precise Elements

Let θ be a partition of L, and p be a prime element. First note that for every two distinct elements of θ, one of them is less than p. Because, if $a \neq b$, $a \wedge b = 0 \leq p$, since p is prime, $a \leq p$ or $b \leq p$. On the other hand, since $p < 1$, there is $a \in \theta$ such that $a \not\leq p$. So, there is a unique $a \in \theta$ such that $a \not\leq p$, we denoted it by a_p. So, we have proved:

Lemma 5.1. *Let θ be a partition of L. If $p \in L$ is a prime element, there is a unique $a_p \in \theta$, such that $a_p \not\leq p$.*

Theorem 5.2. *If $p \in L$ is prime, then*

1. $\underline{\theta}(p) = \bigvee(\theta \setminus \{a_p\})$.
2. *p is a precise element if and only if $a_p \wedge p = 0$.*
3. *$\overline{\theta}(p) = 1$ if and only if $a_p \wedge p \neq 0$.*
4. *p is a precise element if and only if $\overline{\theta}(p) < 1$.*

Proof. By Lemma 5.1, for every $a \in \theta$, $a \leq p$ if and only if $a \neq a_p$, which proves (1). Now, assume that $a_p \wedge p = 0$. Hence, by (1), we have

$$\overline{\theta}(p) = \bigvee\{a \in \theta : a \wedge p \neq 0\} = \bigvee(\theta \setminus \{a_p\}) = \underline{\theta}(p)$$

Conversely, suppose that $a_p \wedge p \neq 0$, and hence

$$\overline{\theta}(p) = \bigvee\{a \in \theta : a \wedge p \neq 0\} = \bigvee \theta = 1 \neq \underline{\theta}(p)$$

These prove both (2) and (3). By (2) and (3) we easily get (4). □

Corollary 5.3. *Let θ be a partition of L. If $p \in L$ is prime and $p \leq x < 1$ such that x is a precise element, then p is a precise element.*

Proof. Since x is a precise element, $\overline{\theta}(x) = x < 1$. By Lemma 3.3, $\overline{\theta}(p) \leq \overline{\theta}(x) < 1$. By Theorem 5.2(4), p is a precise element. □

6. Conclusion

We introduced pointfree form of rough set theory, by taking a complete lattice L with a partition θ, and we defined $\overline{\theta}$ and $\underline{\theta}$, upper and lower pointless approximation maps. The partitions $\overline{\theta}$ and $\underline{\theta}$ on a frame are closure and kernel operators, respectively, and also, the pair $(\overline{\theta}, \underline{\theta})$ is a Galois connection on L. We showed that $\underline{\theta}(x^*)$ and $\overline{\theta}(x^*)$ are complements of $\underline{\theta}(x)$ and $\overline{\theta}(x)$ respectively. We introduced the notion of a precise element and proved that $\underline{\theta}(x)$ and $\overline{\theta}(x)$ are precise elements. Also, we showed that the precise elements are complemented and their complements are precise elements. We gave some equations by the Heyting operation $\rightarrow$ under condition of "$\overline{\theta}$ preserves binary meets", and we concluded that the equation $\underline{\theta}(\overline{\theta}(a) \rightarrow b) = a \rightarrow \underline{\theta}(b)$ is equivalent to $\overline{\theta}$ to be a frame map. We proved that $\overline{\theta}$ is a one-one map if and only if θ is the set of all atoms of L and L is an atomic complete boolean algebra. By the way pointless rough set theory we studied relations between Galois connections and partitions of L. We considered all Galois connections of L, $Gal(L)$, and all partitions of L, $Part(L)$. We proved that both of them are $\vee$-semi lattice, and we showed that the map given by $\theta \rightsquigarrow (\overline{\theta}, \underline{\theta})$ is an order preserving map and a monomorphism, also, it does not preserve binary joins. Also, we proved that $\bigvee_\theta(\overline{\theta}, \underline{\theta}) = (id_L, id_L) = 1_{Gal(L)}$. Finally, we characterized prime precise elements of L, and described $\overline{\theta}(p)$ and $\underline{\theta}(p)$ for a prime element $p \in L$.

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