Original Article
International Journal of Fuzzy Logic and Intelligent Systems
Vol. 15, No. 2, June 2015, pp. 132-136
http://dx.doi.org/10.5391/IJFIS.2015.15.2.132

ISSN(Print) 1598-2645
ISSN(Online) 2093-744X

On Common Fixed Point for Single and Set-Valued Maps Satisfying OWC Property in IFMS using Implicit Relation

Jong Seo Park
Department of Mathematics Education, Chinju National University of Education, Jinju, Korea

Abstract

In this paper, we introduce the notion of single and set-valued maps satisfying OWC property in IFMS using implicit relation. Also, we obtain common fixed point theorems for single and set-valued maps satisfying OWC properties in IFMS using implicit relation.

Keywords: Common fixed point, Occasionally weakly compatible map, Implicit relation.

1. Introduction

Several authors [1–5] studied and developed the various concepts in different direction and proved some fixed point in fuzzy metric space. Also, Jungck [6] introduced the concept of compatible maps, and Vijayaraju and Sajath [7] obtained some common fixed point theorems in fuzzy metric space. Recently, Park et.a.. [8] introduced the intuitionistic fuzzy metric space (IFMS), Park [12, 13] studied the compatible and weakly compatible maps in IFMS, and proved common fixed point theorem in IFMS. Also, Park [9] proved some properties for several types compatible maps, and Park [10] defined occasionally weakly semi-compatible map and obtained some fixed point using this maps in IFMS.

In this paper, we introduce the notion of single and set-valued maps satisfying occasionally weakly compatible (OWC) property in IFMS using implicit relation. Also, we obtain common fixed point theorems for single and set-valued maps satisfying OWC property in IFMS using implicit relation.

2. Preliminaries

In this part, we recall some definitions, properties and known results in the IFMS as follows : Let us recall ([11]) that a continuous $t-$norm is an operation $*:[0,1]\times[0,1]\to[0,1]$ which satisfies the following conditions: (a)$*$ is commutative and associative, (b)$*$ is continuous, (c)$a*1=a$ for all $a\in[0,1]$, (d)$a*b\le c*d$ whenever $a\le c$ and $b\le d$ $(a,b,c,d\in[0,1])$. Also, a continuous $t-$conorm is an operation $\diamond:[0,1]\times[0,1]\to[0,1]$ which satisfies the following conditions: (a)$\diamond$ is commutative and associative, (b)$\diamond$ is continuous, (c)$a\diamond 0=a$ for all $a\in[0,1]$, (d)$a\diamond b\ge c\diamond d$ whenever $a\le c$ and $b\le d$ $(a,b,c,d\in[0,1])$.

Definition 2.1. ([8]) The $5-$tuple $(X,M,N,*,\diamond)$ is said to be an intuitionistic fuzzy metric space (IFMS) if X is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and M,N are fuzzy sets on $X^2\times(0,\infty)$ satisfying the following conditions; for

Received: Jan. 27, 2015
Revised : May 25, 2015
Accepted: May 27, 2015

Correspondence to: Jong Seo Park
(parkjs@cue.ac.kr)

all x, y, z in X and all $s, t \in (0, \infty)$,

(a)$M(x, y, t) > 0$,

(b)$M(x, y, t) = 1$ if and only if $x = y$,

(c)$M(x, y, t) = M(y, x, t)$,

(d)$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(e)$M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous,

(f)$N(x, y, t) > 0$,

(g)$N(x, y, t) = 0$ if and only if $x = y$,

(h)$N(x, y, t) = N(y, x, t)$,

(i)$N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,

(j)$N(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

Note that (M, N) is called an IFM on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively

Through out this paper, X will represent the IFMS and $CB(X)$, the set of all non-empty closed and bounded subsets of X. For $A, B \in CB(X)$ and for every $t > 0$, denote

$$
\begin{aligned}
H(A, B, t) &= \sup\{M(a, b, t); a \in A, b \in B\},\\
h(A, B, t) &= \inf\{N(a, b, t); a \in A, b \in B\},\\
\delta_M(A, B, t) &= \inf\{M(a, b, t); a \in A, b \in B\},\\
\delta^N(A, B, t) &= \sup\{N(a, b, t); a \in A, b \in B\}.
\end{aligned}
$$

If A consists of a single point a, we write

$$\delta_M(A, B, t) = \delta_M(a, B, t), \;\; \delta^N(A, B, t) = \delta^N(a, B, t).$$

Furthermore, if B consists of a single point b, we write

$$\delta_M(A, B, t) = M(a, b, t), \;\; \delta^N(A, B, t) = N(a, b, t).$$

It follows immediately from definition that

$$
\begin{aligned}
\delta_M(A, B, t) &= \delta_M(B, A, t) \geq 0,\\
\delta^N(A, B, t) &= \delta^N(B, A, t) \leq 1.
\end{aligned}
$$

Also, $\delta_M(A, B, t) = 1$ and $\delta^N(A, B, t) = 0$ if and only if $A = B = \{a\}$ for al $A, B \in CB(X)$.

Definition 2.2. Let X be an IFMS, $A : X \to X$ and $B : X \to CB(X)$.

(a) A point $x \in X$ is called a coincidence point of hybrid maps A and B if $x = Ax \in Bx$.

(b) Hybrid maps A and B are said to be compatible if $ABx \in CB(X)$ for all $x \in X$ and

$$
\begin{aligned}
\lim_{n\to\infty} H(ABx_n, BAx_n, t) = 1,\\
\lim_{n\to\infty} h(ABx_n, BAx_n, t) = 0
\end{aligned}
$$

whenever $\{x_n\}$ is a sequence in X such that $Bx_n \to D \in CB(X)$ and $Ax_n \to x \in D$.

(c) Hybrid maps A and B are said to be weakly compatible if $ABx = BAx$ whenever $Ax \in Bx$.

(d) Hybrid maps A and B are said to be occasionally weakly compatible (OWC) if there exists some points $x \in X$ such that $Ax \in Bx$ and $ABx \subseteq BAx$.

Example 2.3. Let $X = [0, \infty)$ with $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and for all $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \;\; N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Define the maps $A : X \to X$ and $B : X \to CB(X)$ by

$$
\begin{aligned}
Ax &= \begin{cases} 0 & \text{if } 0 \leq x < 1,\\ x + 1 & \text{if } 1 \leq x < \infty, \end{cases}\\
Bx &= \begin{cases} \{0\} & \text{if } 0 \leq x < 1,\\ [1, x + 3] & \text{if } 1 \leq x < \infty. \end{cases}
\end{aligned}
$$

Here 1 is a coincidence point of A and B, but A and B are not weakly compatible as $BA(1) = [1, 5] \neq AB(1) = [2, 5]$. Also, A and B are OWC hybrid maps as A and B are weakly compatible at $x = 0$ as $A(0) \in B(0)$ and $0 = AB(0) \subseteq BA(0) = \{0\}$. Hence weakly compatible hybrid maps are OWC, but the converse is not true in general.

3. Main Results

Theorem 3.1. Let X be an IFMS with $t * t = t$ and $t \diamond t = t$ for all $t \in [0, 1]$. Also, let $A, B : X \to X$ and $S, T : X \to CB(X)$ be single and set-valued mappings such that the hybrid pairs (A, S) and (B, T) are OWC satisfying

$$
\begin{aligned}
&\phi\{\delta_M(Sx, Ty, t), M(Ax, By, t),\\
&\quad H(Ax, Sx, t), H(By, Ty, t),\\
&\quad H(Ax, Ty, t) * H(By, Sx, t)\} \geq 0 \qquad (1)\\
&\psi\{\delta^N(Sx, Ty, t), N(Ax, By, t),\\
&\quad h(Ax, Sx, t), h(By, Ty, t),\\
&\quad h(Ax, Ty, t) \diamond h(By, Sx, t)\} \leq 1
\end{aligned}
$$

for every $x, y \in X$, $t > 0$.

Also, let implicit relation $\Phi = \{\phi, \psi\}$ such that $\phi : [0,1]^5 \to [0,1]$ and $\psi : [0,1]^5 \to [0,1]$ continuous functions satisfying

(a) $\phi(t_1, t_2, t_3, t_4, t_5)$ is non-increasing in t_2 and t_5 for all $t > 0$. $\psi(t_1, t_2, t_3, t_4, t_5)$ is non-decreasing in t_2 and t_5 for all $t > 0$.

(b) $\phi(t, t, 1, 1, t) \geq 0$ implies that $t = 1$, and $\psi(t, t, 0, 0, t) \leq 1$ implies that $t = 0$ for all $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

Proof Since the hybrid pairs (A, S) and (B, T) are OWC maps, there exist two elements $u, v \in X$ such that $Au \in Su$, $ASu \subseteq SAu$ and $Bv \in Tv$, $BTv \subseteq TBv$.

First, we prove that $Au = Bv$. As $Au \in Su$ and $Bv \in Tv$, so,

$$\begin{aligned}
M(Au, Bv, t) &\geq \delta_M(Su, Tv, t), \\
M(Au, Tv, t) &\geq \delta_M(Su, Tv, t), \\
M(Bv, Su, t) &\geq \delta_M(Su, Tv, t), \\
N(Au, Bv, t) &\leq \delta^N(Su, Tv, t), \\
N(Au, Tv, t) &\leq \delta^N(Su, Tv, t), \\
N(Bv, Su, t) &\leq \delta^N(Su, Tv, t).
\end{aligned}$$

If $Au \neq Bv$, then $\delta_M(Su, Tv, t) < 1$ and $\delta^N(Su, Tv, t) > 0$. Using (1) for $x = u$ and $y = v$, we have

$$\begin{aligned}
&\phi\{\delta_M(Su, Tv, t), M(Au, Bv, t), 1, 1, \\
&\quad M(Au, Tv, t) * M(Su, Bv, t)\} \geq 0 \\
&\psi\{\delta^N(Su, Tv, t), N(Au, Bv, t), 0, 0, \\
&\quad N(Au, Tv, t) \diamond N(Su, Bv, t)\} \leq 1.
\end{aligned}$$

That is,

$$\begin{aligned}
&\phi\{\delta_M(Su, Tv, t), \delta_M(Su, Tv, t), \\
&\quad 1, 1, \delta_M(Su, Tv, t)\} \geq 0 \\
&\psi\{\delta^N(Su, Tv, t), \delta^N(Au, Bv, t), \\
&\quad 0, 0, \delta^N(Au, Tv, t)\} \leq 1.
\end{aligned}$$

Also, ϕ, ψ satisfies (b), so

$$\delta_M(Su, Tv, t) = 1 \text{ and } \delta^N(Su, Tv, t) = 0.$$

This is a contradiction which gives $Au = Bv$

Now, we prove that $A^2u = Au$. Suppose that $A^2u \neq Au$, then $\delta_M(SAu, Tv, t) < 1$ and $\delta^N(SAu, Tvt) > 0$. Also, using (1) for $x = Au$ and $y = v$, we get

$$\begin{aligned}
&\phi\{\delta_M(SAu, Tv, t), M(AAu, Bv, t), 1, 1, \\
&\quad M(AAu, Tv, t) * M(SAu, Bv, t)\} \geq 0 \\
&\psi\{\delta^N(SAu, Tv, t), N(AAu, Bv, t), 0, 0, \\
&\quad N(AAu, Tv, t) \diamond N(SAu, Bv, t)\} \leq 1.
\end{aligned}$$

Also, $Au \in Su$ and $ASu \in SAu$, so $AAu \in ASu \subseteq SAu$, $Bv \in Tv$ and $BTv \subseteq TBv$, hence

$$\begin{aligned}
M(AAu, Bv, t) &\geq \delta_M(SAu, Tv, t), \\
M(Bv, SAu, t) &\geq \delta_M(SAu, Tv, t), \\
N(AAu, Bv, t) &\leq \delta^N(SSA, Tv, t), \\
N(Bv, SAu, t) &\leq \delta^N(SAu, Tv, t).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\phi\{\delta_M(SAu, Tv, t), \delta_M(SAu, Tv, t), \\
&\quad 1, 1, \delta_M(SAu, Tv, t)\} \geq 0 \\
&\psi\{\delta^N(SAu, Tv, t), \delta^N(SAu, Tv, t), \\
&\quad 0, 0, \delta^N(SAu, Tv, t)\} \leq 1.
\end{aligned}$$

But ϕ, ψ satisfies (b), so,

$$\delta_M(SAu, Tv, t) = 1 \text{ and } \delta^N(SAu, Tv, t) = 0,$$

a contradiction and hence $A^2u = Au = Bv$. Similarly, we can show that $B^2v = Bv$.

Let $Au = Bv = z$, then $Az = z = Bz$, $z \in Sz$ and $z \in Tz$. Therefore z is a fixed point of A, B, S and T.

Finally, we prove the uniqueness of the fixed point. Let $z \neq z_0$ be another fixed point of A, B, S and T, then by (1), we have,

$$\begin{aligned}
&\phi\{\delta_M(Sz, Tz_0, t), \delta_M(Az, Tz_0, t), 1, 1, \\
&\quad \delta_M(Az, Tz_0, t) * \delta_M(Sz, Tz_0, t)\} \geq 0 \\
&\psi\{\delta^N(Sz, Tz_0, t), \delta^N(Az, Tz_0, t), 0, 0, \\
&\quad \delta^N(Az, Tz_0, t) \diamond \delta^N(Sz, Tz_0, t)\} \leq 1.
\end{aligned}$$

From (b), we get

$$\delta_M(Sz, Tz_0, t) = 1, \;\; \delta^N(Sz, Tz_0, t) = 0.$$

This is a contradiction. Hence $z = z_0$. Therefore z is unique common fixed point of A, B, S and T.

Example 3.2. Let X be an IFMS in which $X = R^+$, $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ such that for all $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Define the maps A, B, S and T on X by

$$Ax = \begin{cases} 2x - 1 & \text{if } x \leq 5, \\ 2x & \text{if } x > 5, \end{cases}$$

$$Bx = \begin{cases} 3 - 2x & \text{if } x \leq 1, \\ x + 1 & \text{if } 1 > x, \end{cases}$$

$$Sx = \begin{cases} \{1\} & \text{if } x < 2, \\ [2x, 2x + 5] & \text{if } x \geq 2, \end{cases}$$

$$Tx = \begin{cases} \{1\} & \text{if } x = 1, \\ [x, x + 2] & \text{if otherwise.} \end{cases}$$

Define $\phi : [0, 1] \to [0, 1]$, $\psi : [0, 1] \to [0, 1]$ as

$$\phi(t_1, t_2, t_3, t_4, t_5) = \min\{t_1, t_2, t_3, t_4, t_5\},$$
$$\psi(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\}.$$

Here the pairs (A, S) and (B, T) are OWC and the contractive condition is satisfied. Hence 1 is a unique common fixed point of A, B, S and T.

Corollary 3.3. Let X be an IFMS, $t * t = t$ and $t \diamond t = t$ for all $t \in [0, 1]$ and let $A : X \to X$ and $S, T : X \to CB(X)$ be single and set-valued mappings such that the hybrid pair (A, S) and (A, T) are OWC satisfying

$$\begin{aligned} &\phi\{M(Sx, Ty, t), M(Ax, Ay, t), \\ &\qquad H(Ax, Sx, t), H(Ay, Ty, t), \\ &\qquad H(Ax, Sy, t) * H(Ay, Sx, t)\} \geq 0 \\ &\psi\{N(Sx, Ty, t), N(Ax, Ay, t), \\ &\qquad h(Ax, Sx, t), h(Ay, Ty, t), \\ &\qquad h(Ax, Sy, t) \diamond h(Ay, Sx, t)\} \leq 1 \end{aligned}$$

for every $x, y \in X$, $t > 0$ and ϕ, ψ are satisfies (a) and (b), respectively in Theorem 3.1. Then A, S and T have a unique common fixed point in X.

Proof Suppose that $A = B$ in Eq. (1) of Theorem 3.1, then we get this corollary.

Corollary 3.4. Let X be an IFMS, $t * t = t$ and $t \diamond t = t$ for all $t \in [0, 1]$ and let $A : X \to X$ and $S : X \to CB(X)$ be single and set-valued mappings such that the hybrid pair (A, S) is OWC satisfying

$$\begin{aligned} &\phi\{\delta_M(Sx, Sy, t), M(Ax, Ay, t), \\ &\qquad H(Ax, Sx, t), H(Ay, Sy, t), \\ &\qquad H(Ax, Sy, t) * H(Ay, Sx, t)\} \geq 0 \\ &\psi\{\delta^N(Sx, Sy, t), N(Ax, Ay, t), \\ &\qquad h(Ax, Sx, t), h(Ay, Sy, t), \\ &\qquad h(Ax, Sy, t) \diamond h(Ay, Sx, t)\} \leq 1 \end{aligned}$$

for every $x, y \in X$, $t > 0$ and ϕ, ψ are functions satisfying (a) and (b), respectively in Theorem 3.1. Then A and S have a unique common fixed point in X.

Proof Suppose that $A = B$ and $S = T$ in Eq. (1) of Theorem 3.1, then we get this corollary.

4. Conclusion

Park et.al. [8] introduced the IFMS, and proved common fixed point theorem in IFMS. Also, Park [9] proved some properties for several types compatible maps, and Park [10] defined occasionally weakly semi-compatible map and obtained some fixed point using this maps in IFMS.

In this paper, we introduce the notion of single and set-valued maps satisfying OWC property in IFMS using implicit relation. Also, we obtain common fixed point theorems for single and set-valued maps satisfying OWC property in IFMS using implicit relation.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

Acknowledgments

This author is supported by Chinju National University of Education Research Fund in 2014.

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Jong Seo Park received the B.S., M.S. and Ph.D. degrees in mathematics from Dong-A University, Busan, Korea, in 1983, 1985 and 1995, respectively. He is currently Professor in Chinju National University of Education, Jinju, Korea. His research interests include fuzzy mathematics, fuzzy fixed point theory and fuzzy differential equation, etc.

E-mail: parkjs@cue.ac.kr