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COMMON COUPLED FIXED POINT THEOREM UNDER GENERALIZED MIZOGUCHI-TAKAHASHI CONTRACTION FOR HYBRID PAIR OF MAPPINGS GENERALIZED MIZOGUCHI-TAKAHASHI CONTRACTION

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Abstract. We establish a common coupled fixed point theorem for hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. An example is also given to validate our results. We improve, extend and generalize several known results.

1. Introduction

Let $(X,\ d)$ be a metric space. We denote by 2^X the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed bounded subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. A functional $H : CL(X) \times CL(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be the Pompeiu-Hausdorff generalized metric induced by d is given by

$$H(A,\ B) = \begin{cases} \max\{\sup_{a\in A} D(a,\ B),\ \sup_{b\in B} D(b,\ A)\}, & \text{if maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $A,\ B \in CB(X)$, where $D(x,\ A) = \inf_{a\in A} d(x,\ a)$ denote the distance from x to $A \subset X$. For simplicity, if $x \in X$, we denote $g(x)$ by gx.

Markin [23] initiated to study the existence of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric which was further studied by many authors under different contractive conditions. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics.

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Bhaskar and Lakshmikantham [6] established some coupled fixed point theorems and applied these results to study the existence and uniqueness of solution for periodic boundary value problems, which were later extended by Lakshmikantham and Ciric [19]. For more details, see [5, 7, 8, 9, 10, 16, 17, 21, 22, 27, 30].

Samet et al. [28] claimed that most of the coupled fixed point theorems for single valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

The concepts related to coupled fixed point theory for multivalued mappings were extended by Abbas et al. [2] and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces. Very few researcher gave attention to coupled fixed point problems for hybrid pair of mappings including [1, 2, 11, 12, 13, 14, 15, 20, 29].

In [2], Abbas et al. introduced the following for multivalued mappings:

Definition 1.1. Let X be a nonempty set, $F : X \times X \to 2^X$ (a collection of all nonempty subsets of X) and g be a self-mapping on X. An element $(x, y) \in X \times X$ is called

(1) a *coupled fixed point* of F if $x \in F(x, y)$ and $y \in F(y, x)$.
(2) a *coupled coincidence point* of hybrid pair $\{F, g\}$ if $gx \in F(x, y)$ and $gy \in F(y, x)$.
(3) a *common coupled fixed point* of hybrid pair $\{F, g\}$ if $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F and g by $C(F, g)$. Note that if $(x, y) \in C(F, g)$, then (y, x) is also in $C(F, g)$.

Definition 1.2. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called $w-$*compatible* if $gF(x, y) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.

Definition 1.3. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called $F-$*weakly commuting* at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.

Lemma 1.4 ([26]). *Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in K(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b \in B} d(a, b)$.*

Nadler [25] extended the famous Banach Contraction Principle [4] from single-valued mapping to multivalued mapping. Mizoguchi and Takahashi [24] proved the following generalization of Nadler's fixed point theorem for weak contraction:

Theorem 1.5. *Let $(X,\ d)$ be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Assume that*

$$H(Tx,\ Ty) \leq \psi(d(x,\ y))d(x,\ y),$$

for all $x,\ y \in X$, where ψ is a function from $[0,\ \infty)$ into $[0,\ 1)$ satisfying

$$\limsup_{s \to t+} \psi(s) < 1$$

for all $t \geq 0$. Then T has a fixed point.

Suzuki [31] gave its very simple proof. Amini-Harandi and O'Regan [3] obtained a generalization of Mizoguchi and Takahashi's fixed point theorem.

In [8], Ciric et al. proved coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. Main results of Ciric et al. [8] extended and generalized the results of Bhaskar and Lakshmikantham [6], Du [17] and Harjani et al. [18].

In this paper, we prove a common coupled fixed point theorem for hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. We improve, extend and generalize the results of Amini-Harandi and O'Regan [3], Bhaskar and Lakshmikantham [6], Ciric et al. [8], Du [17], Harjani et al. [18] and Mizoguchi and Takahashi [24]. The effectiveness of our generalization is demonstrated with the help of an example.

2. Main Results

Let Φ denote the set of all functions $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying
(i_φ) φ is non-decreasing,
(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$,
(iii_φ) $\limsup_{t \to 0+} \frac{t}{\varphi(t)} < \infty$.

Let Ψ denote the set of all functions $\psi : [0,\ +\infty) \to [0,\ 1)$ which satisfies $\lim_{r \to t+} \psi(r) < 1$ for all $t \geq 0$. For example, if $\varphi(t) = \ln(t+1)$ and $\psi(t) = \frac{\varphi(t)}{t}$. Obviously, then $\varphi \in \Phi$ and $\psi \in \Psi$, because φ is non-decreasing, positive in $(0,\ +\infty)$,

$\varphi(0) = 0$ and $\limsup_{s\to 0+} \frac{s}{\varphi(s)} = 1 < \infty$. Also, $\lim_{r\to t+} \psi(r) = \lim_{r\to t+} \frac{\varphi(r)}{r} = \lim_{r\to t+} \frac{\ln(r+1)}{r} = \lim_{h\to 0} \frac{\ln(t+h+1)}{t+h} = \frac{\ln(t+1)}{t} < 1$.

Theorem 2.1. *Let (X, d) be a metric space, $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Assume that there exist some $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} &\varphi\left(H(F(x,\ y),\ F(u,\ v))\right) \\ &\quad\leq\ \psi\left(\varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right)\right)\varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right), \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x, y) \in C(F, g)$ and gx and gy are fixed points of g, that is, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

Proof. Let $x_0, y_0 \in X$ be arbitrary. Then $F(x_0, y_0)$ and $F(y_0, x_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$, because $F(X \times X) \subseteq g(X)$. Since $F : X \times X \to K(X)$, therefore by Lemma 1.4, there exist $z_1 \in F(x_1, y_1)$ and $z_2 \in F(y_1, x_1)$ such that

$$\begin{aligned} d(gx_1,\ z_1) &\leq H(F(x_0,\ y_0),\ F(x_1,\ y_1)), \\ d(gy_1,\ z_2) &\leq H(F(y_0,\ x_0),\ F(y_1,\ x_1)). \end{aligned}$$

Since $F(X \times X) \subseteq g(X)$, there exist $x_2, y_2 \in X$ such that $z_1 = gx_2$ and $z_2 = gy_2$. Thus

$$\begin{aligned} d(gx_1,\ gx_2) &\leq H(F(x_0,\ y_0),\ F(x_1,\ y_1)), \\ d(gy_1,\ gy_2) &\leq H(F(y_0,\ x_0),\ F(y_1,\ x_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n, y_n)$ and $gy_{n+1} \in F(y_n, x_n)$ such that

$$\begin{aligned} d(gx_n,\ gx_{n+1}) &\leq H(F(x_{n-1},\ y_{n-1}),\ F(x_n,\ y_n)), \\ d(gy_n,\ gy_{n+1}) &\leq H(F(y_{n-1},\ x_{n-1}),\ F(y_n,\ x_n)), \end{aligned}$$

which, by (i_φ) and (2.1), implies

$$\begin{aligned} & \varphi\left(d(gx_n,\ gx_{n+1})\right) \\ \leq\ & \varphi\left(H(F(x_{n-1},\ y_{n-1}),\ F(x_n,\ y_n))\right) \\ \leq\ & \psi\left(\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right)\right) \\ & \times\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right), \end{aligned}$$

which, by the fact that $\psi < 1$, implies

$$\varphi\left(d(gx_n,\ gx_{n+1})\right) \leq \varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right). \tag{2.2}$$

Similarly

$$\varphi\left(d(gy_n,\ gy_{n+1})\right) \leq \varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right). \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$\begin{aligned} & \max\left\{\varphi\left(d(gx_n,\ gx_{n+1})\right),\ \varphi\left(d(gy_n,\ gy_{n+1})\right)\right\} \\ \leq\ & \varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right). \end{aligned}$$

Since φ is non-decreasing, it follows that

$$\begin{aligned} & \varphi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}\right) \\ \leq\ & \varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right),\ \text{for all } n \geq 0. \end{aligned} \tag{2.4}$$

Now (2.4) shows that $\{\varphi(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\})\}$ is a non-increasing sequence. Therefore, there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \varphi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}\right) = \delta. \tag{2.5}$$

Since $\psi \in \Psi$, we have $\lim_{r\to\delta+}\psi(r) < 1$ and $\psi(\delta) < 1$. Then there exists $\alpha \in [0,\ 1)$ and $\varepsilon > 0$ such that $\psi(r) \leq \alpha$ for all $r \in [\delta,\ \delta+\varepsilon)$. From (2.5), we can take $n_0 \geq 0$ such that $\delta \leq \varphi(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}) \leq \delta+\varepsilon$ for all $n \geq n_0$. Then from (2.1), for all $n \geq n_0$, we have

$$\begin{aligned} & \varphi\left(d(gx_n,\ gx_{n+1})\right) \\ \leq\ & \psi\left(\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right)\right) \\ & \times\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right) \\ \leq\ & \alpha\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right). \end{aligned}$$

Thus, for all $n \geq n_0$, we have

$$\varphi\left(d(gx_n,\ gx_{n+1})\right) \leq \alpha\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right). \tag{2.6}$$

Similarly, for all $n \geq n_0$, we have

$$\varphi\left(d(gy_n,\ gy_{n+1})\right) \leq \alpha\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right). \tag{2.7}$$

Combining (2.6) and (2.7), for all $n \geq n_0$, we get

$$\begin{aligned} & \max\left\{\varphi\left(d(gx_n,\ gx_{n+1})\right),\ \varphi\left(d(gy_n,\ gy_{n+1})\right)\right\} \\ \leq\ & \alpha\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right). \end{aligned}$$

Since φ is non-decreasing, it follows that, for all $n \geq n_0$,

$$\begin{aligned} & \varphi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}\right) \\ \leq\ & \alpha\varphi\left(\max\left\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\right\}\right). \end{aligned} \tag{2.8}$$

Letting $n \to \infty$ in (2.8) and using (2.5), we obtain that $\delta \leq \alpha\delta$. Since $\alpha \in [0,\ 1)$, therefore $\delta = 0$. Thus

$$\lim_{n\to\infty} \varphi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}\right) = 0. \tag{2.9}$$

Since $\{\varphi(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\})\}$ is a non-increasing sequence and φ is non-decreasing, then $\{\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\}$ is also a non-increasing sequence of positive numbers. This implies that there exists $\theta \geq 0$ such that

$$\lim_{n\to\infty} \max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\} = \theta. \tag{2.10}$$

Since φ is non-decreasing, we have

$$\varphi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}\right) \geq \varphi(\theta).$$

Letting $n \to \infty$ in this inequality, by using (2.9), we get $0 \geq \varphi(\theta)$, which, by (ii_φ), implies that $\theta = 0$. Thus, by (2.10), we get

$$\lim_{n\to\infty} \max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\} = 0. \tag{2.11}$$

Suppose that $\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = 0$, for some $n \geq 0$. Then, we have $d(gx_n, gx_{n+1}) = 0$ and $d(gy_n, gy_{n+1}) = 0$ which implies that $gx_n = gx_{n+1} \in F(x_n, y_n)$ and $gy_n = gy_{n+1} \in F(y_n, x_n)$, that is, (x_n, y_n) is a coupled coincidence point of F and g. Now, suppose that $\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} \neq 0$, for all $n \geq 0$. Denote

$$a_n = \varphi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}\right),\ \text{for all } n \geq 0.$$

From (2.8), we have

$$a_n \leq \alpha a_{n-1},\ \text{for all } n \geq n_0.$$

Then, we have

$$\sum_{n=0}^{\infty} a_n \le \sum_{n=0}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} \alpha^{n-n_0} a_{n_0} < \infty. \tag{2.12}$$

On the other hand, by (iii_φ), we have

$$\limsup_{n\to\infty} \frac{\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}}{\varphi\left(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}\right)} < \infty. \tag{2.13}$$

Thus, by (2.12) and (2.13), we have $\sum \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} < \infty$. It means that $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_n\}_{n=0}^{\infty}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, therefore there exist $x,\ y \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \text{ and } \lim_{n\to\infty} gy_n = gy. \tag{2.14}$$

Now, since $gx_{n+1} \in F(x_n,\ y_n)$ and $gy_{n+1} \in F(y_n,\ x_n)$, therefore by using condition (2.1) and (i_φ), we get

$$\begin{aligned} & \varphi\left(D(gx_{n+1},\ F(x,\ y))\right) \\ \le\ & \varphi\left(H(F(x_n,\ y_n),\ F(x,\ y))\right) \\ \le\ & \psi\left(\varphi\left(\max\{d(gx_n,\ gx),\ d(gy_n,\ gy)\}\right)\right) \\ & \times\varphi\left(\max\{d(gx_n,\ gx),\ d(gy_n,\ gy)\}\right) \\ \le\ & \varphi\left(\max\{d(gx_n,\ gx),\ d(gy_n,\ gy)\}\right). \end{aligned}$$

Since φ is non-decreasing, we have

$$D(gx_{n+1},\ F(x,\ y)) \le \max\{d(gx_n,\ gx),\ d(gy_n,\ gy)\}. \tag{2.15}$$

Letting $n \to \infty$ in (2.15), by using (2.14), we obtain

$$D(gx,\ F(x,\ y)) = 0. \text{ Similarly } D(gy,\ F(y,\ x)) = 0,$$

which implies that

$$gx \in F(x,\ y) \text{ and } gy \in F(y,\ x),$$

that is, (x, y) is a coupled coincidence point of F and g. Hence $C(F, g)$ is nonempty.

Suppose now that (a) holds. Assume that for some $(x, y) \in C(F, g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v, \tag{2.16}$$

where $u,\ v \in X$. Since g is continuous at u and v. We have, by (2.16), that u and v are fixed points of g, that is,

$$gu = u \text{ and } gv = v. \tag{2.17}$$

As F and g are $w-$compatible, so

$$(g^n x,\ g^n y) \in C(F,\ g),\ \text{for all } n \geq 1,$$

that is,

$$g^n x \in F(g^{n-1}x,\ g^{n-1}y) \text{ and } g^n y \in F(g^{n-1}y,\ g^{n-1}x),\ \text{for all } n \geq 1. \tag{2.18}$$

Now, by using (2.1) and (2.18), we obtain

$$\begin{aligned}
& \varphi\left(D(g^n x,\ F(u,\ v))\right) \\
\leq\ & \varphi\left(H(F(g^{n-1}x,\ g^{n-1}y),\ F(u,\ v))\right) \\
\leq\ & \psi\left(\varphi\left(\max\left\{d(g^n x,\ gu),\ d(g^n y,\ gv)\right\}\right)\right) \\
& \times\varphi\left(\max\left\{d(g^n x,\ gu),\ d(g^n y,\ gv)\right\}\right) \\
\leq\ & \varphi\left(\max\left\{d(g^n x,\ gu),\ d(g^n y,\ gv)\right\}\right).
\end{aligned}$$

Since φ is non-decreasing, we have

$$D(g^n x,\ F(u,\ v)) \leq \max\left\{d(g^n x,\ gu),\ d(g^n y,\ gv)\right\}.$$

On taking limit as $n \to \infty$ in the above inequality, by using (2.16) and (2.17), we get

$$D(gu,\ F(u,\ v)) = 0. \text{ Similarly } D(gv,\ F(v,\ u)) = 0,$$

which implies that

$$gu \in F(u,\ v) \text{ and } gv \in F(v,\ u). \tag{2.19}$$

Now, from (2.17) and (2.19), we have

$$u = gu \in F(u,\ v) \text{ and } v = gv \in F(v,\ u),$$

that is, $(u,\ v)$ is a common coupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x,\ y) \in C(F,\ g)$, g is $F-$weakly commuting, that is $g^2x \in F(gx,\ gy)$, $g^2y \in F(gy,\ gx)$ and $g^2x = gx$, $g^2y = gy$. Thus $gx = g^2x \in F(gx,\ gy)$ and $gy = g^2y \in F(gy,\ gx)$, that is, $(gx,\ gy)$ is a common coupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x, y) \in C(F, g)$ and for some $u, v \in X$,

$$\lim_{n\to\infty} g^n u = x \text{ and } \lim_{n\to\infty} g^n v = y.$$

Since g is continuous at x and y, then x and y are fixed points of g, that is,

$$gx = x \text{ and } gy = y.$$

Since $(x, y) \in C(F, g)$, so we obtain

$$x = gx \in F(x,\ y) \text{ and } y = gy \in F(y,\ x),$$

that is, (x, y) is a common coupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C(F, g)) = \{(x, x)\}$. Then $\{x\} = \{gx\} = F(x, x)$. Hence (x, x) is a common coupled fixed point of F and g. $\square$

Example. Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F : X \times X \to K(X)$ be defined as

$$F(x,\ y) = \begin{cases} \{0\}, \text{ for } x,\ y = 1 \\ \left[0,\ \frac{x^4}{4}\right], \text{ for } x,\ y \in [0,\ 1), \end{cases}$$

and $g : X \to X$ be defined as

$$g(x) = x^2, \text{ for all } x \in X.$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \begin{cases} \ln(t+1), \text{ for } t \neq 1 \\ \frac{3}{4}, \text{ for } t = 1, \end{cases}$$

and $\psi : [0, +\infty) \to [0, 1)$ defined by

$$\psi(t) = \frac{\varphi(t)}{t}, \text{ for all } t \geq 0.$$

Now, for all $x, y, u, v \in X$ with $x, y, u, v \in [0, 1)$, we have

Case (a). If $x = u$, then

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ = & \frac{u^4}{4} \\ \leq & \ln(u^2+1) \\ \leq & \ln(\max\{x^2,\ u^2\}+1) \\ \leq & \ln(d(gx,\ gu)\ +1) \\ \leq & \ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}+1\right), \end{aligned}$$

which implies that

$$\begin{aligned}
& \varphi\left(H(F(x,\ y),\ F(u,\ v))\right) \\
= \ & \ln\left(H(F(x,\ y),\ F(u,\ v)) + 1\right) \\
\leq \ & \ln\left(\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right) + 1\right) \\
\leq \ & \frac{\ln\left(\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right) + 1\right)}{\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right)} \\
& \times \ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right) \\
\leq \ & \psi\left(\varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right)\right) \\
& \times \varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right).
\end{aligned}$$

Case (b). If $x \neq u$ with $x < u$, then

$$\begin{aligned}
& H(F(x,\ y),\ F(u,\ v)) \\
= \ & \frac{u^4}{4} \\
\leq \ & \ln(u^2 + 1) \\
\leq \ & \ln(\max\{x^2,\ u^2\} + 1) \\
\leq \ & \ln(d(gx,\ gu)\ + 1) \\
\leq \ & \ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
& \varphi\left(H(F(x,\ y),\ F(u,\ v))\right) \\
= \ & \ln\left(H(F(x,\ y),\ F(u,\ v)) + 1\right) \\
\leq \ & \ln\left(\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right) + 1\right) \\
\leq \ & \frac{\ln\left(\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right) + 1\right)}{\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right)} \\
& \times \ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\} + 1\right) \\
\leq \ & \psi\left(\varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right)\right) \\
& \times \varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right).
\end{aligned}$$

Similarly, we obtain the same result for $u < x$. Thus the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v \in [0,\ 1)$. Again, for all $x,\ y,\ u,\ v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$, we have

$$H(F(x,\ y),\ F(u,\ v))$$

$$
\begin{aligned}
&= \frac{x^4}{4} \\
&\leq \ln(x^2+1) \\
&\leq \ln(\max\{x^2,\ u^2\}+1) \\
&\leq \ln(d(gx,\ gu)\ +1) \\
&\leq \ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}+1\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \varphi\left(H(F(x,\ y),\ F(u,\ v))\right) \\
= \ & \ln\left(H(F(x,\ y),\ F(u,\ v))+1\right) \\
\leq \ & \ln\left(\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}+1\right)+1\right) \\
\leq \ & \frac{\ln\left(\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}+1\right)+1\right)}{\ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}+1\right)} \\
& \times \ln\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}+1\right) \\
\leq \ & \psi\left(\varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right)\right) \\
& \times \varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right).
\end{aligned}
$$

Thus the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$. Similarly, we can see that the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v = 1$. Hence, the hybrid pair $\{F,\ g\}$ satisfies the contractive condition (2.1), for all $x,\ y,\ u,\ v \in X$. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = (0,\ 0)$ is a common coupled fixed point of hybrid pair $\{F,\ g\}$. The function $F : X \times X \to K(X)$ involved in this example is not continuous at the point $(1,\ 1) \in X \times X$.

Remark 2.2. We improve, extend and generalize the results of Ciric et al. [8] in the sense that

(i) We prove our result for hybrid pair of mappings.

(ii) We prove our result in the framework of non-complete metric space $(X,\ d)$ and the product set $X \times X$ is not empowered with any order.

(iii) We prove our result without the assumption of continuity and mixed g-monotone property for mapping $F : X \times X \to K(X)$.

(iv) The functions $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ and $\psi : [0,\ +\infty) \to [0,\ 1)$ involved in our theorem and example are discontinuous.

If we put $g = I$ (the identity mapping) in Theorem 2.1, we get the following result:

Corollary 2.3. *Let (X, d) be a complete metric space, $F : X \times X \to K(X)$ be a mapping. Assume that there exist some $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} & \varphi (H(F(x,\ y),\ F(u,\ v))) \\ \leq\ & \psi (\varphi (\max \{d(x,\ u),\ d(y,\ v)\}))\, \varphi (\max \{d(x,\ u),\ d(y,\ v)\}), \end{aligned}$$

for all $x, y, u, v \in X$. Then F has a coupled fixed point.

If we put $\psi(t) = 1 - \frac{\widetilde{\psi}(t)}{t}$ for all $t \geq 0$ in Theorem 2.1, then we get the following result:

Corollary 2.4. *Let (X, d) be a metric space, $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Assume that there exist some $\varphi \in \Phi$ and $\widetilde{\psi} \in \Psi$ such that*

$$\begin{aligned} & \varphi (H(F(x,\ y),\ F(u,\ v))) \\ \leq\ & \varphi (\max \{d(gx,\ gu),\ d(gy,\ gv)\}) - \widetilde{\psi} (\varphi (\max \{d(gx,\ gu),\ d(gy,\ gv)\})), \end{aligned}$$

for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x, y) \in C(F, g)$ and gx and gy are fixed points of g, that is, $g^2 x = gx$ and $g^2 y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

If we put $g = I$ (the identity mapping) in the Corollary 2.4, we get the following result:

Corollary 2.5. *Let (X, d) be a complete metric space, $F : X \times X \to K(X)$ be a mapping. Assume that there exist some $\varphi \in \Phi$ and $\widetilde{\psi} \in \Psi$ such that*

$$\begin{aligned} & \varphi (H(F(x,\ y),\ F(u,\ v))) \\ \leq\ & \varphi (\max \{d(x,\ u),\ d(y,\ v)\}) - \widetilde{\psi} (\varphi (\max \{d(x,\ u),\ d(y,\ v)\})), \end{aligned}$$

for all $x, y, u, v \in X$. Then F has a coupled fixed point.

If we put $\varphi(t) = 2t$ for all $t \geq 0$ in Theorem 2.1, then we get the following result:

Corollary 2.6. *Let (X, d) be a metric space, $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Assume that there exists some $\psi \in \Psi$ such that*

$$\begin{aligned} & H(F(x, y), F(u, v)) \\ \leq \ & \psi(2\max\{d(gx, gu), d(gy, gv)\})\max\{d(gx, gu), d(gy, gv)\}, \end{aligned}$$

for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x, y) \in C(F, g)$ and gx and gy are fixed points of g, that is, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

If we put $g = I$ (the identity mapping) in the Corollary 2.6, we get the following result:

Corollary 2.7. *Let (X, d) be a complete metric space, $F : X \times X \to K(X)$ be a mapping. Assume that there exists some $\psi \in \Psi$ such that*

$$H(F(x, y), F(u, v)) \leq \psi(2\max\{d(x, u), d(y, v)\})\max\{d(x, u), d(y, v)\},$$

for all $x, y, u, v \in X$. Then F has a coupled fixed point.

If we put $\psi(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.6, then we get the following result:

Corollary 2.8. *Let (X, d) be a metric space. Assume $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings satisfying*

$$H(F(x, y), F(u, v)) \leq k\max\{d(gx, gu), d(gy, gv)\},$$

for all $x, y, u, v \in X$, where $0 < k < 1$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x, y) \in C(F, g)$ and gx and gy are fixed points of g, that is, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

If we put $g = I$ (the identity mapping) in the Corollary 2.8, we get the following result:

Corollary 2.9. *Let (X, d) be a complete metric space. Assume $F : X \times X \to K(X)$ be a mapping satisfying*

$$H(F(x, y), F(u, v)) \leq k \max \{d(x, u), d(y, v)\},$$

for all $x, y, u, v \in X$, where $0 < k < 1$. Then F has a coupled fixed point.

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