J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2015.22.3.275
Volume 22, Number 3 (August 2015), Pages 275–283

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

SPECTRAL PROPERTIES OF k-QUASI-2-ISOMETRIC OPERATORS

Junli Shen [a] and Fei Zuo [b,*]

Abstract. Let T be a bounded linear operator on a complex Hilbert space H. For a positive integer k, an operator T is said to be a k-quasi-2-isometric operator if $T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k = 0$, which is a generalization of 2-isometric operator. In this paper, we consider basic structural properties of k-quasi-2-isometric operators. Moreover, we give some examples of k-quasi-2-isometric operators. Finally, we prove that generalized Weyl's theorem holds for polynomially k-quasi-2-isometric operators.

1. Introduction

Let H be an infinite dimensional separable Hilbert space. We denote by $B(H)$ the algebra of all bounded linear operators on H, write $N(T)$ and $R(T)$ for the null space and range space of T, and also, write $\sigma(T)$, $\sigma_a(T)$ and iso$\sigma(T)$ for the spectrum, the approximate point spectrum and the isolated point spectrum of T, respectively.

An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim H/R(T) < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The Weyl spectrum $w(T)$ of T is defined by $w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$. Following [12], we say that Weyl's theorem holds for T if $\sigma(T)\backslash w(T) = \pi_{00}(T)$, where $\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \dim N(T - \lambda) < \infty\}$.

More generally, Berkani investigated B-Fredholm theory (see [4, 5, 6]). An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty$.

Received by the editors July 04, 2015. Accepted July 14, 2015.
2010 *Mathematics Subject Classification.* 47A10, 47B20.
Key words and phrases. k-quasi-2-isometric operator, polaroid, generalized Weyl's theorem.
*Corresponding author.

Similarly, a B-Fredholm operator T is called B-Weyl if $i(T_{[n]}) = 0$. The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\}$. We say that generalized Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if generalized Weyl's theorem holds for T, then so does Weyl's theorem [5].

In [1] Agler obtained certain disconjugacy and Sturm-Lioville results for a subclass of the Toeplitz operators. These results were suggested by the study of operators T which satisfies the equation,

$$T^{*2}T^2 - 2T^*T + I = 0.$$

Such T are natural generalizations of isometric operators ($T^*T = I$) and are called 2-isometric operators. It is known that an isometric operator is a 2-isometric operator. 2-isometric operators have been studied by many authors and they have many interesting properties (see [2, 3, 7, 8, 9, 14]).

In order to extend 2-isometric operators we introduce k-quasi-2-isometric operators defined as follows:

Definition 1.1. For a positive integer k, an operator T is said to be a *k-quasi-2-isometric operator* if

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k = 0.$$

It is clear that each 2-isometric operator is a k-quasi-2-isometric operator and each k-quasi-2-isometric operator is a $(k+1)$-quasi-2-isometric operator.

In this paper we give a necessary and sufficient condition for T to be a k-quasi-2-isometric operator. Moreover, we study characterizations of weighted shift operators which are k-quasi-2-isometric operators. Finally, we prove polynomially k-quasi-2-isometric operators satisfy generalized Weyl's theorem.

2. Main Results

We begin with the following theorem which is the essence of this paper; it is a structure theorem for k-quasi-2-isometric operators.

Theorem 2.1. *If T^k does not have a dense range, then the following statements are equivalent:*

(1) *T is a k-quasi-2-isometric operator;*

(2) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ *on* $H = \overline{R(T^k)} \oplus N(T^{*k})$, *where* T_1 *is a* 2*-isometric operator and* $T_3^k = 0$. *Furthermore,* $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. (1) $\Rightarrow$ (2) Consider the matrix representation of T with respect to the decomposition $H = \overline{R(T^k)} \oplus N(T^{*k})$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Let P be the projection onto $\overline{R(T^k)}$. Since T is a k-quasi-2-isometric operator, we have

$$P(T^{*2}T^2 - 2T^*T + I)P = 0.$$

Therefore

$$T_1^{*2}T_1^2 - 2T_1^*T_1 + I = 0.$$

On the other hand, for any $x = (x_1, x_2) \in H$, we have

$$(T_3^k x_2, x_2) = (T^k(I-P)x, (I-P)x) = ((I-P)x, T^{*k}(I-P)x) = 0,$$

which implies $T_3^k = 0$.

Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$, where M is the union of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ by Corollary 7 of [11], and $\sigma(T_1) \cap \sigma(T_3)$ has no interior point and T_3 is nilpotent, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) $\Rightarrow$ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$, where $T_1^{*2}T_1^2 - 2T_1^*T_1 + I = 0$ and $T_3^k = 0$. Since

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix},$$

we have

$$\begin{aligned} & T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \\ = & \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*k} \end{aligned}$$

$$\times\left(\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*2}\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{2} - 2\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*}\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} + I\right)$$

$$\times\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{k}$$

$$= \begin{pmatrix} T_1^{*k}DT_1^k & T_1^{*k}D\sum_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j} \\ (\sum_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j})^*DT_1^k & (\sum_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j})^*D\sum_{j=0}^{k-1}T_1^jT_2T_3^{k-1-j} \end{pmatrix}$$

where $D = T_1^{*2}T_1^2 - 2T_1^*T_1 + I$. It follows that $T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k = 0$ on $H = \overline{R(T^{*k})} \oplus N(T^k)$. Thus T is a k-quasi-2-isometric operator. □

Corollary 2.2. *If $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is a k-quasi-2-isometric operator and T_1 is invertible, then T is similar to a direct sum of a 2-isometric operator and a nilpotent operator.*

Proof. Since T_1 is invertible, we have $\sigma(T_1) \cap \sigma(T_3) = \phi$. Then there exists an operator S such that $T_1S - ST_3 = T_2$ [15]. Since $\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix}$, it follows that

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}.$$

□

Corollary 2.3. *If T is a k-quasi-2-isometric operator and $R(T^k)$ is dense, then T is a 2-isometric operator.*

Proof. This is a result of Theorem 2.1. □

Corollary 2.4. *If T is a k-quasi-2-isometric operator, then so is T^n for every natural number n.*

Proof. We decompose T as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Then by Theorem 2.1, $T_1^{*2}T_1^2 - 2T_1^*T_1 + I = 0$. Hence T_1 is a 2-isometric operator, by [14, Theorem 2.1], T_1^n is a 2-isometric operator. Since

$$T^n = \begin{pmatrix} T_1^n & \sum_{j=0}^{n-1}T_1^jT_2T_3^{n-1-j} \\ 0 & T_3^n \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}),$$

T^n is a k-quasi-2-isometric operator for every natural number n by Theorem 2.1. □

Lemma 2.5. *T is a k-quasi-2-isometric operator if and only if*

$$||T^{k+2}x||^2 + ||T^k x||^2 = 2||T^{k+1}x||^2$$

for every $x \in H$.

Theorem 2.6. *Let T be a k-quasi-2-isometric operator and M be an invariant subspace for T. Then the restriction $T|_M$ is also a k-quasi-2-isometric operator.*

Proof. For $x \in M$, we have

$$\begin{aligned} 2||(T|_M)^{k+1}x||^2 =& 2||T^{k+1}x||^2 \\ =& ||T^{k+2}x||^2 + ||T^k x||^2 = ||(T|_M)^{k+2}x||^2 + ||(T|_M)^k x||^2. \end{aligned}$$

Thus $T|_M$ is a k-quasi-2-isometric operator. □

Example 2.7. Given a bounded sequence $\alpha : \alpha_0, \alpha_1, \alpha_2, \ldots$ (called weights), the unilateral weighted shift W_α associated with α is the operator on l_2 defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthogonal basis for l_2 and $|\alpha_n| \neq 0$ for each $n \geq 0$. Then the following statement holds: W_α is a k-quasi-2-isometric operator if and only if

$$W_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \alpha_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \alpha_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where

$$|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0 \ (n = k, k+1, k+2, \cdots).$$

Proof. By calculation, $W_\alpha^* W_\alpha = |\alpha_0|^2 \oplus |\alpha_1|^2 \oplus |\alpha_2|^2 \oplus \cdots$ and $W_\alpha^{*2} W_\alpha^2 = |\alpha_0|^2|\alpha_1|^2 \oplus |\alpha_1|^2|\alpha_2|^2 \oplus |\alpha_2|^2|\alpha_3|^2 \oplus \cdots$, by definition, W_α is a k-quasi-2-isometric operator if and only if $|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0$ $(n = k, k+1, k+2, \cdots)$. □

Remark 2.8. Let W_α be the unilateral weighted shift with weight sequence $(\alpha_n)_{n\geq 0}$ and $|\alpha_n| \neq 0$ for each $n \geq 0$. From Example 2.7 we obtain the following characterizations:

1. W_α is a k-quasi-2-isometric operator if and only if

$$|\alpha_n|^2 = \frac{(n-k+1)|\alpha_k|^2 - (n-k)}{(n-k)|\alpha_k|^2 - (n-k-1)}$$

for $n \geq k$.

2. $\{|\alpha_n|\}$ is a decreasing sequence of real numbers converging to 1 for $n \geq k$.

3. $\sqrt{2} \geq |\alpha_n| \geq 1$ for $n \geq k+1$.

4. Let $2 = |\alpha_k|, 1 = |\alpha_{k+1}| = |\alpha_{k+2}| = |\alpha_{k+3}| = \cdots$. Then W_α is a $(k+1)$-quasi-2-isometric operator but not a k-quasi-2-isometric operator.

In the sequel, we focus on polynomially k-quasi-2-isometric operators.

We say that T is a polynomially k-quasi-2-isometric operator if there exists a nonconstant complex polynomial p such that $p(T)$ is a k-quasi-2-isometric operator. It is clear that a k-quasi-2-isometric operator is a polynomially k-quasi-2-isometric operator. The following example provides an operator which is a polynomially k-quasi-2-isometric operator but not a k-quasi-2-isometric operator.

Example 2.9. *Let* $T = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \in B(l_2 \oplus l_2)$. *Then* T *is a polynomially* k*-quasi-2-isometric operator but not a* k*-quasi-2-isometric operator.*

Proof. Since
$$T^* = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix},$$
we have
$$T^{*2}T^2 - 2T^*T + I = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}.$$
Then
$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix} \neq 0.$$
Therefore T is not a k-quasi-2-isometric operator.

On the other hand, consider the complex polynomial $h(z) = (z-1)^2 + 1$. Then $h(T) = I$, and hence T is a polynomially k-quasi-2-isometric operator. □

Recall that an operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid, then it is isoloid. However, the converse is not true.

Theorem 2.10. *Let* T *be a polynomially* k*-quasi-2-isometric operator. Then* T *is polaroid.*

Proof. We first show that a k-quasi-2-isometric operator is polaroid. We consider

the following two cases: Case I: If the range of T^k is dense, then T is a 2-isometric operator, T is polaroid. Since an invertible 2-isometric operator is a unitary operator by [2, Proposition 1.23], and if T is a non-invertible 2-isometric operator, then iso$\sigma(T)$ is empty.
Case II: If the range of T^k is not dense, by Theorem 2.1, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Let $\lambda \in \text{iso}\sigma(T)$. Suppose that T_1 is a non-invertible 2-isometric operator. Then $\sigma(T) = D$, where D is the closed unit disk. Since $\sigma(T) = \sigma(T_1) \cup \{0\}$, we have iso$\sigma(T)$ is empty; thus T_1 is a invertible 2-isometric operator and $\lambda \in \text{iso}\sigma(T_1)$ or $\lambda = 0$, T_1 is a unitary operator, T_3 is nilpotent. It is easy to prove that $T - \lambda$ has finite ascent and descent, i.e., λ is a pole of the resolvent of T, therefore T is polaroid.

Next we show that a polynomially k-quasi-2-isometric operator is polaroid. If T is a polynomially k-quasi-2-isometric operator, then $p(T)$ is a k-quasi-2-isometric operator for some nonconstant polynomial p. Hence it follows from the first part of the proof that $p(T)$ is polaroid. Now apply [10, Lemma 3.3] to conclude that $p(T)$ polaroid implies T polaroid. □

Corollary 2.11. *Let T be a polynomially k-quasi-2-isometric operator. Then T is isoloid.*

An operator T is said to has the single valued extension property (abbreviated SVEP) if, for every open subset G of $\mathbb{C}$, any analytic function $f : G \to H$ such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G.

Theorem 2.12. *Let T be a polynomially k-quasi-2-isometric operator. Then T has SVEP.*

Proof. We first suppose that T is a k-quasi-2-isometric operator. We consider the following two cases:
Case I: If the range of T^k is dense, then T is a 2-isometric operator, T has SVEP by [8, Theorem 2].
Case II: If the range of T^k is not dense, by Theorem 2.1, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Suppose $(T - z)f(z) = 0$, $f(z) = f_1(z) \oplus f_2(z)$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$. Then we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1(z) + T_2 f_2(z) \\ (T_3 - z)f_2(z) \end{pmatrix} = 0.$$

And T_3 is nilpotent, T_3 has SVEP, hence $f_2(z) = 0$, $(T_1 - z)f_1(z) = 0$. Since T_1 is a 2-isometric operator, T_1 has SVEP by [8, Theorem 2], then $f_1(z) = 0$. Consequently, T has SVEP.

Now suppose that T is a polynomially k-quasi-2-isometric operator. Then $p(T)$ is a k-quasi-2-isometric operator for some nonconstant complex polynomial p, and hence $p(T)$ has SVEP. Therefore, T has SVEP by [13, Theorem 3.3.9]. □

Since the SVEP for T entails that generalized Browder's theorem holds for T, i.e. $\sigma_{BW}(T) = \sigma_D(T)$, where $\sigma_D(T)$ denotes the Drazin spectrum, a sufficient condition for an operator T satisfying generalized Browder's theorem to satisfy generalized Weyl's theorem is that T is polaroid. In [14], Patel showed that Weyl's theorem holds for 2-isometric operator. Then we have the following result:

Theorem 2.13. *If T is a polynomially k-quasi-2-isometric operator, then generalized Weyl's theorem holds for T, so does Weyl's theorem.*

Proof. It is obvious from Theorem 2.10, Theorem 2.12 and the statements of the above. □

REFERENCES

1. J. Agler: A disconjugacy theorem for Toeplitz operators. *Amer. J. Math.* **112** (1990), no. 1, 1-14.
2. J. Agler & M. Stankus: m-isometric transformations of Hilbert space. I. *Integral Equ. Oper. Theory* **21** (1995), no. 4, 383-429.
3. T. Bermudez, A. Martinon & E. Negrin: Weighted shift operators which are m-isometry. *Integral Equ. Oper. Theory* **68** (2010), 301-312.
4. M. Berkani & A. Arroud: Generalized Weyl's theorem and hyponormal operators. *J. Austra. Math. Soc.* **76** (2004), no. 2, 291-302.
5. M. Berkani & J.J. Koliha: Weyl type theorems for bounded linear operators. *Acta Sci. Math.(Szeged)* **69** (2003), no. 1-2, 359-376.
6. M. Berkani & M. Sarih: On semi B-Fredholm operators. *Glasgow Math. J.* **43** (2001), no. 3, 457-465.
7. M. Chō, S. Ôta, & K. Tanahashi: Invertible weighted shift operators which are m-isometries. *Proc. Amer. Math. Soc.* **141** (2013), no. 12, 4241-4247.

8. M. Chō, S. Ôta, K. Tanahashi & A. Uchiyama: Spectral properties of m-isometric operators. *Funct. Anal. Approx. Comput.* **4** (2012), no. 2, 33-39.
9. B.P. Duggal: Tensor product of n-isometries. *Linear Algebra Appl.* **437** (2012), 307-318.
10. ______: Polaroid operators, SVEP and perturbed Browder, Weyl theorems. *Rendiconti Circ Mat di Palermo LVI* **56** (2007), 317-330.
11. J.K. Han & H.Y. Lee: Invertible completions of $2*2$ upper triangular operator matrices. *Proc. Amer. Math. Soc.* **128** (1999), 119-123.
12. R.E. Harte & W.Y. Lee: Another note on Weyl's theorem. *Trans. Amer. Math. Soc.* **349** (1997), no. 5, 2115-2124.
13. K.B. Laursen & M.M. Neumann: *Introduction to Local Spectral Theory.* Clarendon Press, Oxford, 2000.
14. S.M. Patel: 2-isometry operators. *Glasnik Mat.* **37** (2002), no. 57, 143-147.
15. M.A. Rosenblum: On the operator equation $BX - XA = Q$. *Duke Math. J.* **23** (1956), 263-269.

[a]College of Computer and Information Technology, Henan Normal University, Xinxiang 453007, China
Email address: shenjunli08@126.com

[b]Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, School of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
Email address: zuofei2008@sina.com