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Weyl Type Theorems for Unbounded Hyponormal Operators

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Abstract. If T is an unbounded hyponormal operator on an infinite dimensional complex Hilbert space H with $\rho(\mathrm{T}) \neq \phi$, then it is shown that T satisfies Weyl's theorem, generalized Weyl's theorem, Browder's theorem and generalized Browder's theorem. The equivalence of generalized Weyl's theorem with generalized Browder's theorem, property (gw) with property (gb) and property (w) with property (b) have also been established. It is also shown that a-Browder's theorem holds for T as well as its adjoint T^*.

1. Introduction

Throughout this paper H will be an infinite dimensional complex Hilbert space and $C(H)$ denotes the set of all closed linear operators from H to H. By $\mathcal{D}(\mathrm{T})$, $\mathcal{R}(\mathrm{T})$ and $\mathcal{N}(\mathrm{T})$ we denote the domain, range and null space of T, respectively. If the range of an operator $\mathrm{T} \in C(H)$ is closed and nullity of T, $\alpha(\mathrm{T}) = \dim \mathcal{N}(\mathrm{T}) < \infty$ (respectively, defect of T, $\beta(\mathrm{T}) = \operatorname{codim} \mathcal{R}(\mathrm{T}) < \infty$) then T is called an *upper semi-Fredholm* (respectively, *lower semi-Fredholm*) operator. A *semi-Fredholm operator* is an upper or lower semi-Fredholm operator. If both $\alpha(\mathrm{T})$ and $\beta(\mathrm{T})$ are finite then T is called a *Fredholm operator*. By $SF_+(H)$ (respectively, $SF_-(H)$) we denote the

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class of upper (respectively, lower) semi-Fredholm operators. For T $\in SF_+(H) \cup SF_-(H)$, index of T is defined as ind(T) = α(T) - β(T). An operator $T \in C(H)$ is called *Weyl* if it is Fredholm of index 0 and the *Weyl spectrum* of T is defined as $\sigma_W(\mathrm{T}) = \{\lambda \in \mathbb{C} : \mathrm{T} - \lambda\mathrm{I}$ is not Weyl$\}$. Also we have

$$SF_+^-(H) = \{T \in C(H) : T \in SF_+(H) \text{ and } ind(T) \leqslant 0\},$$
$$SF_-^+(H) = \{T \in C(H) : T \in SF_-(H) \text{ and } ind(T) \geqslant 0\}$$

and these operators generate the following spectrum

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SF_+^-(H)\}$$
$$\sigma_{SF_-^+}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SF_-^+(H)\}.$$

The concept of Fredholm operators was generalized by Berkani [5] to the class of B-Fredholm operators in the following way: Let T $\in C(H)$ and let

$$\Delta(\mathrm{T}) = \{n \in \mathbb{N} : \forall m \in \mathbb{N},\ m \geqslant n \Rightarrow \mathcal{R}(\mathrm{T}^n) \cap \mathcal{N}(\mathrm{T}) \subseteq \mathcal{R}(\mathrm{T}^m) \cap \mathcal{N}(\mathrm{T})\}.$$

Then the *degree of stable iteration* of T is defined as dis(T) = inf Δ(T) where dis(T) = ∞ if $\Delta(\mathrm{T}) = \phi$. Let T $\in C(H)$ be densely defined on H. We say that T $\in C(H)$ is a *semi B-Fredholm operator* if it is either upper or lower semi B-Fredholm operator, where T is an upper (respectively, lower) semi B-Fredholm operator if there exists an integer $d \in \Delta(\mathrm{T})$ such that $\mathcal{R}(\mathrm{T}^d)$ is closed and dim $\{\mathcal{N}(\mathrm{T}) \cap \mathcal{R}(\mathrm{T}^d)\} < \infty$ (respectively, codim $\{\mathcal{R}(\mathrm{T}) + \mathcal{N}(\mathrm{T}^d)\} < \infty$).

In this case, index of T is defined as the number

$$ind(\mathrm{T}) = \dim \{\mathcal{N}(\mathrm{T}) \cap \mathcal{R}(\mathrm{T}^d)\} \ - \ \mathrm{codim}\ \{\mathcal{R}(\mathrm{T}) + \mathcal{N}(\mathrm{T}^d)\}.$$

Also, we say that T is a *B-Fredholm operator* if T is both upper and lower semi B-Fredholm operator, that is, there exists an integer $d \in \Delta(\mathrm{T})$ such that T satisfies the following conditions:

(i) dim $\{\mathcal{N}(\mathrm{T}) \cap \mathcal{R}(\mathrm{T}^d)\} < \infty$

(ii) codim $\{\mathcal{R}(\mathrm{T}) + \mathcal{N}(\mathrm{T}^d)\} < \infty$.

Let $SBF_+(H)$ denote the class of all upper semi B-Fredholm operators. Then

$$SBF_+^-(H) = \{T \in C(H) : T \in SBF_+(H) \text{ and } ind(T) \leqslant 0\} \quad \text{and}$$
$$\sigma_{SBF_+^-}(H) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(H)\}.$$

An operator T $\in C(H)$ is said to be *B-Weyl* if it is a B-Fredholm operator of index zero and the *B-Weyl spectrum* of T is then defined as $\sigma_{BW}(\mathrm{T}) = \{\lambda \in \mathbb{C} : \mathrm{T} - \lambda\mathrm{I}$ is not B-Weyl$\}$.

The *ascent* p(T) and *descent* q(T) of an operator $T \in C(H)$ are given by

$$p(T) = inf\{n : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\} \quad \text{and}$$
$$q(T) = inf\{n : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}.$$

An operator $T \in C(H)$ is said to be *upper semi-Browder* (respectively, *lower semi-Browder*) if T is upper semi-Fredholm with $p(T) < \infty$ (respectively, lower semi-Fredholm with $q(T) < \infty$). If T is both upper and lower semi-Browder, that is, if T is a Fredholm operator with ascent and descent both finite, then T is *Browder*. The upper-Browder, lower-Browder and Browder spectra are defined as

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not upper semi-Browder}\},$$
$$\sigma_{lb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not lower semi-Browder}\} \text{ and}$$
$$\sigma_{b}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not Browder}\}, \text{ respectively.}$$

Clearly, $\sigma_{SF_+^-}(T) \subseteq \sigma_{ub}(T)$ and $\sigma_w(T) \subseteq \sigma_b(T)$.

In [15], Weyl asserts that if T is a hermitian operator then $\sigma_w(T)$ consists precisely of all points in $\sigma(T)$ except the isolated eigenvalues of finite multiplicity. Weyl's theorem has been extended from hermitian operators to the class of bounded normal, hyponormal and Toeplitz operators [8]. Further in [3], Berkani proved that if T is a bounded normal operator acting on a Hilbert space H, then $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, where E(T) is the set of all isolated eigenvalues of T. This gives the generalization of the Weyl's theorem. Also, in [4] he proved this generalized version of classical Weyl's theorem for bounded hyponormal operators.

In this paper, we denote by $\Re(H) = \{T \in C(H)$: T is an unbounded hyponormal operator with resolvent set $\rho(T) \neq \phi\}$ and we study several Weyl-type theorems and properties for operators $T \in \Re(H)$. The second section deals with the spectrum of an unbounded hyponormal operator where we show that operators $T \in \Re(H)$ are polaroid (every isolated spectral point is a pole) with ascent less than or equal to 1. In the third section, we show that T satisfies Weyl's theorem, generalized Weyl's theorem and Browder's theorem. Also, it is shown that a-Browder's theorem holds for T as well as its adjoint T^*. In the fourth section, the results of section 2 are used to establish the equivalence of generalized Weyl's theorem with generalized Browder's theorem, Weyl's theorem with Browder's theorem and property (gw) ([2]) with property (gb) ([7]). Moreover, the two variants of Weyl's theorem, property (w) ([13]) and property (b) ([7]) are also shown to be equivalent. Finally, as a conclusion, we summarize the relations between Weyl-type theorems, Browder-type theorems and various properties in a diagram.

2. Spectrum of an Unbounded Hyponormal Operator

A linear operator T is called *hyponormal* if (i) T is closed, (ii) $\mathcal{D}(T) = \mathcal{D}(T^*)$, $\mathcal{D}(T)$ is dense in H, and (iii) $T^*T - TT^* \geqslant 0$. Also, we have a characterization for unbounded hyponormal operators, namely, an operator T is *hyponormal* iff it satisfies (i) T^* exists, $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $\mathcal{D}(T)$ is dense in H and (ii) $\|Tx\| \geqslant \|T^*x\|$, for all $x \in \mathcal{D}(T) = \mathcal{D}(T^*)$.

Theorem 2.1. *Let $T \in C(H)$ be an unbounded hyponormal operator. Then $p(T - \lambda I) = 0$ or 1 for every $\lambda \in \mathbb{C}$.*

Proof. Since T is hyponormal, so is $T - \lambda I$. Then for $x \in \mathcal{D}(T - \lambda I)^2$, we have

$$\begin{aligned} \|(T-\lambda I)x\|^2 &= |\langle (T-\lambda I)x, (T-\lambda I)x\rangle| \\ &= |\langle x, (T-\lambda I)^*(T-\lambda I)x\rangle| \\ &\leqslant \|x\|\, \|(T-\lambda I)^*(T-\lambda I)x\| \\ &\leqslant \|x\|\, \|(T-\lambda I)^2x\|. \end{aligned}$$

Therefore, $\|x\|\|(T-\lambda I)^2x\| \geqslant \|(T-\lambda I)x\|^2$, and hence $\mathcal{N}(T - \lambda I)^2 \subseteq \mathcal{N}(T - \lambda I)$. Since the reverse inclusion always holds, we have $p(T - \lambda I) \leqslant 1$. □

One of the interesting properties in Fredholm theory is the single valued extension property (SVEP). This property was first introduced by Dunford [9]. Mainly we concern with the localized version of SVEP, the SVEP at a point, introduced by Finch [10] and relate it to the finiteness of the ascent of an operator. Let $T : \mathcal{D}(T) \subset H \to H$ be a closed linear mapping and let λ_o be a complex number. The operator T has the *single valued extension property* (SVEP) at λ_o if $f = 0$ is the only solution to $(T- \lambda I)f(\lambda) = 0$ that is analytic in a neighborhood of λ_o. Also, T has SVEP if it has this property at every point λ_o in the complex plane.

Theorem 2.2.([10]) *Let $T \in C(H)$.*

(i) *If $p(T - \lambda I)$ is finite for some $\lambda \in \mathbb{C}$, then T has SVEP at λ.*

(ii) *If T is onto and not one-one, then T does not have SVEP at $\lambda = 0$.*

The second part of this theorem can also be rephrased as "If T has SVEP, then T is invertible whenever it is onto, that is, $\sigma(T) = \sigma_s(T)$, where $\sigma_s(T)$ is the surjective spectrum of T".

It is well known that the resolvent operator $R_\lambda(T) = (T - \lambda I)^{-1}$ is an analytic operator-valued function for all $\lambda \in \rho(T)$ and the isolated points of $\sigma(T)$ are either poles or essential singularities of $R_\lambda(T)$. For $T \in C(H)$, an isolated point $\lambda \in \sigma(T)$ is said to be a pole of order p if $p = p(T - \lambda I) < \infty$ and $q(T - \lambda I) < \infty$ [12].

Theorem 2.3.([12]) *Let T be a closed linear operator with $\rho(T) \neq \phi$. If $\lambda_o \in \sigma(T)$ and there exists two closed subspaces M and N such that $T - \lambda_o I$ is one-one mapping of $\mathcal{D}(T) \cap M$ onto M, $T - \lambda_o I|_N$ is nilpotent of order p and $H = M \oplus N$, then $M = \mathcal{R}(T - \lambda_o I)^p$, $N = \mathcal{N}(T - \lambda_o I)^p$ and λ_o is a pole of the resolvent $R_\lambda(T)$ of order p. The above condition is also necessary.*

Lemma 2.4. *If $T \in \Re(H)$, then λ is an isolated point of $\sigma(T)$ iff λ is a simple pole of the resolvent of T.*

Proof. If λ is a pole of resolvent of T then λ is an isolated point of $\sigma(T)$. □

Conversely, suppose λ is an isolated point of $\sigma(T)$. Then $\{\lambda\}$ is a spectral set of T and by [14, §V.10], there is a corresponding spectral projection operator E_o such that $H = \mathcal{N}(E_o) \oplus \mathcal{R}(E_o) =$ X$_1\oplus$ X$_2$, say. We have, T is completely reduced by the pair of closed subspaces X$_1$ and X$_2$ and if $T_1 = T|_{X_1}$ and $T_2 = T|_{X_2}$, then $\sigma(T_1) = \sigma(T) \setminus \{\lambda\}$ and $\sigma(T_2) = \{\lambda\}$ so that $T - \lambda I$ is a one-one mapping of X$_1$ onto itself.

$$\begin{aligned} \text{Also,}\quad \mathcal{R}(T-\lambda I) &= X_1 \oplus (T-\lambda I)X_2 \\ &= X_1 \oplus 0 \qquad (\because \text{p(T - }\lambda\text{I)} \leqslant 1 \text{ by Theorem 2.1)}. \\ \text{Then,}\quad \mathcal{R}(T-\lambda I)^2 &= (T-\lambda I)X_1 \oplus 0 \\ &= X_1 \oplus 0 = \mathcal{R}(T-\lambda I). \end{aligned}$$

Thus, q(T - λI) = p(T - λI) $\leqslant 1$ and hence λ is a simple pole of the resolvent operator $R_\lambda(T)$. □

Let σ_a(T) be the approximate point spectrum of T. By E(T) and E$_o$(T) we denote the set of all isolated eigenvalues of T and the set of all isolated eigenvalues of finite multiplicities in σ(T), respectively. Also, let π(T) and π_o(T) denote the set of all poles and the set of all poles of finite multiplicities of the resolvent of T, respectively. We say that T $\in C(H)$ satisfies:

(i) Weyl's theorem if σ(T) $\setminus$ σ_w(T) = E$_o$(T).

(ii) Generalized Weyl's theorem if σ(T) $\setminus$ σ_{BW}(T) = E(T).

(iii) Browder's theorem if σ_w(T) = σ_b(T).

(iv) Generalized Browder's theorem if σ(T) $\setminus$ σ_{BW}(T) = π(T).

(v) a-Browder's theorem if $\sigma_{SF_+^-}$(T) = σ_{ub}(T).

(vi) property (w) if σ_a(T) $\setminus$ $\sigma_{SF_+^-}$(T) = E$_o$(T).

(vii) property (gw) if σ_a(T) $\setminus$ $\sigma_{SBF_+^-}$(T) = E(T).

(viii) property (b) if σ_a(T) $\setminus$ $\sigma_{SF_+^-}$(T) = σ(T) $\setminus\sigma_b$(T).

(ix) property (gb) if σ_a(T) $\setminus$ $\sigma_{SBF_+^-}$(T) = π(T).

3. Weyl-Type Theorems

In this section we prove several Weyl-type theorems for T $\in \Re(H)$.

Theorem 3.1. *If $T \in \Re(H)$, then Weyl's theorem holds for T.*

Proof. Let $\lambda \in \sigma$(T) $\setminus$ σ_w(T). Then $\mathcal{R}$(T - λI) is closed and α(T - λI) = β(T - λI) $< \infty$. If α(T - λI) = β(T - λI) = 0, T - λI is a one-one mapping of $\mathcal{D}$(T) onto all of H. The inverse (T - λI)$^{-1}$ is then closed and hence bounded, thus $\lambda \notin \sigma$(T), which is a contradiction. Hence, α(T - λI) > 0. Also, since p(T - λI) $= 1 < \infty$, [1, Theorem 3.4(iv)] gives q(T - λI) = p(T - λI) $< \infty$. Therefore, $\lambda \in$ E$_o$(T).

Conversely, suppose $\lambda \in$ E$_o$(T). Then λ is a pole of order p(T - λI) = 1 so that $H = \mathcal{R}$(T - λI) $\oplus$ $\mathcal{N}$(T - λI). Therefore, β(T - λI) = α(T - λI) $< \infty$ and hence, $\lambda \in \sigma$(T) $\setminus$ σ_w(T). Thus, Weyl's theorem holds for T. □

Theorem 3.2. *If $T \in \Re(H)$, then T satisfies generalized Weyl's theorem.*

Proof. Suppose $\lambda \in \sigma$(T) $\setminus$ σ_{BW}(T). Then T - λI is a B-Fredholm operator of index 0. By [5, theorem 2.4], there exist two closed invariant subspaces M and N of H such that $H = M \oplus N$, T$_o$ = (T - λI)$|_M$ is a closed Fredholm operator of index 0, and T$_1$ = (T - λI)$|_N$ is a nilpotent operator. Since T is hyponormal and M is a closed invariant subspace of T, so S = T$|_M$ is a hyponormal operator. Also (S - λI)$|_M$ = T$_o$ is a Fredholm operator of index 0 so that $\lambda \notin \sigma_w$(S). We have the following two cases:

Case (i): $\lambda \in \sigma$(S).

S being hyponormal satisfies Weyl's theorem and $\lambda \notin \sigma_w$(S), therefore $\lambda \in$ E$_o$(S). In particular, λ is isolated in σ(S) or 0 is isolated in σ(T$_o$). Since T - λI = T$_o$ $\oplus$ T$_1$ and T$_1$ is a nilpotent operator, so σ(T$_o$) $\setminus$ $\{0\}$ = σ(T - λI) $\setminus$ $\{0\}$. Therefore, 0 is isolated in σ(T - λI) or equivalently, λ is isolated in σ(T). Also λ, being an eigenvalue of S, is an eigenvalue of T. Hence, $\lambda \in$ E(T).

Case (ii): $\lambda \notin \sigma$(S).

In this case, T - λI = T$_o$ $\oplus$ T$_1$ = (S - λI)$|_M$ $\oplus$ T$_1$ where (S - λI)$|_M$ is a one-one mapping of $\mathcal{D}$(T) $\cap$ M onto M and T$_1$ is a nilpotent operator. Therefore, λ is a pole of the resolvent operator R$_\lambda$(T). Hence, λ is an isolated eigenvalue of T so that $\lambda \in$ E(T).

Therefore, σ(T) $\setminus$ σ_{BW}(T) $\subseteq$ E(T).

Conversely, let $\lambda \in$ E(T). Then, α(T - λI) > 0 and p(T - λI) = 1. λ being an isolated point is a pole of order p(T - λI) = 1. Thus, $H = \mathcal{R}$(T - λI) $\oplus$ $\mathcal{N}$(T - λI) = $M \oplus N$, say. Now, (T - λI)$|_M$ is a one-one operator from $\mathcal{D}$(T) $\cap$ M onto M, and hence a closed Fredholm operator of index 0, and (T - λI)$|_N$ is a nilpotent operator of index 1. By [5, Theorem 2.4], T - λI is a B-Fredholm operator of index 0. Therefore, $\lambda \in \sigma$(T) $\setminus$ σ_{BW}(T). □

Theorem 3.3. *Every unbounded hyponormal operator satisfies Browder's theorem.*
Proof. Since for every unbounded hyponormal operator T we have p(T - λI) $< \infty$, Browder's theorem follows from [1, Theorem 3.4]. □

Theorem 3.4. *If T is an unbounded hyponormal operator, then T and T^* satisfy a-Browder's theorem.*

Proof. Suppose that $\lambda \notin \sigma_{ub}$(T). Then T - λI is upper semi-Fredholm operator and p(T - λI) $< \infty$, [1, Theorem 3.4(i)] gives ind(T - λI) $\leqslant 0$ so that $\lambda \notin \sigma_{SF_+^-}$(T). Thus, $\sigma_{SF_+^-}$(T) $\subseteq \sigma_{ub}$(T).

Conversely, since p(T - λI) $< \infty$ for all $\lambda \in \mathbb{C}$, so σ_{ub}(T) $\subseteq \sigma_{SF_+^-}$(T) and hence T satisfies a-Browder's theorem.

Now suppose $\lambda \notin \sigma_{SF_+^-}(\mathrm{T}^*) = \sigma_{SF_-^+}(\mathrm{T})$. Then T - λI is lower semi-Fredholm with ind(T - λI) $\geqslant 0$. Since T - λI is hyponormal, we have p(T - λI) $< \infty$ so that by [1, Theorem 3.4(i)], ind(T - λI) $\leqslant 0$. Thus, ind(T - λI) = 0 and hence, q(T - λI) = p(T - λI) $< \infty$. Therefore, $\lambda \notin \sigma_{lb}(\mathrm{T}) = \sigma_{ub}(\mathrm{T}^*)$. Since the reverse inclusion holds for every operator, we have that T* satisfies a-Browder's theorem. □

The following example of an operator T $\in \Re(H)$ illustrates all the above theorems:

Example 3.5. Let $H = l^2$ and let T be defined as:

$$\begin{aligned} T(x_1, x_2, x_3, \dots) &= (0, x_1, 2x_2, 3x_3, 4x_4, \dots) \\ &= (0, a_1x_1, a_2x_2, a_3x_3, a_4x_4, \dots) \end{aligned}$$

where, $a_n = n$ for all $n \in \mathbb{N}$ and $\mathcal{D}(\mathrm{T}) = \left\{ (x_n) \in l^2 : \sum_{j=1}^{\infty} |a_j x_j|^2 < \infty \right\}$.
If c_{oo} =$\{x = (x_n) : x_n \neq 0$ for only finitely many $n \in \mathbb{N}\}$, then c_{oo} is dense in l^2. Since $c_{oo} \subseteq \mathcal{D}(\mathrm{T})$, so that $\mathcal{D}(\mathrm{T})$ is dense in H. Also, since $|a_n| \leqslant |a_{n+1}|$, T is an unbounded hyponormal operator.

Consider an operator S on H defined as:

$$\begin{aligned} S(x_1, x_2, x_3, \dots) &= (x_2, 2x_3, 3x_4, 4x_5, \dots) \\ &= (\overline{a_1}x_2, \overline{a_2}x_3, \overline{a_3}x_4, \overline{a_4}x_5, \dots) \end{aligned}$$

with $\mathcal{D}(\mathrm{S}) = \left\{ (x_n) \in l^2 : \sum_{j=1}^{\infty} |\overline{a_j} x_j|^2 < \infty \right\}$. Then S=T*, the adjoint of T.

Clearly, $\sigma_p(\mathrm{T}) = \phi$. Therefore, E(T) = E$_o$(T) = ϕ.

From [11], since $M = \lim_{n\to\infty} |a_n| = \infty$, we have
$\sigma(\mathrm{T}) = \{\lambda : |\lambda| \leqslant M\}$ i.e. $\sigma(\mathrm{T}) = \mathbb{C} \cup \{\infty\}$, the extended complex plane,
$\sigma_a(\mathrm{T}) = \{\lambda : |\lambda| = M\}$ i.e. $\sigma_a(\mathrm{T}) = \{\infty\}$ and
$\sigma_p(\mathrm{T}^*) = \{\lambda : |\lambda| < M\}$ i.e. $\sigma_p(\mathrm{T}^*) = \mathbb{C}$.

Case (i): $\lambda \in \sigma(\mathrm{T})$, $\lambda \neq \infty$.

Then, $\lambda \notin \sigma_a(\mathrm{T})$ implies that α(T - λI) = 0, hence p(T - λI) = 0, and $\mathcal{R}$(T - λI) is closed so that $\lambda \notin \sigma_{SF_+^-}(\mathrm{T})$ and $\lambda \notin \sigma_{ub}(\mathrm{T})$. Also, $\lambda \in \sigma_p(\mathrm{T}^*)$ implies that β(T - λI) = α(T* - λI) $\neq 0$ so that ind(T - λI) $\neq 0$ and hence $\lambda \in \sigma_w(\mathrm{T}) \subseteq \sigma_b(\mathrm{T})$.

Further, α(T - λI) = 0 implies that dim $\{\mathcal{N}(\mathrm{T} - \lambda\mathrm{I}) \cap \mathcal{R}(\mathrm{T} - \lambda\mathrm{I})^d\} = 0$. Also p(T - λI) = 0 implies that codim $\{\mathcal{R}(\mathrm{T} - \lambda\mathrm{I}) + \mathcal{N}(\mathrm{T} - \lambda\mathrm{I})^d\}$ = codim $\mathcal{R}$(T - λI) = β(T - λI) $\neq 0$. Thus B-Fredholm index of T - λI is non-zero and hence $\lambda \in \sigma_{BW}(\mathrm{T})$.

Case (ii): $\lambda = \infty$.

Then, $\lambda \notin \sigma_p$(T) and $\lambda \in \sigma_a$(T) implies that α(T - λI) = 0 but $\mathcal{R}$(T - λI) is not closed. Hence, $\lambda = \infty$ belongs to $\sigma_{SF_+^-}$(T), σ_{ub}(T), σ_w(T) and σ_b(T).

Also, codim $\{\mathcal{R}$(T - λI) + $\mathcal{N}$(T - λI)$^d\}$ = codim $\mathcal{R}$(T - λI) = β(T - λI) $\nless \infty$ (because otherwise $\mathcal{R}$(T - λI) must be closed). Hence, $\lambda \in \sigma_{BW}$(T).

From the above cases, we have

σ_w(T) = $\mathbb{C} \cup \{\infty\}$ = σ_b(T) and hence T satisfies Browder's theorem,
$\sigma_{SF_+^-}$(T) = $\{\infty\}$ = σ_{ub}(T) and hence T satisfies a-Browder's theorem,
σ(T) $\backslash\sigma_w$(T) = ϕ = E_o(T) and hence T satisfies Weyl's theorem, and
σ(T) $\backslash\sigma_{BW}$(T) = ϕ = E(T) and hence T satisfies generalized Weyl's theorem.

4. Equivalent Variants of Weyl-Type Theorems and Hyponormal Operators

It is known that property (gw) implies property (gb) and property (w) implies property (b) for every T $\in B(H)$, but the converses of these results do not hold true in general [7]. In this section we prove the equivalence of property (gw) with (gb) and property (w) with (b) for an operator T $\in \Re(H)$. Further, we also prove that for such a T, generalized Weyl's theorem is equivalent to generalized Browder's theorem and Weyl's theorem is equivalent to Browder's theorem.

Theorem 4.1. *Let $T \in \Re(H)$. Then property* (b) *holds for T iff property* (w) *holds for T.*

Proof. Suppose T satisfies property (b).

Let $\lambda \in \sigma_a$(T) $\setminus$ $\sigma_{SF_+^-}$(T). Since T satisfies property (b), $\lambda \in \sigma$(T) $\setminus$ σ_b(T) so that T - λI is a Fredholm operator of index 0, p(T - λI) $< \infty$ and q(T - λI) $< \infty$. Then, λ is an isolated point of σ(T). Also $0 < \alpha$(T - λI) $< \infty$ (otherwise α(T - λI) = β(T - λI) = 0 and $\lambda \notin \sigma$(T)). Therefore, $\lambda \in E_o$(T).

Conversely, suppose $\lambda \notin \sigma_a$(T) $\setminus$ $\sigma_{SF_+^-}$(T). Since T is hyponormal, p(T - λI) $< \infty$. By [1, Theorem 3.4(i)], *ind*(T - λI) $\leqslant 0$. We have to show that $\lambda \notin E_o$(T). We have the following two cases:

Case (i): α(T - λI) = ∞.

Then $\lambda \notin E_o$(T).

Case (ii): α(T - λI) $< \infty$.

Since T - λI is not upper semi-Fredholm operator, therefore, $\mathcal{R}$(T - λI) is not closed. If $\lambda \in E_o$(T), then λ is a pole of order p(T - λI) = 1. Then, $H = \mathcal{R}$(T - λI) $\oplus$ $\mathcal{N}$(T - λI) so that $\mathcal{R}$(T - λI) is closed which is a contradiction. Therefore, $\lambda \notin E_o$(T).

Thus, E_o(T) $\subseteq \sigma_a$(T) $\setminus$ $\sigma_{SF_+^-}$(T) and hence T satisfies property (w).

Conversely, suppose T satisfies property (w).

Let $\lambda \in \sigma$(T) $\setminus$ σ_b(T). By definition, σ(T) $\setminus$ σ_b(T) $\subseteq E_o$(T). Since T satisfies property (w), $\lambda \in E_o$(T) = σ(T) $\setminus$ $\sigma_{SF_+^-}$(T).

Conversely, suppose $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$. Since T satisfies property (w), $\lambda \in \mathrm{E}_o(\mathrm{T})$. Then, λ is a pole of order $\mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = 1$ so that $H = \mathcal{R}(\mathrm{T} - \lambda\mathrm{I}) \oplus \mathcal{N}(\mathrm{T} - \lambda\mathrm{I})$ and hence $\beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$. By [1, Theorem 3.4(iv)], $\mathrm{q}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = 1$. Therefore, $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_b(\mathrm{T})$ so that T satisfies property (b). □

The following theorem is a direct consequence of Lemma 2.4.

Theorem 4.2. *For every $T \in \Re(H)$, $E(\mathrm{T}) = \pi(\mathrm{T})$. In particular, $E_o(\mathrm{T}) = \pi_o(\mathrm{T})$.*

Remark 4.3. Since $\sigma(\mathrm{T}) \setminus \sigma_b(\mathrm{T}) = \pi_o(\mathrm{T})$ for every $\mathrm{T} \in C(H)$, the above theorem implies that Weyl's theorem is equivalent to Browder's theorem for every $\mathrm{T} \in \Re(H)$. Now, Theorem 3.3 can be viewed as a corollary to Theorem 3.1.

Corollary 4.4. *Let $T \in \Re(H)$. Then:*

(i) *generalized Weyl's theorem is equivalent to generalized Browder's theorem*

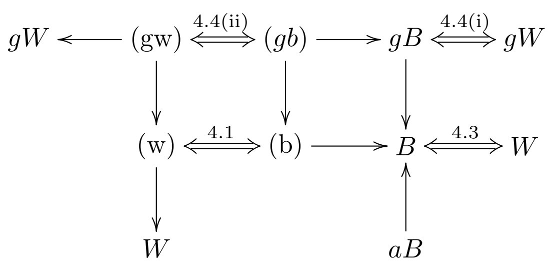
(ii) *property* (gw) *is equivalent to property* (gb).

Remark 4.5. By above corollary and Theorem 3.2, we get that every $\mathrm{T} \in \Re(H)$ satisfies generalized Browder's theorem.

5. Conclusion

To summarize, we use the abbreviations gW, W, (gw) and (w) to signify that an operator $\mathrm{T} \in \Re(H)$ satisfies generalized Weyl's theorem, Weyl's theorem, property (gw), property (w), respectively. Analogous abbreviations aB, gB, B, (gb) and (b) have been used with respect to a-Browder's theorem, generalized Browder's theorem, Browder's theorem, property (gb) and property (b), respectively.

The following diagram shows the relations between Weyl-type theorems, Browder-type theorems and properties for an operator $\mathrm{T} \in \Re(H)$. The arrows signify implications between the theorems and properties. The numbers near the arrows are references to the results proved in the present paper. We notice that several one-sided implications that hold for an operator $\mathrm{T} \in B(H)$ become equivalences when we consider $\mathrm{T} \in \Re(H)$.



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