J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2015.22.3.263 ISSN(Online) 2287-6081
Volume 22, Number 3 (August 2015), Pages 263–273

A SHARP SCHWARZ LEMMA AT THE BOUNDARY

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Abstract. In this paper, a boundary version of Schwarz lemma is investigated. For the function holomorphic $f(z) = a + c_p z^p + c_{p+1} z^{p+1} + ...$ defined in the unit disc satisfying $|f(z) - 1| < 1$, where $0 < a < 2$, we estimate a module of angular derivative at the boundary point b, $f(b) = 2$, by taking into account their first nonzero two Maclaurin coefficients. The sharpness of these estimates is also proved.

1. Introduction

In recent years, a boundary version of Schwarz lemma was investigated in D. M. Burns and S. G. Krantz [8], R. Osserman [11], V. N. Dubinin [3, 4], M. Jeong [6, 7], H. P. Boas [1], X. Tang, T. Liu and J. Lu [10], D. Chelst [2], X. Tang and T. Liu [9] and other's studies.

The classical Schwarz lemma states that an holomorphic function f mapping the unit disc $D = \{z : |z| < 1\}$ into itself, with $f(0) = 0$, satisfies the inequality $|f(z)| \leq |z|$ for any point $z \in D$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \lambda z$, $|\lambda| = 1$ [5, p.329].

Let $f(z) = a + c_p z^p + c_{p+1} z^{p+1} + ...$, $c_p \neq 0$, be a holomorphic function in the unit disc D, and $|f(z) - 1| < 1$ for $|z| < 1$, where $0 < a < 2$. Consider the functions

$$h(z) = f(z) - 1$$

and

$$\varphi(z) = \frac{h(z) - h(0)}{1 - \overline{h(0)} h(z)}.$$

Received by the editors May 08, 2015. Accepted August 07, 2015.
2010 *Mathematics Subject Classification.* 30C80.
Key words and phrases. Schwarz lemma on the boundary, angular limit and derivative, Julia-Wolff-Lemma, holomorphic function.
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Then $h(z)$ and $\varphi(z)$ are holomorphic functions on D with $|h(z)|<1$ and $|\varphi(z)|<1$ for $|z|<1$ and $\varphi(0)=0$. Therefore, from the Schwarz lemma, we obtain

$$|f(z)| \leq \frac{a\,(1+|z|^p)}{1-|1-a|\,|z|^p} \tag{1.1}$$

and

$$|c_p| \leq a\,(2-a)\,. \tag{1.2}$$

The equality in (1.1) for some nonzero $z \in D$ or in (1.2) holds if and only if

$$f(z)=\frac{a\left(1+z^p e^{i\theta}\right)}{1+(a-1)\,z^p e^{i\theta}},$$

where θ is a real number (see [6]).

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b|=1$, and if $|f(b)|=1$ and $f'(b)$ exists, then $|f'(b)| \geq 1$, which is known as the Schwarz lemma on the boundary.

R. Osserman [11] considered the case that only one boundary fixed point of f is given and obtained a sharp estimate based on the values of the function. He has first showed that

$$\left|f'(b)\right| \geqslant \frac{2}{1+|f'(0)|} \tag{1.3}$$

and

$$\left|f'(b)\right| \geqslant 1, \tag{1.4}$$

under the assuumption $f(0)=0$ where f is a holomorphic function mapping the unit disc into itself and b is a boundary point to which f extends continuously and $|f(b)|=1$. In addition, the equality in (1.3) holds if and only if f is of the form

$$f(z)=ze^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z},$$

where θ is a real number and $\alpha \in D$ satisfies $\arg\alpha=\arg b$. Also, the equality in (1.4) holds if and only if $f(z)=ze^{i\theta}$, where θ is a real number.

If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)|=1$, then by the Julia-Wolff lemma the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$ (see [14]).

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [5], [14]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1], [3], [4], [6], [7], [9], [10], [11], [13] and references therein).

Vladimir N. Dubinin [2] has continued this line of research and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + ...$, with a zero set $\{a_k\}$.

X. Tang, T. Liu and J. Lu [10] established a new type of the classical boundary Schwarz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Also, M. Jeong [6] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality.

2. Main Results

In this section, we can obtain more general results on the angular derivatives of holomorphic function on the unit disc at boundary by taking into account c_p, c_{p+1} and zeros of $f(z) - a$ if we know the first and the second coefficient in the expansion of the function $f(z) = a + c_p z^p + c_{p+1} z^{p+1} + ...$ We obtain a sharp lower bound of $|f'(b)|$ at the point b, where $|b| = 1$.

Theorem 2.1. *Let $f(z) = a + c_p z^p + c_{p+1} z^{p+1} + ...$, $c_p \neq 0$ be a holomorphic function in the unit disc D, and $|f(z) - 1| < 1$ for $|z| < 1$, where $0 < a < 2$. Assume that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = 2$. Then*

$$\left|f'(b)\right| \geqslant \frac{2-a}{a}\left(p + \frac{2\left(a(2-a) - |c_p|\right)^2}{a^2(2-a)^2 - |c_p|^2 + a(2-a)\left|c_{p+1}\right|}\right). \tag{1.5}$$

Moreover, the equality in (1.5) *occurs for the function*

$$f(z) = \frac{a\left(1 + z^p\right)}{1 + (a-1)\, z^p}.$$

Proof. Consider the functions

$$\varphi(z) = \frac{h(z) - h(0)}{1 - \overline{h(0)}h(z)},\ \ B(z) = z^p.$$

$\varphi(z)$ and $B(z)$ are holomorphic functions in D, and $|\varphi(z)| < 1$, $|B(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in D$, we have

$$|\varphi(z)| \leq |B(z)|\,.$$

Therefore,

$$p(z)=\frac{\varphi(z)}{B(z)}$$

is holomorphic function in D and $|p(z)|\leq 1$ for $|z|<1$. In particular, we have

$$|p(0)|=\frac{|c_p|}{a(2-a)}\leq 1 \tag{1.6}$$

and

$$\left|p'(0)\right|=\frac{|c_{p+1}|}{a(2-a)}.$$

Moreover, it can be seen that for $b\in\partial D$

$$\frac{b\varphi'(b)}{\varphi(b)}=\left|\varphi'(b)\right|\geq\left|B'(b)\right|=\frac{bB'(b)}{B(b)}.$$

The function

$$\phi(z)=\frac{p(z)-p(0)}{1-\overline{p(0)}p(z)}$$

is holomorphic in the unit disc D, $|\phi(z)|<1$, $\phi(0)=0$ and $|\phi(b)|=1$ for $b\in\partial D$. From (1.3), we obtain

$$\begin{aligned}\frac{2}{1+|\phi'(0)|} &\leq \left|\phi'(b)\right|=\frac{1-|p(0)|^2}{\left|1-\overline{p(0)}p(b)\right|^2}\left|p'(b)\right|\\ &\leq \frac{1+|p(0)|}{1-|p(0)|}\left|\frac{\varphi'(b)}{B(b)}-\frac{\varphi(b)B'(b)}{B^2(b)}\right|\\ &= \frac{1+|p(0)|}{1-|p(0)|}\left|\frac{\varphi(b)}{bB(b)}\right|\left|\frac{b\varphi'(b)}{\varphi(b)}-\frac{bB'(b)}{B(b)}\right|\\ &= \frac{1+|p(0)|}{1-|p(0)|}\left\{\left|\varphi'(b)\right|-\left|B'(b)\right|\right\}\end{aligned}$$

and since

$$\phi'(z)=\frac{1-|p(0)|^2}{\left(1-\overline{p(0)}p(z)\right)^2}p'(z)$$

$$\begin{aligned}\phi'(0) &= \frac{1-|p(0)|^2}{\left(1-|p(0)|^2\right)^2}p'(0)\\ &= \frac{p'(0)}{1-|p(0)|^2}\end{aligned}$$

$$\left|\phi'(0)\right|=\frac{|p'(0)|}{1-|p(0)|^2}=\frac{\frac{|c_{p+1}|}{a(2-a)}}{1-\left(\frac{|c_p|}{a(2-a)}\right)^2}=\frac{a(2-a)\,|c_{p+1}|}{a^2(2-a)^2-|c_p|^2},$$

we obtain

$$\frac{2}{1+\frac{a(2-a)|c_{p+1}|}{a^2(2-a)^2-|c_p|^2}} \leq \frac{1+\frac{|c_p|}{a(2-a)}}{1-\frac{|c_p|}{a(2-a)}} \left\{ \frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(b)\right|^2} \left|h'(b)\right| - p \right\}.$$

Since $h(0) = f(0) - 1 = a - 1$ and $h(b) = 1$ for $b \in \partial D$,

$$\begin{aligned}
\frac{2\left(a^2(2-a)^2-|c_p|^2\right)}{a^2(2-a)^2-|c_p|^2+a(2-a)\,|c_{p+1}|} &\leq \frac{a(2-a)+|c_p|}{a(2-a)-|c_p|}\left\{\frac{1-(a-1)^2}{|1-(a-1)|^2}\left|f'(b)\right|-p\right\} \\
&= \frac{a(2-a)+|c_p|}{a(2-a)-|c_p|}\left\{\frac{a(2-a)}{(2-a)^2}\left|f'(b)\right|-p\right\} \\
&= \frac{a(2-a)+|c_p|}{a(2-a)-|c_p|}\left\{\frac{a}{(2-a)}\left|f'(b)\right|-p\right\}
\end{aligned}$$

So, we take the inequality (1.5).

Now, we shall show that the inequality (1.5) is sharp. Let

$$f(z) = \frac{a\left(1+z^p\right)}{1+(a-1)\,z^p}.$$

Then

$$f'(z) = \frac{apz^{p-1}\left(1+(a-1)\,z^p\right) - pz^{p-1}(a-1)a\left(1+z^p\right)}{\left(1+(a-1)\,z^p\right)^2}$$

and

$$f'(1) = \frac{a^2p-2ap(a-1)}{a^2} = p\frac{2-a}{a}.$$

Also,

$$f(z) = \frac{a(1+z^p)}{1+(a-1)z^p} = a + \frac{a(2-a)z^p}{1+(a-1)z^p},$$

$$\frac{f(z)-a}{z^p} = \frac{a(2-a)}{1+(a-1)z^p},$$

$$\frac{f(z)-f(0)}{z^p} = \frac{a(2-a)}{1+(a-1)z^p}$$

and

$$|c_p| = a(2-a).$$

Since $|c_p| = a(2-a)$, (1.5) is satisfied with equality. □

If $f(z) - a$ have no zeros different from $z = 0$ in Theorem 2.1, the inequality (1.5) can be further strengthened. This is given by the following Theorem.

Theorem 2.2. *Let $f(z) = a + c_p z^p + c_{p+1} z^{p+1} + ...$, $c_p > 0$ be a holomorphic function in the unit disc D, $f(z) - a$ has no zeros in D except $z = 0$ and $|f(z) - 1| < 1$ for $|z| < 1$, where $0 < a < 2$. Assume that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = 2$. Then*

$$\left|f'(b)\right| \geqslant \frac{2-a}{a}\left(p - \frac{2\left|c_p\right| \ln^2\left(\frac{|c_p|}{a(2-a)}\right)}{2\left|c_p\right| \ln\left(\frac{|c_p|}{a(2-a)}\right) + \left|c_{p+1}\right|}\right) \tag{1.7}$$

and

$$\left|c_{p+1}\right| \leq 2\left|c_p \ln\left(\frac{|c_p|}{a(2-a)}\right)\right|. \tag{1.8}$$

In addition, the equality in (1.7) *occurs for the function*

$$f(z) = \frac{a\left(1+z^p\right)}{1+(a-1)\, z^p}$$

and the equality in (1.8) *occurs for the function*

$$f(z) = a\frac{1 + z^p e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{a(2-a)}\right)}}{1 + (a-1) z^p e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{a(2-a)}\right)}},$$

where $0 < c_p < 1$ and $\ln\left(\frac{c_p}{a(2-a)}\right) < 0$.

Proof. Let $c_p > 0$. Let $\varphi(z)$ and $p(z)$ be as in the proof of Theorem2.1. Having inequality (1.6) in mind , we denote by $\ln p(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln p(0) = \ln\left(\frac{c_p}{a(2-a)}\right) < 0.$$

The composite function

$$\Phi(z) = \frac{\ln p(z) - \ln p(0)}{\ln p(z) + \ln p(0)}$$

is holomorphic in the unit disc D, $|\Phi(z)| < 1$ for $|z| < 1$, $\Phi(0) = 0$ and $|\Phi(b)| = 1$ for $b \in \partial D$. From (1.3), we obtain

$$\begin{aligned}
\frac{2}{1 + |\Phi'(0)|} &\leq \left|\Phi'(b)\right| = \frac{|2\ln p(0)|}{|\ln p(b) + \ln p(0)|^2}\left|\frac{p'(b)}{p(b)}\right| \\
&= \frac{|2\ln p(0)|}{|\ln p(b) + \ln p(0)|^2}\left|p'(b)\right|
\end{aligned}$$

$$= \frac{|2\ln p(0)|}{|\ln p(b)+\ln p(0)|^2}\left|\frac{\varphi'(b)}{B(b)}-\frac{\varphi(b)B'(b)}{B^2(b)}\right|$$
$$= \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(b)}\left\{\left|\varphi'(b)\right|-\left|B'(b)\right|\right\}$$

and

$$\frac{2}{1+|\Phi'(0)|}\leq \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(b)}\left\{\left|\varphi'(b)\right|-\left|B'(b)\right|\right\}.$$

It can be seen that

$$\Phi'(z)=\frac{2\ln p(0)}{(\ln p(z)+\ln p(0))^2}\frac{p'(z)}{p(z)},$$
$$\Phi'(0)=\frac{2\ln p(0)}{(2\ln p(0))^2}\frac{p'(0)}{p(0)},$$

and

$$\left|\Phi'(0)\right|=\frac{1}{|2\ln p(0)|}\left|\frac{p'(0)}{p(0)}\right|=\frac{1}{-2\ln\left(\frac{|c_p|}{a(2-a)}\right)}\frac{|c_{p+1}|}{|c_p|}.$$

Therefore, we obtain

$$\frac{2}{1-\frac{1}{2\ln\left(\frac{|c_p|}{a(2-a)}\right)}\frac{|c_{p+1}|}{|c_p|}}\leq \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(b)}\left(\frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(b)\right|^2}\left|h'(b)\right|-p\right).$$

Replacing $\arg^2 p(b)$ *by zero, we take*

$$\frac{2}{1-\frac{1}{2\ln\left(\frac{|c_p|}{a(2-a)}\right)}\frac{|c_{p+1}|}{|c_p|}} \leq \frac{-2}{\ln p(0)}\left(\frac{a}{(2-a)}\left|f'(b)\right|-p\right)$$
$$= \frac{-2}{\ln\left(\frac{|c_p|}{a(2-a)}\right)}\left(\frac{a}{(2-a)}\left|f'(b)\right|-p\right),$$
$$p-\frac{2\left|c_p\right|\ln^2\left(\frac{|c_p|}{a(2-a)}\right)}{2\ln\left(\frac{|c_p|}{a(2-a)}\right)-|c_{p+1}|}\leq \frac{a}{(2-a)}\left|f'(b)\right|,$$

and we obtain (1.7) with an obvious equality case. Similary, function $\Phi(z)$ satisfies the assumptions of the Schwarz lemma, we obtain

$$1\geq \left|\Phi'(0)\right|=\frac{|2\ln p(0)|}{|\ln p(0)+\ln p(0)|^2}\left|\frac{p'(0)}{p(0)}\right|$$

and

$$1\geq \frac{-1}{2\ln\left(\frac{|c_p|}{a(2-a)}\right)}\frac{|c_{p+1}|}{|c_p|}.$$

Therefore, we have the inequality (1.8).

Now, we shall show that the inequality (1.8) is sharp. Let

$$f(z) = a + z^p g(z),$$

where

$$g(z) = a(2-a)\frac{e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{a(2-a)}\right)}}{1+(a-1)z^p e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{a(2-a)}\right)}}.$$

Then

$$g'(0) = c_{p+1}.$$

Under the simple calculations, we take

$$c_{p+1} = 2c_p \ln\left(\frac{c_p}{a(2-a)}\right).$$

Therefore, we obtain

$$c_{p+1} = 2\left|c_p \ln\left(\frac{|c_p|}{a(2-a)}\right)\right|.$$

□

If $f(z) - a$ have zeros different from $z = 0$, taking into account these zeros, the inequality (1.5) can be strengthened in another way. This is given by the following Theorem.

Theorem 2.3. *Let $f(z) = a + c_p z^p + c_{p+1} z^{p+1} + \dots$, $c_p \neq 0$ be a holomorphic function in the unit disc D, and $|f(z) - 1| < 1$ for $|z| < 1$, where $0 < a < 2$. Assume that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = 2$. Let $a_1, a_2, \dots, a_n$ be zeros of the function $f(z) - a$ in D that are different from zero. Then we have the inequality*

(1.9)

$$\left|f'(b)\right| \geq \frac{2-a}{a}\left(p + \sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2} + \frac{2\left(a(2-a)\prod_{k=1}^{n}|a_k| - |c_p|\right)^2}{\left(a(2-a)\prod_{k=1}^{n}|a_k|\right)^2 - |c_p|^2 + a(2-a)\prod_{k=1}^{n}|a_k|\,|c_{p+1}|}\right).$$

In addition, the equality in (1.9) occurs for the function

$$f(z) = a\frac{1 + z^p \prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1 + (a-1)z^p \prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}},$$

where $a_1, a_2, \dots, a_n$ are positive real numbers.

Proof. Let $\varphi(z)$ be as in the proof of Theorem2.1 and $a_1, a_2, ..., a_n$ be zeros of the function $f(z) - a$ in D that are different from zero.

$$B_0(z) = z^p \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z}$$

is a holomorphic function in D and $|B_0(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in D$, we have

$$|\varphi(z)| \leq |B_0(z)|.$$

The function

$$k(z) = \frac{\varphi(z)}{B_0(z)}$$

is holomorphic in D and $|k(z)| \leq 1$ for $|z| < 1$. In particular, we have

$$|k(0)| = \frac{|c_p|}{a(2-a) \prod\limits_{k=1}^{n} |a_k|} \leq 1$$

and

$$\left|k'(0)\right| = \frac{|c_{p+1}|}{a(2-a) \prod\limits_{k=1}^{n} |a_k|}.$$

Moreover, it can be seen that

$$\frac{b\varphi'(b)}{\varphi(b)} = \left|\varphi'(b)\right| \geq \left|B_0'(b)\right| = \frac{bB_0'(b)}{B_0(b)}.$$

Besides, with the simple calculations, we take

$$\left|B_0'(b)\right| = \frac{bB_0'(b)}{B_0(b)} = p + \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|b - a_k|^2}.$$

The auxiliary function

$$\Upsilon(z) = \frac{k(z) - k(0)}{1 - \overline{k(0)} k(z)}$$

is holomorphic in the unit disc D, $|\Upsilon(z)| < 1$ for $|z| < 1$, $\Upsilon(0) = 0$ and $|\Upsilon(b)| = 1$ for $b \in \partial D$. From (1.3), we obtain

$$\begin{aligned} \frac{2}{1 + |\Upsilon'(0)|} &\leq \left|\Upsilon'(b)\right| = \frac{1 - |k(0)|^2}{\left|1 - \overline{k(0)} k(b)\right|^2} \left|k'(b)\right| \leq \frac{1 + |k(0)|}{1 - |k(0)|} \left(\left|\varphi'(b)\right| - \left|B_0'(b)\right|\right) \\ &= \frac{1 + \frac{|c_p|}{a(2-a) \prod\limits_{k=1}^{n} |a_k|}}{1 - \frac{|c_p|}{a(2-a) \prod\limits_{k=1}^{n} |a_k|}} \left(\frac{2-a}{a} \left|f'(b)\right| - \left(p + \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|b - a_k|^2} \right) \right). \end{aligned}$$

Since

$$\Upsilon'(z) = \frac{1-|k(0)|^2}{\left(1-\overline{k(0)}k(z)\right)^2}k'(z)$$

and

$$\left|\Upsilon'(0)\right| = \frac{|k'(0)|}{1-|k(0)|^2} = \frac{\frac{|c_{p+1}|}{a(2-a)\prod_{k=1}^{n}|a_k|}}{1-\left(\frac{|c_p|}{a(2-a)\prod_{k=1}^{n}|a_k|}\right)^2} = \frac{a(2-a)\prod_{k=1}^{n}|a_k|\,|c_{p+1}|}{\left(a(2-a)\prod_{k=1}^{n}|a_k|\right)^2-|c_p|^2}$$

we take

$$\frac{2}{1+\frac{a(2-a)\prod_{k=1}^{n}|a_k||c_{p+1}|}{\left(a(2-a)\prod_{k=1}^{n}|a_k|\right)^2-|c_p|^2}} \leq \frac{1+\frac{|c_p|}{a(2-a)\prod_{k=1}^{n}|a_k|}}{1-\frac{|c_p|}{a(2-a)\prod_{k=1}^{n}|a_k|}}\left(\frac{2-a}{a}\left|f'(b)\right|-\left(p+\sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2}\right)\right).$$

Thus, we take the inequality (1.9) with an obvious equality case. □

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