KYUNGPOOK Math. J. 55(2015), 541-548
http://dx.doi.org/10.5666/KMJ.2015.55.3.541
pISSN 1225-6951 eISSN 0454-8124

Quasi-Normal Relations - a New Class of Relations

Daniel Abraham Romano
Faculty of Education Bijeljina, East Sarajevo Univestity, Bijeljina 76300, Semberski ratari street b. b., Bosnia and Herzegobina
e-mail : bato49@hotmail.com

Abstract. In this paper, concepts of quasi-normal and dually quasi-normal relations are introduced. Characterizations of these relations are obtained. In addition, particulary we show that the anti-order relation $\nleqslant$ $(= \leqslant^C)$ is a (dually) quasi-normal relation if and only if the partially ordered set $(X, \leqslant)$ is an anti-chain.

1. Introduction and Preliminaries

The regularity of binary relations was first characterized by Zareckiĭ ([9],[10]). Further criteria for regularity were given by Hardy and Petrich ([3]), Markowsky ([7]), Schein ([8]) and Xu Xiao-quan and Liu Yingming ([11]) (see also [1] and [2]). The concepts of conjugative relations, dually conjugative relations and dually normal relations were introduced by Guanghao Jiang and Luoshan Xu ([4], [5]), and a characterization of normal relations was introduced and analyzed by Jiang Guanghao, Xu Luoshan, Cai Jin and Han Guiwen ([6]). In this paper, we introduce and analyze two new classes of relations in sets - class of *quasi-normal relations* and class of *dually quasi-normal* relations on sets.

Notions and notations which aren't explicitly exposed but are used in this article, readers can find them from texts [3] and [11], for an example.

For a set X, we call ρ a binary relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, we define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $(\mathcal{B}(X), \circ)$ is a semigroup. For a binary relation α on a set X, we define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = (X \times X) \setminus \alpha$.

Let A and B be subsets of X. For $\alpha \in \mathcal{B}(X)$, we set

Received June 19, 2014; revised October 31, 2014; accepted November 27, 2014.
2010 Mathematics Subject Classification: 20M20, 03E20, 06B11.
Key words and phrases: relations, quasi-normal relations, dually quasi-normal relations.

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\} \text{ and } \alpha B = \{x \in X : (\exists b \in B)((x, b) \in \alpha)\}.$$

Specially, we put $a\alpha$ instead of $\{a\}\alpha$ and αb instead of $\alpha\{b\}$.

The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated:
– *normal* ([5]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ (\alpha^C)^{-1}.$$

– *dually normal* ([4]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

– *conjugative* ([3]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

– *dually conjugative* ([3]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

Put $\alpha^1 = \alpha$. It is easy to see that $(\alpha^{-1})^C = (\alpha^C)^{-1}$ holds. Previous description gives equality

$$\alpha = (\alpha^a)^i \circ \beta \circ (\alpha^b)^j$$

for some $\beta \in \mathcal{B}(X)$ where $i, j \in \{-1, 1\}$ and $a, b \in \{1, C\}$. We should investigate all other possibilities since some of possibilities given in the previous equation have been investigated.

Diverse descriptions of regular elements of $\mathcal{B}(X)$ can be found in [7], [8], and [9]. For any $\alpha \in \mathcal{B}(X)$, Zaretskiĭ ([9], Section 3.2) (See, also, paper [3]) introduced the following relation in his study of regular elements of $\mathcal{B}(X)$
:

$$\alpha^+ = \{(x, y) \in X \times X : \alpha \circ \{(x, y)\} \circ \alpha \subseteq \alpha\}.$$

Schein in [8], Theorem 1, proved that $\alpha^+ = (\alpha^{-1} \circ \alpha^C \circ \alpha^{-1})^C$ is the maximal element in the family of all elements $\beta \in \mathcal{B}(X)$ such that $\alpha \circ \beta \circ \alpha \subseteq \alpha$.

2. Quasi-Normal and Dually Quasi-Normal Relations

In the following definition we introduce two new classes of elements in $\mathcal{B}(X)$.

Definition 2.1. (a) For relation $\alpha \in \mathcal{B}(X)$ we say that it is a *quasi-normal* relation on X if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^C \circ \beta \circ (\alpha^C)^{-1}.$$

(b) For relation $\alpha \in \mathcal{B}(X)$ we say that it is a *dually quasi-normal* relation on X if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha^C.$$

Remark 2.1. The family of quasi-normal relations is not empty. Let $\alpha \in \mathcal{B}(X)$ be a relation such that $\alpha^C \circ (\alpha^C)^{-1} = Id_X$. Then, we have

$\alpha = Id_X \circ \alpha \circ Id_X = (\alpha^C \circ (\alpha^C)^{-1}) \circ \alpha \circ (\alpha^C \circ (\alpha^C)^{-1}) =$
$\alpha^C \circ ((\alpha^C)^{-1} \circ \alpha \circ \alpha^C) \circ (\alpha^C)^{-1} = \alpha^C \circ \beta \circ (\alpha^C)^{-1}.$

So, such relation α is a quasi-normal relation on X. Analogously, for relation $\alpha \in \mathcal{B}(X)$ such that $(\alpha^C)^{-1} \circ \alpha^C = Id_X$, we have

$\alpha = Id_X \circ \alpha \circ Id_X = ((\alpha^C)^{-1} \circ \alpha^C) \circ \alpha \circ ((\alpha^C)^{-1} \circ \alpha^C) =$
$(\alpha^C)^{-1} \circ (\alpha^C \circ \alpha \circ (\alpha^C)^{-1}) \circ \alpha^C = (\alpha^C)^{-1} \circ \beta \circ \alpha^C.$

Therefore, this relation α is a dually quasi-normal relation. So, the family of dually quasi-normal is not empty, either.

Particulary, for the relation $\nabla = (Id_X)^C$, since we have

$$(\nabla^C)^{-1} \circ \nabla^C = Id_X \circ Id_X = Id_X = \nabla^C \circ (\nabla^C)^{-1},$$

we conclude that it is a quasi-normal relation on X and it is a dually quasi-normal relation on X as well.

In following proposition we give a connection between quasi-normal and dually quasi-normal relations.

Proposition 2.1. *Relation α^{-1} is a dually quasi-normal relation on X if and only if α is a quasi-normal relation on X.*

Proof. Let α be a quasi-normal relation on set X. Then there exists a relation β on X such that $\alpha = \alpha^C \circ \beta \circ (\alpha^C)^{-1}$. Thus,

$$\alpha^{-1} = (\alpha^C \circ \beta \circ (\alpha^C)^{-1})^{-1} = ((\alpha^{-1})^C)^{-1} \circ \beta^{-1} \circ (\alpha^{-1})^C.$$

So, the relation α^{-1} is a dually quasi-normal if α is a quasi-normal relation. The second statement we demonstrate by analogy of the previous statement. □

Our second proposition is an adaptation of Schein's concept exposed in [8], Theorem 1 (See, also, [2], Lemma 1.) for our needs.

Theorem 2.1. *For a binary relation $\alpha \in \mathcal{B}(X)$, relation*

$$\alpha^* = ((\alpha^C)^{-1} \circ \alpha^C \circ \alpha^C)^C$$

is the maximal element in the family of all relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha.$$

Proof. First, remember ourself that

$$max\{\beta \in \mathcal{B}(X) : \alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha\} = \cup\{\beta \in \mathcal{B}(X) : \alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha\}.$$

Let $\beta \in B(X)$ be an arbitrary relation such that $\alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha$. We will prove that $\beta \subseteq \alpha^*$. If not, there is $(x, y) \in \beta$ such that $\neg((x, y) \in \alpha^*)$. The last gives:

$(x, y) \in (\alpha^C)^{-1} \circ \alpha^C \circ \alpha^C \Longleftrightarrow$

$(\exists u, v \in X)((x, u) \in \alpha^C \wedge (u, v) \in \alpha^C \wedge (v, y) \in (\alpha^C)^{-1}) \Longleftrightarrow$

$(\exists u, v \in X)((u, x) \in (\alpha^C)^{-1} \wedge (u, v) \in \alpha^C \wedge (y, v) \in \alpha^C) \Longrightarrow$

$(\exists u, v \in X)((u, x) \in (\alpha^C)^{-1} \wedge (x, y) \in \beta \wedge (y, v) \in \alpha^C \wedge (u, v) \in \alpha^C) \Longrightarrow$

$(\exists u, v) \in X)((u, v) \in \alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha \wedge (u, v) \in \alpha^C)$.

We got a contradiction. So, must be $\beta \subseteq \alpha^*$.

On the other hand, we should prove that

$$\alpha^C \circ \alpha^* \circ (\alpha^C)^{-1} \subseteq \alpha.$$

Let $(x, y) \in \alpha^C \circ \alpha^* \circ (\alpha^C)^{-1}$ be an arbitrary element. Then, there are elements $u, v \in X$ such that $(x, u) \in (\alpha^C)^{-1}$, $(u, v) \in \alpha^*$ and $(v, y) \in \alpha^C$. So, from

$$(x, u) \in (\alpha^C)^{-1},\ \neg((u, v) \in (\alpha^C)^{-1} \circ \alpha^C \circ \alpha^C),\ (v, y) \in \alpha^C,$$

we have $\neg((x, y) \in \alpha^C)$. Suppose that $(x, y) \in \alpha^C$. Then, we have $(u, v) \in (\alpha^C)^{-1} \circ \alpha^C \circ \alpha^C$, which is impossible. Hence, we have $(x, y) \in \alpha$ and, therefore, $\alpha^C \circ \alpha^* \circ (\alpha^C)^{-1} \subseteq \alpha$.

Finally, we conclude that α^* is the maximal element of the family of all relations $\beta \in \mathcal{B}(X)$ such that $\alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha$. □

We have the following proposition by dual process to previous theorem:

Theorem 2.2. *For a binary relation $\alpha \in \mathcal{B}(X)$, relation*

$$\alpha_* = (\alpha^C \circ \alpha^C \circ (\alpha^C)^{-1})^C$$

is the maximal element in the family of all relation $\beta \in \mathcal{B}(X)$ such that

$$(\alpha^C)^{-1} \circ \beta \circ \alpha^C \subseteq \alpha.$$

Some properties of relations α_* and α^* and connection between them are given in the following proposition.

Proposition 2.2. *For relation $\alpha \in \mathcal{B}(X)$ we have:*

(a) $\alpha^* = \{(x, y) \in X \times X : \alpha^C \circ \{(x, y)\} \circ (\alpha^C)^{-1} \subseteq \alpha\}$
$= \{(x, y) \in X \times X : x\alpha^C \times y\alpha^C \subseteq \alpha\}$.

(b) $\alpha_* = \{(x, y) \in X \times X : (\alpha^C)^{-1} \circ \{(x, y)\} \circ \alpha^C \subseteq \alpha\}$
$= \{(x, y) \in X \times X : \alpha^C x \times \alpha^C y \subseteq \alpha\}$.

(c) $(\alpha^*)^{-1} = (\alpha^{-1})_*$;
(d) $(\alpha_*)^{-1} = (\alpha^{-1})^*$.

Proof. (a) By Theorem 2.1, it is straightforward to show that

$$\alpha^* = \{(x,y) \in X \times X : \alpha^C \circ \{(x,y)\} \circ (\alpha^C)^{-1} \subseteq \alpha\}.$$

Furthermore, we have

$$\begin{aligned}(u,v) \in \alpha^C \circ \{(x,y)\} \circ (\alpha^C)^{-1} &\Longleftrightarrow (u,x) \in (\alpha^C)^{-1} \wedge (y,v) \in \alpha^C \\ &\Longleftrightarrow (x,u) \in \alpha^C \wedge (y,v) \in \alpha^C \\ &\Longleftrightarrow u \in x\alpha^C \wedge v \in y\alpha^C \\ &\Longleftrightarrow (u,v) \in x\alpha^C \times y\alpha^C.\end{aligned}$$

(b) Also, without difficulty, we can prove that

$$\begin{aligned}\alpha_* &= \{(x,y) \in X \times X : (\alpha^C)^{-1} \circ \{(x,y)\} \circ \alpha^C \subseteq \alpha\} \\ &= \{(x,y) \in X \times X : \alpha^C x \times \alpha^C y \subseteq \alpha\}\end{aligned}$$

holds.

(c)
$$\begin{aligned}(\alpha^*)^{-1} &= \{(x,y) \in X \times X : x\alpha^C \times y\alpha^C \subseteq \alpha\}^{-1} \\ &= \{(y,x) \in X \times X : x\alpha^C \times y\alpha^C \subseteq \alpha\} \\ &= \{(x,y) \in X \times X : y\alpha^C \times x\alpha^C \subseteq \alpha\} \\ &= \{(x,y) \in X \times X : x\alpha^C \times y\alpha^C \subseteq \alpha^{-1}\} \\ &= \{(x,y) \in X \times X : (\alpha^C)^{-1}x \times (\alpha^C)^{-1}y \subseteq \alpha^{-1}\} \\ &= \{(x,y) \in X \times X : (\alpha^{-1})^C x \times (\alpha^{-1})^C y \subseteq \alpha^{-1}\} \\ &= (\alpha^{-1})_* \,.\end{aligned}$$

(d) The proof of this proposition can be given analogically by previous assertion (c). □

In the following proposition we give an essential characterization of dually quasi-normal relations. It is our adaptation of concept exposed in [3], Theorem 7.2.

Theorem 2.3. *For a binary relation α on a set X, the following conditions are equivalent:*

(1) *α is a dually quasi-normal relation.*
(2) *For all $x, y \in X$, if $(x,y) \in \alpha$, there exists $u, v \in X$ such that:*
  (a) $(x,u) \in \alpha^C \wedge (y,v) \in \alpha^C$
  (b) $(\forall s,t \in X)((s,u) \in \alpha^C \wedge (t,v) \in \alpha^C \Longrightarrow (s,t) \in \alpha)$.
(3) $\alpha \subseteq (\alpha^C)^{-1} \circ \alpha_* \circ \alpha^C$.

Proof. (1) $\Longrightarrow$ (2). Let α be a dually quasi-normal relation, i.e. let there exist a relation β such that $\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha^C$. Let $(x,y) \in \alpha$. Then there exist elements $u, v \in X$ such that

$$(x,u) \in \alpha^C,\ (u,v) \in \beta,\ (v,y) \in (\alpha^C)^{-1}.$$

From previous, it follows that there exist elements $u, v \in X$ such that $(x, u) \in \alpha^C$ and $(y, v) \in \alpha^C$. This proves condition (a). Now we check the condition (b). Let $s, t \in X$ be arbitrary elements such that $(s, u) \in \alpha^C$ and $(t, v) \in \alpha^C$. Now from $(s, u) \in \alpha^C$, $(u, v) \in \beta$ and $(v, t) \in (\alpha^C)^{-1}$ follows $(s, t) \in (\alpha^C)^{-1} \circ \beta \circ \alpha^C = \alpha$.

(2) $\Longrightarrow$ (1). Let us define a binary relation

$$\alpha' = \{(u, v) \in X \times X : (\forall s, t \in X)((s, u) \in \alpha^C \wedge (t, v) \in \alpha^C \Longrightarrow (s, t) \in \alpha)\}$$

and show that $\alpha^C \circ \alpha' \circ (\alpha^C)^{-1} = \alpha$ is valid. Let $(x, y) \in \alpha$. Then there exist elements $u, v \in X$ such that the conditions (a) and (b) hold. We have $(u, v) \in \alpha'$ by definition of relation α'.

Further, from $(x, u) \in \alpha^C$, $(u, v) \in \alpha'$ and $(v, y) \in (\alpha^C)^{-1}$ follows $(x, y) \in (\alpha^C)^{-1} \circ \alpha' \circ \alpha^C$. Hence, we have $\alpha \subseteq (\alpha^C)^{-1} \circ \alpha' \circ \alpha^C$. Contrary, let $(x, y) \in (\alpha^C)^{-1} \circ \alpha' \circ \alpha^C$ be an arbitrary pair. There exist elements $u, v \in X$ such that $(x, u) \in \alpha^C$, $(u, v) \in \alpha'$ and $(v, y) \in (\alpha^C)^{-1}$. From previous we have $(x, u) \in \alpha^C$ and $(y, v) \in \alpha^C$. Hence, by definition of relation α', follows $(x, y) \in \alpha$ since $(u, v) \in \alpha'$. Therefore, $(\alpha^C)^{-1} \circ \alpha' \circ \alpha^C \subseteq \alpha$. So, the relation α is a dually quasi-normal relation on X since there exists a relation α' such that $(\alpha^C)^{-1} \circ \alpha' \circ \alpha^C = \alpha$.

(1) $\Longleftrightarrow$ (3). Let α be a dually quasi-normal relation. Then there is a relation β such that $\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha^C$. Since $\alpha_* = \bigcup\{\beta \in \mathcal{B}(X) : (\alpha^C)^{-1} \circ \beta \circ \alpha^C \subseteq \alpha\}$, we have $\beta \subseteq \alpha_*$ and $\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha^C \subseteq (\alpha^C)^{-1} \circ \alpha_* \circ \alpha^C$. Contrary, let $\alpha \subseteq (\alpha^C)^{-1} \circ \alpha_* \circ \alpha^C$, for a relation α. Then, we have $\alpha \subseteq (\alpha^C)^{-1} \circ \alpha_* \circ \alpha^C \subseteq \alpha$. So, the relation α is a dually quasi-normal relation on set X. □

Corollary 2.1. *Let $(X, \leq)$ be a poset. Relation $\leq^C$ is a dually quasi-normal relation on X if and only if $(X, \leqslant)$ is an anti-chain.*

Proof. Let $\leq^C$ be a dually quasi-normal relation on set X, and let $x, y \in X$ be elements such that $x \leq^C y$. Then, by previous theorem, there exist elements $u, v \in X$ such that:

(a) $x \leq u \wedge y \leq v$;

(b) $(\forall s, t \in X)((s \leq u \wedge t \leq v) \Longrightarrow s \leq^C t)$.

Let z be an arbitrary element and if we put $z = s = t$ in formula (b) we have

$$(z \leq u \wedge z \leq v) \Longrightarrow z \leq^C z,$$

which is a contradiction. Hence $\neg(z \leq u \wedge z \leq v)$, it follows

$$z \leq^C u \vee z \leq^C v.$$

However, one can observe that these conditions are satisfied only for the partially ordered sets witch are anti-chain. Indeed, let X have comparable elements, for example, $y < x$. Then, $x \leqslant^C y$. If for elements $u, v \in X$ the following $u \geqslant x$ and $v \geqslant y$ hold, then $y \leqslant u \wedge y \leqslant v$. Therefore, there exists z in X such that $z \leqslant u \wedge z \leqslant v$ (namely, $z = y$). It contradicts to the condition (b). So, there is no different comparable elements in X.

Contrary, let $x, y \in X$ be arbitrary elements such that $x \leq^C y$. There exist elements $u, v \in X$ such that

(a') $x \leq u \wedge y \leq v$ and

(b') $(\forall z \in X)(z \leq^C u \vee z \leq^C z)$.

Let $s, t \in X$ be arbitrary elements such that $s \leq u$ and $t \leq v$. At the second hand, from proposition (b'), for $z = s$, in this case, $s \leq^C u \vee s \leq^C v$, since the second option $s \leq^C u$ is impossible, the following is $s \leq^C v$. Thus, we have $s \leq^C t \vee t \leq^C v$. Since the second option is also impossible, we have to have $s \leq^C t$. So, elements u and v satisfy condition (b) of Theorem 3.1. So, the relation $\leq^C$ is a dually quasi-normal relation. □

Remark 2.2. Corollary 2.1 can be also verified as follows:

Assume first that the relation $\leq^C$ is quasi-normal. It means $\leq^C = \leq^{-1} \circ \beta \circ \leq$ for a relation β. If $x, y \in X$ with $x < y$ then $(y, x) \in \leqslant^C$, and so the previous equation implies $x \leqslant c$ and $d \leqslant^{-1} y$ for some $c, d \in X$ such that $(c, d) \in \beta$. Hence $x \leqslant c \beta d \leqslant^{-1} x$ follows since $d \leqslant^{-1} y$ and $x < y$. By the above equation, we deduce $(x, x) \in \leqslant^C$ which contradicts reflexivity of $\leqslant$. This shows that there are no elements $x, y \in X$ with $x < y$, and so $(X, \leqslant)$ is an anti-chain.

Conversely, if $(X, \leqslant)$ is an anti-chain, $\leqslant$ is the equality relation and then we have $\leqslant^{-1} \circ \beta \circ \leqslant = \beta$ for any relation β. Hence, choosing β to be $\leqslant^C$, we obtain that $\leqslant^C$ is quasi- normal.

Assertions, analogous to previous theorem and corollary which regards on quasi-normal relations, are presented below.

Theorem 2.4. *For a binary relation α on a set X, the following conditions are equivalent:*

(1) *α is a quasi-normal relation.*

(2) *For all $x, y \in X$, if $(x, y) \in \alpha$, there exists $u, v \in X$ such that:*

  (a) $(u, x) \in \alpha^C \wedge (v, y) \in \alpha^C$

  (b) $(\forall s, t \in X)((u, s) \in \alpha^C \wedge (v, t) \in \alpha^C \Longrightarrow (s, t) \in \alpha)$.

(3) $\alpha \subseteq \alpha^C \circ \alpha^* \circ (\alpha^C)^{-1}$.

Corollary 2.3. *Let $(X, \leq)$ be a poset. Relation $\leq^C$ is a quasi-normal relation on X if and only if $(X, \leqslant)$ is an anti-chain.*

References

[1] H. J. Bandelt, *Regularity and complete distributivity*, Semigroup Forum, **19**(1980), 123-126

[2] H. J. Bandelt, *On regularity classes of binary relations*, In : Universal Algebra and Applications, Banach Center Publications, **9**(1982), 329-333

[3] D. Hardy and M. Petrich, *Binary relations as lattice isomorphisms*, Ann. Mat. Pura Appl, **177(1)**(1999), 195-224

[4] Jiang Guanghao and Xu Luoshan, *Conjugative Relations and Applications*, Semigroup Forum, **80(1)**(2010), 85-91.

[5] Jiang Guanghao and Xu Luoshan, *Dually normal relations on sets*, Semigroup Forum, **85(1)**(2012), 75-80

[6] Jiang Guanghao, Xu Luoshan, Cai Jin and Han Guiwen, *Normal Relations on Sets and Applications*, Int. J. Contemp. Math. Sciences, **6(15)**(2011), 721 - 726

[7] G. Markowsky, *Idempotents and product representations with applications to the semigroup of binary relations*, Semigroup Forum, **5**(1972), 95-119

[8] B. M. Schein, *Regular elements of the semigroup of all binary relations*, Semigroup Forum, **13**(1976), 95-102

[9] K. A. Zaretskii, *Regular elements of the semigroup of binary relations*, Uspehi Matem. Nauk, **3(105)**(1962), 177-179.

[10] K. A. Zareckii, *The semigroup of binary relations*, Mat. Sb., **61**(1963), 291-305.

[11] Xu Xiao-quan and Liu Yingming, *Relational representations of hypercontinuous lattices*, In : Domain Theory, Logic, and Computation, Kluwer Academic Publisher, 2003, 65-74.