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Partial Backordering Inventory Model under Purchase Dependence

Changkyu Park*
College of Business Administration, University of Ulsan, Ulsan, Korea

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ABSTRACT

Purchase dependence is a frequent phenomenon in retail shops and is characterized by the purchase of certain items together due to their unknown interior associations. Although this concept has been significantly examined in the marketing field (e.g. market basket analysis), it has largely remained unaddressed in operations management. Since purchase dependence is an important factor in designing inventory replenishment policies, this paper demonstrates the means of applying it to the partial backordering inventory model. Through computational analyses, this paper compares the performance of inventory models that either consider or ignore purchase dependence; the results demonstrate that inventory models that ignore purchase dependence incur more average cost per unit time than the model that considers purchase dependence, and the impact of purchase dependence can increase in significance as the item set becomes more closely correlated with regard to order demand.

Keywords: Inventory, EOQ, Partial Backordering, Correlated Demand, Purchase Dependence

* Corresponding Author, E-mail: ckparkuou@ulsan.ac.kr

1. INTRODUCTION

The classic square-root economic order quantity (EOQ) model was developed by Harris (1913) and this is by far the best known model. Subsequently, hundreds of papers and books have published inventory models under a wide variety of conditions and assumptions. One unrealistic assumption within Harris's model that has been subject to criticism is the notion that stockouts are impermissible. Relaxation of this assumption has led to the development of an inventory model with shortages. When inventory is out of stock, there are two basic cases for stockouts: backorders and lost sales. Backorders represent cases in which unsatisfied customers are willing to wait and therefore their demand will be filled by subsequent stock replenishment. Lost sales represent cases in which impatient customers may buy their desired products from other suppliers, resulting in a loss of the unmet demand.

Research in this area would benefit from the development of models combining these two basic cases of backorders and lost sales. Such a combined case would consider that some unsatisfied customers may be willing to wait and backorder their unmet demands, while others may choose another vendor. This would result in partial backordering.

Montgomery *et al*. (1973) published the first paper developing a model for the basic EOQ with partial backordering as well as a solution procedure. In the subsequent four decades, many authors have developed models that have relaxed additional assumptions of the basic EOQ with partial backordering, including elements such as time- or backlog-dependent backordering probabilities, inventory deterioration, time- or inventory-dependent demand functions, and quantity discounts. Pentico and Drake (2011) conducted an excellent survey of deterministic models for the EOQ with partial backordering.

Although numerous studies have focused on the deterministic EOQ model with partial backordering, the majority have been concerned only with single-item inventory problems. Most assumed no correlation in sales and therefore their models applied independent demands across items. Nevertheless, Zhang *et al*. (2011) considered a two-item inventory system, wherein the demand of a minor item is correlated with that of a major item due to cross-selling. Cross-selling is defined as a situation in which the purchase of one item is related to the purchase of another item. In other words, the sale of a major item may lead to additional demand for minor items. Thus, these minor items can either be sold independently or promoted through joint sales with the major item, and demand for the minor items will decrease when the major item is stocked out. Zhang (2012) expanded this model in order to address the case of multiple minor items.

In this paper, we consider purchase dependence. Purchase dependence concerns the purchase behavior of customers, wherein some items are purchased together due to their unknown interior associations. For example, if a group of certain items has the attribute of purchase dependence, there is a tendency for these items to be ordered together. Under this purchase pattern, if all items are in stock, there is no difficulty in satisfying customer orders. However, if any one item is not in stock, while all other items are in stock, the situation is the same as if all items are out of stock. This situation can result in either total lost sales or partial backorders. In addition, there might be two possibilities of partial backorders: partial backordering with no shipment until all items are available, which is assumed by Zhang *et al*. (2011), and partial backordering with shipment as the items become available, which is assumed by this paper.

Purchase dependence is a frequent phenomenon in retail shops and supermarkets. Bala (2008, 2012) and Bala *et al*. (2010) identified purchase dependence in retail sales. Park and Seo (2013) recognized purchase dependence while analyzing the inventory operations practice of a ship engine and generator spare parts distributor, and developed approximate continuous and periodic review models for the case of total lost sales under purchase dependence. However, this paper develops the EOQ model with partial backordering by taking purchase dependence into consideration.

The phenomenon of purchase dependence with partial backorders can be observed in Internet shopping malls that conduct e-tailing businesses with apparel, books, CDs, toys, and so on. Web customers shop for their desired items, which are placed into a market basket. If customers push the purchase button after shopping, the website immediately shows which items in the market basket are deliverable and which are out-of-stock (if any). When some items are not in stock, impatient customers may cancel the purchase order and go to other shopping malls. However, some patient customers may receive immediately deliverable items first and be willing to wait out-of-stock items for a short time.

Purchase dependence differs from demand dependence caused by cross-selling, as purchase dependence does not discern a major item from a minor item. Additionally, purchase dependence does not limit directional dependence by one purchase on another. Consequently, demand dependence created through cross-selling is a special case of purchase dependence. This fact will be explained in detail in the latter part of Section 6.

The remainder of this paper is organized as follows. The following section discusses related literature. Section 3 summarizes notations used in this paper. Section 4 briefly reviews a single-item EOQ model with partial backordering. Section 5 proposes the two-item EOQ model with partial backordering (when purchase dependence exists) where the order cycles of two items are assumed to be identical. Section 6 expands the two-item EOQ model to a multiple-item EOQ model with partial backordering. Section 7 describes the numerical analysis conducted to illustrate the newly developed model and to examine behaviors of the optimal policy on different backordering rates. The importance of considering purchase dependence is also illustrated by showing its impact on inventory operations costs. Section 8 presents our conclusion.

2. DISCUSSION OF RELATED LITERATURE

Purchase dependence differs from demand dependence. Purchase dependence concerns the purchase behavior of customers, whereas demand dependence concerns the correlation among demands. This section first reviews three types of demand dependence: correlated demand across items, correlated demand in time, and correlated demand across locations. We provide a summary of those studies focused on designing inventory replenishment policies in cases of demand dependence in Table 1. In terms of demand correlation between different inventory items, Liu and Yuan (2000) and Larsen (2009) developed models to compute optimal inventory replenishment policies. They advocated a multi-item inventory system with coordinated replenishments when demands follow a compound-correlated Poisson process (defined as one in which customers arrive after a Poisson process; however item demand is specified by a nonnegative, integer-valued random variable with a given probability distribution). Liu and Yuan (2000) and Larsen (2009) deployed the can-order policy and the Q(s, S) policy, respectively, for joint replenishment problems.

Urban (2000, 2005), Dong and Lee (2003), and Lee and Chew (2005) developed periodic review models for product demands that are auto-correlated (serially-correlated or time-correlated) but independent of each other. With regard to a multi-item inventory system, Lee and Chew (2005) assumed that product demand follows an auto-regressive process, but is independent of other product demands. In order to express auto-correlated demands for a single item, Urban (2000, 2005) utilized auto-regressive demand processes, while Dong and Lee (2003) used the Martingale model of forecasting evolu-

Table 1. Inventory replenishment studies that have considered demand dependence

Type of demand dependence	Research	System	Order type	Inventory Replenishment policy	Demand process
Correlated Demand across Items	Lin and Yuan (2000)	Multi-item inventory system	Joint replenishment	Can-order policy	Compound correlated Poisson process
	Larsen (2009)			Q(s, S) policy	
Correlated Demand in time	Lee and Chew (2005)	Multi-item inventory system	Joint replenishment	Dynamic periodic Review policy	Auto-regressive process
	Urban (2000, 2005)	Single-item inventory system	Single-item replenishment	Periodic review policy	Auto-regressive process
	Dong and Lee (2003)	Serial multiechelon inventory system			Martingale model of forecase evolution
Correlated demand across locations	Corbett and Rajaram (2006)	Single-item inventory system with a multilocation	Single-period replenishment	Multi-location newsboy problem	Correlation coefficient
	Yan *et al*. (2009)	Tow-echelon supply chain (one distributor and multiple retailers)	Single-item replenishment (same period at all retailers)	Periodic review policy	Equicorrelated multivariate Poisson distribution

tion. The latter model offers a powerful descriptive framework incorporating past demands as well as other influential factors in order to characterize forecast processes.

With regard to spatially-correlated demand, Corbett and Rajaram (2006) considered a single-item, single-period inventory system with multiple locations (i.e. a multi-location newsboy problem), while Yan *et al*. (2009) considered a two-echelon supply chain consisting of one distributor and multiple retailers. The inventory replenishment in Yan *et al*. (2009) followed a periodic review policy. In addition, recent literature such as Hsieh and Dye (2010), Lee and Dye (2012), and Pando *et al*. (2012) considered stock-dependent demands in developing an inventory model.

Contrary to demand dependence, scant analysis has been conducted in purchase dependence. Bala (2012) simulated various inventory replenishment policies on synthetic data for a particular purchase pattern. He performed a cost-benefit analysis of all applicable inventory replenishment policies and selected the best one for implementation. Park and Seo (2013) developed approximate continuous and periodic review models concerning purchase dependence, assuming that unmet demand orders are immediately and entirely lost.

3. NOTATION

This section summarizes notations used in this paper.

Parameters

D_i the demand rate of item *i* (units/unit time)

C_{oi} the ordering cost of item *i* for placing and receiving an order ($/order)

C_{hi} the cost of holding one unit of item *i* for one unit time ($/unit time/unit)

C_{bi} the cost of keeping one unit of item *i* backordered for one unit time ($/unit time/unit)

C_{li} the cost of a lost sale of item *i*, including the lost profit and any goodwill loss ($/unit)

β_i the backordering rate (i.e. the fraction of stockouts that will be backordered) of item *i* when all items are out of stock.

Variables

T the order cycle, i.e. the time interval between two replenishments

F_i the fill rate of item *i*, i.e. the percentage of demand that is filled from stock

Q_i the order quantity of item *i*

4. SINGLE-ITEM EOQ MODEL WITH PARTIAL BACKORDERING

In this section, we summarize the single-item EOQ model with partial backordering, which was developed by Pentico and Drake (2009). We employ this particular approach as it generates a set of equations that are simpler to use and have a more understandable form. Supposing the backordering rate (of item 1) is a constant higher than 0 and lower than 1 ($0 < \beta_1 < 1$), then the inventory level of EOQ with partial backordering is shown as Figure 1.

The average cost per unit time comprises the ordering costs, the cost of carrying inventory, the cost of

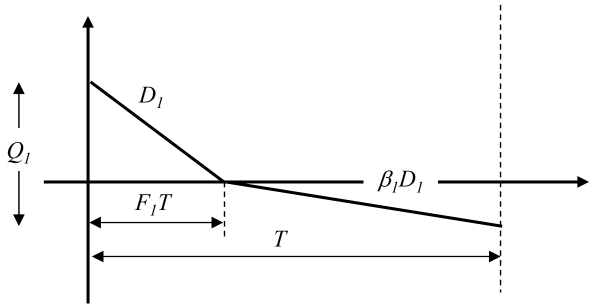

Figure 1. The inventory level of EOQ with partial backordering.

backorders, and the cost of lost sales. Ordering costs are summed up by the number of orders placed. The costs of carrying and backorders are calculated by the triangular areas above and below the time-axis in Figure 1, respectively. The cost of lost sales is calculated through the number of demands that are not backordered. Therefore the average cost per unit time is described as

$$\Gamma(T,F_1)=\frac{C_{o1}}{T}+\frac{C_{h1}D_1TF_1^2}{2}+\frac{\beta_1C_{b1}D_1T(1-F_1)^2}{2}+C_{l1}D_1(1-\beta_1)(1-F_1) \quad (1)$$

Taking the first partial derivatives of $\Gamma(T, F_1)$ with respect to T and F_1 respectively and setting them equal to 0 result in

$$F_1^*=\frac{(1-\beta_1)C_{l1}+\beta_1C_{b1}T^*}{T^*(C_{h1}+\beta_1C_{b1})} \quad (2)$$

and

$$T^*=\sqrt{\frac{2C_{o1}}{D_1C_{h1}}\left[\frac{C_{h1}+\beta_1C_{b1}}{\beta_1C_{b1}}\right]-\frac{\left[(1-\beta_1)C_{l1}\right]^2}{\beta_1C_{h1}C_{b1}}} \quad (3)$$

The order quantity is determined by

$$Q_1=D_1F_1^*T^*+\beta_1D_1(1-F_1^*)T^* \quad (4)$$

In order for the T^* and F_1^* equations to be optimal solutions, the backordering rate should be greater than or equal to the critical value of β^*, as follows

$$\beta_1\ge\beta^*=1-\sqrt{\frac{2C_{o1}C_{h1}}{D_1C_{l1}^2}} \quad (5)$$

If $\beta_1 > \beta^*$, then the values of F_1^* and T^* determined by Eq. (2) and Eq. (3) yield the global minimum of Eq. (1). If $\beta_1 \le \beta^*$, then the optimal strategy is to either lose no sales or not stock and lose all demand-whichever option is less costly.

5. TWO-ITEM PARTIAL BACKORDERING EOQ MODEL UNDER PURCHASE DEPENDENCE

When purchase dependence exists, the purchase of one item can be dependent on the availability of another. Customers can request one or two items in one order. If two items are in stock, any customer orders can be satisfied immediately. When customer requests two items, however, if one item is not in stock, even though the other is in stock, the situation is the same as if all items are out of stock. This can result in either total lost sales or partial backorders.

If some customers are willing to backorder their demand and wait for the next replenishment, the inventory decision can be made by the EOQ model with partial backordering. In this situation, the demand for the item in stock will be influenced by the item that is out of stock; therefore a joint inventory policy should be pursued for the two items in order to maximize the profit from inventory management. When modeling the problem, we make the following assumptions.

(1) All parameters are known and constant over an infinite time horizon.
(2) Replenishment is instantaneous with a zero lead time.
(3) The ordering cost to place and receive an order is constant, independent of the size of the order.
(4) When partially backordering, the item in stock is delivered immediately and the items not in stock are filled by the next replenishment.
(5) A common order cycle is used for all items to facilitate regular communication and easier scheduling of operators.

The inventory levels over the course of an order cycle and their relationships with demand are demonstrated by Figure 2. Note that levels of inventory and backorders for items influence each other when they are out of stock. The order quantities are determined by

$$Q_1=D_1F_1T+\beta_1^{(2)}D_1(F_2-F_1)T+\beta_1D_1(1-F_2)T \quad (6)$$
$$Q_2=D_2F_1T+\alpha_2^{(2)}D_2(F_2-F_1)T+\beta_2D_2(1-F_2)T$$

where $\beta_1^{(2)}$ is the backordering rate of item 1 when item 2 is still in stock, and $\alpha_2^{(2)}$ is the demand change rate of item 2. Note that items in Figure 2 are arranged so that $F_1 \le F_2$.

5.1 Backordering Rate and Demand Change Rate

In Figure 2, when item 1 is out of stock, the demand rate of item 2 is changed to the $\alpha_2^{(2)}$ multiple of the demand rate while item 2 is in stock. In order to calculate the demand change rate $\alpha_2^{(2)}$, we perform a careful examination of the demand for item 2, which reveals that it consists of an independent demand and a joint demand

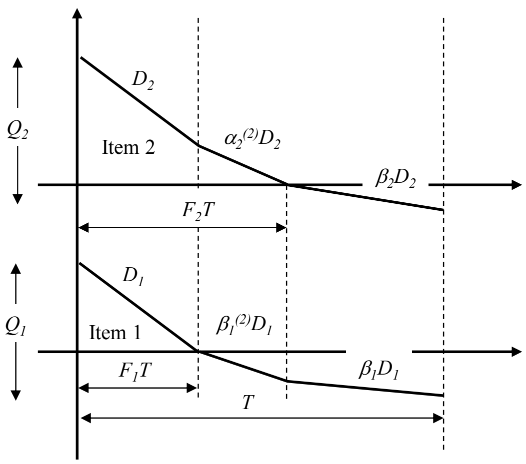

Figure 2. The inventory levels of two items.

with item 1. In other words, customers can order either only item 2, or both items. The independent demand for item 2 is not influenced by the inventory level of item 1.

Table 2 shows twelve possible cases constructed through a combination of four inventory statuses and three order types. Percentage p_i shows the proportion of order case i in the total orders. The b_i is the backordering rate of order case i. Note that in some cases, backorders cannot occur (e.g. b_1, b_2, b_3, b_4, and b_8) (Let $b_i = 1$ when the order case i can be satisfied immediately). The backordering rates β_1, β_2, $\beta_1^{(2)}$ and the demand change rate $\alpha_2^{(2)}$ are then determined by the weighted averages as follows. When $F_1 \le F_2$,

$$\beta_1 = \frac{p_{10}b_{10} + p_{12}b_{12}}{p_{10} + p_{12}} \tag{7}$$
$$\beta_2 = \frac{p_{11}b_{11} + p_{12}b_{12}}{p_{11} + p_{12}}$$
$$\beta_1^{(2)} = \frac{p_7b_7 + p_9b_9}{p_7 + p_9}$$
$$\alpha_2^{(2)} = \frac{p_8b_8 + p_9b_9}{p_8 + p_9} = \frac{p_8 + p_9b_9}{p_8 + p_9}$$

Otherwise, items 1 and 2 are exchanged and then

$$\beta_1 = \frac{p_{11}b_{11} + p_{12}b_{12}}{p_{11} + p_{12}} \tag{8}$$
$$\beta_2 = \frac{p_{10}b_{10} + p_{12}b_{12}}{p_{10} + p_{12}}$$
$$\beta_1^{(2)} = \frac{p_5b_5 + p_6b_6}{p_5 + p_6}$$
$$\alpha_2^{(2)} = \frac{p_4b_4 + p_6b_6}{p_4 + p_6} = \frac{p_4 + p_6b_6}{p_4 + p_6}$$

5.2 The EOQ Model with Partial Backordering ($0 \le F_1 \le F_2 \le 1$)

From Figure 2, the average cost per unit time can be expressed as

$$\begin{aligned}\Gamma(T, F_1, F_2) = {} & \frac{C_{o1}}{T} + \frac{C_{h1}D_1TF_1^2}{2} + \frac{\beta_1^{(2)}C_{b1}D_1T(F_2 - F_1)^2}{2} \\ & + \frac{\beta_1 C_{b1}D_1T(1 - F_2)^2}{2} \\ & + \beta_1^{(2)}C_{b1}D_1T(F_2 - F_1)(1 - F_2) \\ & + C_{l1}D_1\left[(1 - F_1) - \beta_1^{(2)}(F_2 - F_1) - \beta_1(1 - F_2)\right] \\ & + \frac{C_{o2}}{T} + \frac{C_{h2}D_2TF_1^2}{2} + \alpha_2^{(2)}C_{h2}D_2T(F_2 - F_1)F_1 \\ & + \frac{\alpha_2^{(2)}C_{h2}D_2T(F_2 - F_1)^2}{2}\end{aligned} \tag{9}$$

Table 2. Order cases made by inventory status and order type

Order case	Inventory status		Order status		Percentage	Backordering rate
	Item 1	Item 2	Item 1	Item 2		
1	Y	Y	1	0	p_1	$b_1 = 1$
2	Y	Y	0	1	p_2	$b_2 = 1$
3	Y	Y	1	1	p_3	$b_3 = 1$
4	Y	N	1	0	p_4	$b_4 = 1$
5	Y	N	0	1	p_5	b_5
6	Y	N	1	1	p_6	b_6
7	N	Y	1	0	p_7	b_7
8	N	Y	0	1	p_8	$b_8 = 1$
9	N	Y	1	1	p_9	b_9
10	N	N	1	0	p_{10}	b_{10}
11	N	N	0	1	p_{11}	b_{11}
12	N	N	1	1	p_{12}	b_{12}

$$+\frac{\beta_2 C_{b2} D_2 T(1-F_2)^2}{2}$$
$$+C_{l2}D_2\left[(1-F_1)-\alpha_2^{(2)}(F_2-F_1)-\beta_2(1-F_2)\right]$$

In order to simplify this, the average cost per unit time can be expressed as

$$\Gamma(T,F_1,F_2)=\frac{G_{01}}{T}+G_{11}TF_1^2-2G_{12}TF_1-G_{13}F_1 \quad (10)$$
$$+G_{21}TF_2^2-2G_{22}TF_2-G_{23}F_2+G_{02}T+G_{03}$$

where Gs are presented in Appendix 1.

Finally, we can summarize the problem statement as follows.

P1 *minimize* $\Gamma(T, F_1, F_2)$
subject to $T > 0$ and $0 \le F_1 \le F_2 \le 1$

5.2.1 Solution Procedure Using Lagrange Multiplier Method

From the theory of Lagrange multipliers, we know (by converting minimization to maximization) that we can form the Lagrangian

$$L(T, F_1, F_2, \lambda_1, \lambda_2, \lambda_3)=-\frac{G_{01}}{T}-G_{11}TF_1^2+2G_{12}TF_1 \quad (11)$$
$$+G_{13}F_1-G_{21}TF_2^2+2G_{22}TF_2+G_{23}F_2-G_{02}T$$
$$-G_{03}+\lambda_1(-F_1)+\lambda_2(F_2-F_1)+\lambda_3(1-F_2)$$

where λ_i are Lagrange multipliers. Then optimality conditions are given as follows.

$$\frac{G_{01}}{T^2}-G_{11}F_1^2+2G_{12}F_1-G_{21}F_2^2+2G_{22}F_2-G_{02}=0 \quad (12)$$
$$-2G_{11}TF_1+2G_{12}T+G_{13}-\lambda_1-\lambda_2=0$$
$$-2G_{21}TF_2+2G_{22}T+G_{23}+\lambda_2-\lambda_3=0$$
$$\lambda_1(-F_1)=0$$
$$\lambda_2(F_2-F_1)=0$$
$$\lambda_3(1-F_2)=0$$
$$0\le F_1\le F_2\le 1$$
$$\lambda_1,\lambda_2,\lambda_3\ge 0$$

In general, to solve Eq. (12), we begin with complementarity and note that either $\lambda_i = 0$ or corresponding parenthesis is zero. Since there are three complementarity conditions, there are eight cases to check (The eight cases are presented in Appendix 2). Among the eight cases, we come up with one or more feasible solutions. Then, we choose the best one that minimizes the average cost per unit time of Eq. (10).

The solution procedure is not completed yet. In fact, there is another situation that switches the sequence of items. We repeat the above solution procedure for the situation. Then we compare the two best solutions and the global optimal solution is the one that has the lesser average cost per unit time.

6. EXTENDED PARTIAL BACKORDERING EOQ MODEL UNDER PURCHASE DEPENDENCE

This section extends the two-item EOQ model in Section 5 to the multiple-item EOQ model with partial backordering when purchase dependence exists. Although Figure 2 shows inventory levels over the course of an order cycle and their relationships with demand concerning two items, we can easily imagine a similar figure for any k items. Note that $F_1 \le F_2 \le \cdots \le F_i \le \cdots \le F_k$. The order quantity of item j is determined by

$$Q_j=D_jF_1T+\sum_{i=1}^{j-1}\alpha_j^{(k-i+1)}(F_{i+1}-F_i)D_jT$$
$$+\sum_{i=1}^{k-j}\beta_j^{(i+1)}(F_{k-i+1}-F_{k-i})D_jT+\beta_j(1-F_k)D_jT$$

Due to the assumption of the fill rates such that $F_1 \le F_2 \le \cdots \le F_i \le \cdots \le F_k$, there is $k!$ possible sequences of items. In order to find the global optimal solution, we should repeat the solution procedure for all sequences of items. If there are many items, it would be impractical to repeat the solution procedure for all sequences of items. However, a field experience shows that although there would be considerable numbers of groups of items having purchase dependence, most groups contain three or four items. The biggest group contains six items (Park and Seo, 2013).

In order to deal with large-scale problems, Song *et al*. (1999) proposed two approximate methods as follows. (i) One method only focuses on several major demands due to the Pareto phenomenon. Although the total number of potential demand types can be large, by the Pareto phenomenon, often a large portion of the total dollar volume of sales is accounted for by a small number of demand types. (ii) The other method divides the multiple items into several disjoint sets that are either independent or weakly dependent.

The average cost per unit time for any k items can be extended as

$$\Gamma(T, F_1, F_2, \cdots, F_k)=\frac{G_{01}}{T} \quad (13)$$
$$+\sum_{i=1}^{k}(G_{i1}TF_i^2-2G_{i2}TF_i-G_{i3}F_i)+G_{02}T+G_{03}$$

where Gs are presented in Appendix 3.

Thus, the problem statement for any k items can be summarized as

P2 *minimize* $\Gamma(T, F_1, F_2, \cdots, F_k)$
subject to $T > 0$ and $0 \le F_1 \le F_2 \le \cdots \le F_k \le 1$

Similar to P1, we can form the Lagrangian (by converting minimization to maximization)

$$L(T, F_1, F_2, \cdots, F_k, \lambda_1, \lambda_2, \cdots, \lambda_{k+1}) \tag{14}$$
$$= -\frac{G_{01}}{T} - \sum_{i=1}^{k} (G_{i1}TF_i^2 - 2G_{i2}TF_i - G_{i3}F_i)$$
$$-G_{02}T - G_{03} + \lambda_1(-F_1) + \sum_{i=2}^{k} \lambda_i(F_i - F_{i-1}) + \lambda_{k+1}(1 - F_k)$$

where λ_i are Lagrange multipliers. Then optimality conditions are given as follows.

$$\frac{G_{01}}{T^2} - \sum_{i=1}^{k} (G_{i1}F_i^2 - 2G_{i2}F_i) - G_{02} = 0 \tag{15}$$
$$-2G_{11}TF_1 + 2G_{12}T + G_{13} - \lambda_1 - \lambda_2 = 0$$
$$-2G_{i1}TF_i + 2G_{i2}T + G_{i3} + \lambda_i - \lambda_{i+1} = 0, \quad i = 2, \cdots, k$$
$$\lambda_1(-F_1) = 0$$
$$\lambda_i(F_i - F_{i-1}) = 0, \quad i = 2, \cdots, k$$
$$\lambda_{k+1}(1 - F_k) = 0$$
$$0 \le F_1 \le \cdots \le F_k \le 1$$
$$\lambda_1, \cdots, \lambda_{k+1} \ge 0$$

In Eq. (15), there are k+1 complementarity conditions. Thus, there are 2^{k+1} cases to check. For each case, we can easily calculate F_i and λ_i. However, the order cycle T requires some algebraic operations. The value of T can be calculated as follows.

Let J_j be the jth group that has the same value of F_i. For example, case $\lambda_1 = 0$, $\lambda_2 = 0$, $F_2 = F_3$, $\lambda_4 = 0$ (for 3 items) has two groups such as $J_1 = \{F_1\}$ and $J_2 = \{F_2, F_3\}$. For each group J_j, let

$$G'_{il} = \sum_{j \in J_j} G_{jl}, \; i = 1, 2, \cdots, k, \; l = 1, 2, 3$$

Then $T = \sqrt{\frac{A}{B}}$, where

$$A = G_{01} - \sum_{j=1}^{J} \frac{(G'_{j3})^2}{4G'_{j1}} I_j$$
$$B = G_{02} - \sum_{j=1}^{J} \frac{(G'_{j2})^2}{G'_{j1}} I_j + \sum_{j=1}^{J} (G'_{j1} - 2G'_{i2})\delta_j$$
$$I_j = \begin{cases} 0 & \text{if value of } J_j = 0 \text{ or } 1 \\ 1 & \text{otherwise} \end{cases}$$
$$I_j = \begin{cases} 0 & \text{if value of } J_j = 1 \\ 1 & \text{otherwise} \end{cases}$$

Table 3 shows some exemplary cases for 3 items

Table 3. Some exemplary cases for 3 items

Case	Solution
$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$	$F_1 = \frac{2G_{12}T + G_{13}}{2G_{11}T}$, $F_2 = \frac{2G_{22}T + G_{23}}{2G_{21}T}$, $F_3 = \frac{2G_{32}T + G_{33}}{2G_{31}T}$ $T = \sqrt{\frac{A}{B}}$, $A = G_{01} - \frac{G_{13}^2}{4G_{11}} - \frac{G_{23}^2}{4G_{21}} - \frac{G_{33}^2}{4G_{31}}$, $B = G_{02} - \frac{G_{12}^2}{G_{11}} - \frac{G_{22}^2}{G_{21}} - \frac{G_{32}^2}{G_{31}}$
$\lambda_1 = 0, \lambda_2 = 0, F_2 = F_3 = F, \lambda_4 = 0$	$F_1 = \frac{2G_{12}T + G_{13}}{2G_{11}T}$, $F = \frac{2(G_{22} + G_{32})T + (G_{23} + G_{33})}{2(G_{21} + G_{31})T}$, $\lambda_3 = -2G_{21}TF + 2G_{22}T + G_{23}$ $T = \sqrt{\frac{A}{B}}$, $A = G_{01} - \frac{G_{13}^2}{4G_{11}} - \frac{(G_{23} + G_{33})^2}{4(G_{21} + G_{31})}$, $B = G_{02} - \frac{G_{12}^2}{G_{11}} - \frac{(G_{22} + G_{32})^2}{(G_{21} + G_{31})}$
$\lambda_1 = 0, \lambda_2 = 0, F_2 = 1, F_3 = 1$	$F_1 = \frac{2G_{12}T + G_{13}}{2G_{11}T}$, $\lambda_3 = -2G_{21}T + 2G_{22}T + G_{23}$, $\lambda_4 = -2G_{31}T + 2G_{32}T + G_{33} + \lambda_3$ $T = \sqrt{\frac{A}{B}}$, $A = G_{01} - \frac{G_{13}^2}{4G_{11}}$, $B = G_{02} - \frac{G_{12}^2}{G_{11}} + (G_{21} + G_{31}) - 2(G_{22} + G_{32})$
$\lambda_1 = 0, F_1 = F_2 = F, \lambda_3 = 0, F_3 = 1$	$F = \frac{2(G_{12} + G_{22})T + (G_{13} + G_{23})}{2(G_{11} + G_{21})T}$, $\lambda_2 = -2G_{11}TF + 2G_{12}T + G_{13}$, $\lambda_4 = -2G_{31}T + 2G_{32}T + G_{33}$ $T = \sqrt{\frac{A}{B}}$, $A = G_{01} - \frac{(G_{13} + G_{23})^2}{4(G_{11} + G_{21})}$, $B = G_{02} - \frac{(G_{12} + G_{22})^2}{G_{11} + G_{21}} + G_{31} - 2G_{32}$
$F_1 = 0, \lambda_2 = 0, F_2 = F_3 = F, \lambda_4 = 0$	$F = \frac{2(G_{22} + G_{32})T + (G_{23} + G_{33})}{2(G_{21} + G_{31})T}$, $\lambda_1 = 2G_{12}T + G_{13}$, $\lambda_3 = -2G_{21}TF + 2G_{22}T + G_{23}$ $T = \sqrt{\frac{A}{B}}$, $A = G_{01} - \frac{(G_{23} + G_{33})^2}{4(G_{21} + G_{31})}$, $B = G_{02} - \frac{(G_{22} + G_{32})^2}{G_{21} + G_{31}}$

In summary, for any k items, there is $k!$ sequences of items. For each sequence of items, there are 2^{k+1} cases to check. First we calculate the best solutions for each sequence of items. Then we choose the global optimal solution that is the best one among the best solutions.

It is useful to note here that in Section 1, we mentioned that the demand dependence caused by cross-selling is a special case of purchase dependence. In order to explain the cross-selling effect, Zhang *et al*. (2011) consider the example of an electronic supermarket (where major items such as computers, and minor items such as expanded memory, are sold). They assert that it is common for customers buying a computer to simultaneously purchase additional expanded memory in order to improve the computer's performance. To model this problem, they assume that the major item can be partially backordered and the minor item is not stocked out.

Thus, the average cost of the partial backordering problem with the cross-selling effect can be expressed by Eq. (13), in which $F_2 = F_3 = \cdots = F_k = 1$. As a result, the partial backordering problem with a cross-selling effect can be solved by the partial backordering problem with purchase dependence by setting $F_2 = F_3 = \cdots = F_k = 1$. However, the model proposed by this paper cannot be directly compared with those proposed by Zhang *et al*. (2011) and Zhang (2012) because of their different assumptions. This paper assumes that when partially backordered, the item in stock is delivered immediately and the items not in stock are filled by the next replenishment. However Zhang *et al*. (2011) and Zhang (2012) assume that when the backordered demand of a major item is satisfied at the replenishment point, the sale quantity of the minor item caused by the cross-selling effect is sold instantaneously.

It is also useful to note here that the inventory model that considers purchase dependence can be equal to the independent multi-item inventory model by setting $\beta_i^{(\cdot)} = \beta_i$ and $\alpha_i^{(\cdot)} = 1$. These settings remove the influence of other items' stock on the demand.

7. COMPUTATIONAL ANALYSIS

7.1 Numerical Example for Illustration

In order to illustrate the application of the partial backordering EOQ model under purchase dependence, we use a numerical example consisting of two items. The parameters of the numerical example are shown in Table 4 (we utilize items 1 and 2). We assume that $\beta_1 = 0.75$, $\beta_2 = 0.80$ and $\beta_1^{(2)} = 0.85$, $\alpha_2^{(2)} = 0.90$ when $F_1 \le F_2$. Otherwise, $\alpha_1^{(2)} = 0.90$, $\beta_2^{(2)} = 0.85$.

Table 5 summarizes the results of each case for two sequences. The best solutions are case 3 for $F_1 \le F_2$ and case 1 for other sequence. Between them, the case 1 for $F_1 \ge F_2$ costs less than the case 3 for other sequence. Thus, the case 1 for $F_1 \ge F_2$ is the global optimal solution. By Eq. (6), the order quantities are Q_1 = 468.27 ≈ 468 and Q_2 = 72.76 ≈ 73.

Table 4. Parameter values of the numerical example

Parameters	Item		
	1	2	3
D_i	2,000	300	1,000
C_{oi}	$650	$1,000	$600
C_{hi}	$42	$350	$35
C_{bi}	$12	$100	$10
C_{li}	$12	$105	$15

Table 5. Results of the solution procedure for 2 items

		Case						
		1	2	3	4	5	6	7
$F_1 \le F_2$	T	0.28	0.16	**0.28**	0.13	0.28	0.17	0.28
	F_1	0.39	0.54	**0.37**	1.00	0.00	0.00	0.00
	F_2	0.36	1.00	**0.37**	1.00	0.36	1.00	0.00
	λ_1	0.00	0.00	**0.00**	0.00	12,516.36	10,207.22	24,072.85
	λ_2	0.00	0.00	**495.75**	-5,736.99	0.00	0.00	-11,604.61
	λ_3	0.00	-9,714.58	**0.00**	-12,673.99	0.00	-10,465.07	0.00
	Γ	Infeasible	Infeasible	**$19,597.47**	Infeasible	$22,003.44	Infeasible	Infeasible
$F_1 \ge F_2$	T	**0.28**	0.17	0.28	0.13	0.28	0.18	0.28
	F_1	**0.38**	1.00	0.37	1.00	0.38	1.00	0.00
	F_2	**0.37**	0.48	0.37	1.00	0.00	0.00	0.00
	λ_1	**0.00**	0.00	0.00	0.00	14,204.30	11,732.04	24,072.85
	λ_2	**0.00**	0.00	-202.88	-7,859.39	0.00	0.00	-9,800.05
	λ_3	**0.00**	-8,013.94	0.00	-1,273.99	0.00	-8,483.51	0.00
	Γ	$19,596.13	Infeasible	Infeasible	Infeasible	$22,212.26	Infeasible	Infeasible

7.2 Analysis of the Optimal Policy Depending on the Backordering Rate

In order to examine the behavior of the optimal policy with regard to different backordering rates, we investigate the performance of the three-item EOQ model with different backordering rates (see Table 4). By setting β_3 = 0.7, Figure 3 plots the contours of optimal average cost per unit time, the optimal order cycle, and the optimal fill rates against the backorder rate combinations of β_1 and β_2. We assume that the backordering rate and the demand change rate increase 0.05 each segment. In order to figure out a general pattern of curve for the backordering rate, we plot the optimal average cost per unit time, the optimal fill rates, and the optimal order cycle against the backordering rate in Figure 4 for $\beta_1 = \beta_2 = \beta_3$.

Figure 3 and Figure 4 demonstrate that optimal average costs monotonically decrease as the backordering rate increases, which implies that an increase in the number of customers willing to backorder their demand is linked to an ability to realize greater profits. Figure 4 also shows that if the backordering rate is small, the optimal policy is to meet demands without stockouts (i.e. $F^* = 1$). On the other hand, if the backordering rates satisfy the critical condition, the optimal policy is to meet demands through partial backordering.

7.3 Comparison of Approaches for Backordering Rates

There may be several ways of determining the backordering rate b_i of order case i. In this section, we consider three approaches for backordering rates. Approach 1 treats all possible order cases constructed through a combination of inventory statuses and order types. Then the backordering rate b_i of order case i is calculated from the historical data. Table 2 shows an example of two-item case. Approach 2 calculates each average backordering rate for each order type from the historical data. Then, $b_7 = b_{10}$, $b_5 = b_{11}$, and $b_6 = b_9 = b_{12}$ in Table 2. Approach 3 calculates a total average backordering rate $\bar{b}$ for all order cases from the historical data.

In order to compare three approaches, we examine the performance of the three-item EOQ model using different approaches (see Table 4). There are 56 possible order cases constructed through a combination of 8 (= 2^3) inventory statuses and 7 (= ${}_3C_1 + {}_3C_2 + {}_3C_3$) order types. The backordering rate b_i of order case i is assigned by a value between 0.65 and 0.75, resulted in a total average backordering rate of 0.7. Percentage p_i of order case i is assigned according to dissimilarity (Section 7.4 explains it in detail).

Table 6 shows the optimal inventory policies for each approach. Based on the assumption that the approach 1 determines most accurately backordering rates b_i because it considers all possible order cases, Table 6 also shows the increased average costs per unit time of other approaches. We can infer that the cost increment is a price for using the simplified information. Because we think that the cost increment is not high, we will use the approach 3 in Section 7.4.

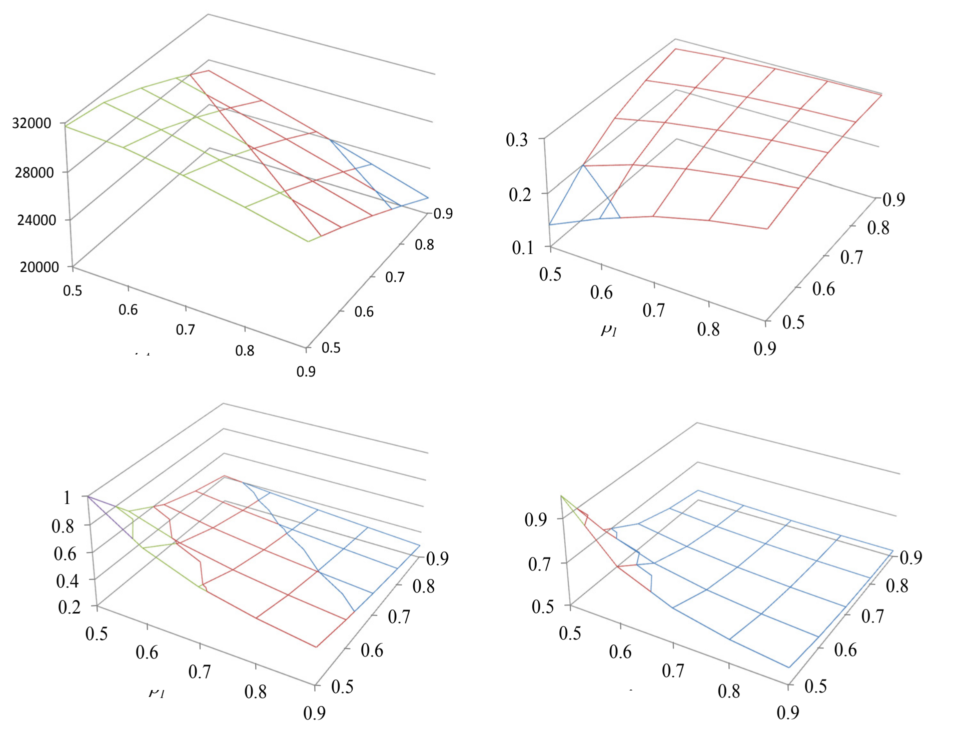

Figure 3. Contours of cost, order cycle, and fill rates.

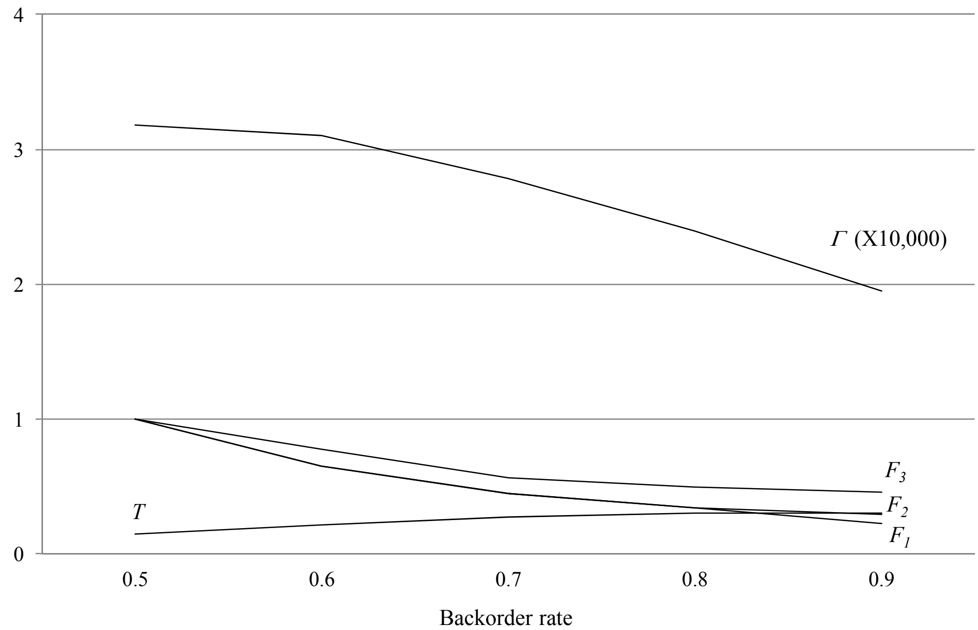

Figure 4. The optimal decisions on different backordering rates.

Table 6. Comparison of three approaches

Approach	Dissimilarity = 0.3					Dissimilarity = 0.5					Dissimilarity = 0.8				
	T	F_1	F_2	F_3	Γ	T	F_1	F_2	F_3	Γ	T	F_1	F_2	F_3	Γ
1	0.26	0.50	0.50	0.50	$29,103.74	0.26	0.49	0.49	0.49	$28,935.54	0.27	0.48	0.48	0.48	$28,570.57
2	0.27	0.45	0.45	0.45	$29,145.78	0.28	0.45	0.45	0.45	$28,967.43	0.28	0.45	0.45	0.45	$28,585.99
3	0.28	0.45	0.45	0.45	$29,151.24	0.28	0.45	0.45	0.45	$28,974.92	0.28	0.44	0.44	0.44	$28,593.24

7.4 Impacts of Purchase Dependence

This section illustrates the significance of considering purchase dependence in inventory management by demonstrating its impact on inventory operations costs. We can explain the impact of purchase dependence by comparing the performance of the inventory model that considers purchase dependence with the inventory model that ignores purchase dependence. Again, we utilize the three-item EOQ model using Table 4. Suppose that customer orders are composed of seven order types as shown in Table 7. For the purpose of simplicity without loss of generality, we assume that the percentage of each order type retains the same value under every inventory status.

Table 7. Order types for three items

Order type	Item 1	Item 2	Item 3	Percentage
1	1			q_1
2		1		q_2
3			1	q_3
4	1	1		q_4
5	1		1	q_5
6		1	1	q_6
7	1	1	1	q_7

For various purchase dependence degrees, by changing the total average backordering rate value from 0.6 to 0.9 with a step of 0.1, we compare the optimal inventory policies determined by the inventory models that consider and ignore purchase dependence. The multiple-item inventory model that ignores purchase dependence can be developed by independently adding multiple single-item EOQ models with partial backordering. The average cost per unit time is described as

$$\Gamma(T, F_1, F_2, \cdots, F_k) = \sum_{i=1}^{k}\left[\frac{C_{oi}}{T} + \frac{C_{hi}D_iTF_i^2}{2} + \frac{\beta_i C_{bi}D_iT(1-F_i)^2}{2} + C_{li}D_i(1-\beta_i)(1-F_i)\right] \quad (16)$$

Taking the first partial derivatives of Eq. (16) with respect to F_i and setting them equal to 0 results in the following

$$F_i^* = \frac{(1-\beta_i)C_{li} + \beta_i C_{bi} T^*}{T^*(C_{hi} + \beta_i C_{bi})} \quad \text{for } i = 1, 2, \cdots, k \qquad (17)$$

The order cycle T^* can be determined by taking the first partial derivatives of Eq. (16) with respect to *T*, replacing F_i by Eq. (17), and setting them equal to 0.

$$T^* = \sqrt{\frac{2\sum_{i=1}^{k} C_{oi} - \sum_{i=1}^{k} \frac{D_i(1-\beta_i)^2 C_{li}^2}{C_{hi} + \beta_i C_{bi}}}{\sum_{i=1}^{k} \frac{D_i \beta_i C_{hi} C_{bi}}{C_{hi} + \beta_i C_{bi}}}} \qquad (18)$$

In order to investigate the impact of purchase dependence on different degrees of purchase dependence, we utilize the concept of dissimilarity proposed by Tsai *et al*. (2009) as a measure of the degree of purchase dependence. Tsai *et al*. (2009) developed the association clustering algorithm, defining the dissimilarity value as (1 - the support value). The support is defined in an association rule that is a type of data mining technique. The support of item set *X* is defined as the percentage of orders in an order database that contains item set *X*, i.e. the support value of item set *X* is equal to |*item set X*| / |*order database*|, where |item *set X*| is the number of the element in item set *X* and |*order database*| is the number of the element in the order database that contains item set *X*. The dissimilarity value ranges from 0 to 1, since the support value is between 1 and 0. A dissimilarity value near 0 indicates that the item set is highly correlated in terms of order demand. However, a dissimilarity value near 1 indicates that the item set has a very low demand relationship.

We consider three cases of dissimilarity. For each case, we assign the value of (1-the dissimilarity value) to q_7 in Table 7, and the remainders are equally assigned to each q_i, in order to avoid any bias. Based on seven order types, as shown in Table 7, and an assumption of the same percentages of order types for each inventory status, the backordering rates and demand change rates are determined as follows (the total average backordering rate $\bar{b}$ by the approach 3 for all order cases is used).

$$\beta_1^{(3)} = \beta_1^{(2)} = \beta_1 = \bar{b}$$

$$\beta_2^{(2)} = \beta_2 = \bar{b}$$

$$\alpha_2^{(3)} = \frac{q_2 + q_4\bar{b} + q_6 + q_7\bar{b}}{q_2 + q_4 + q_6 + q_7}$$

$$\beta_3 = \bar{b}$$

$$\alpha_3^{(2)} = \frac{q_3 + q_5\bar{b} + q_6\bar{b} + q_7\bar{b}}{q_3 + q_5 + q_6 + q_7}$$

$$\alpha_3^{(3)} = \frac{q_3 + q_5\bar{b} + q_6 + q_7\bar{b}}{q_3 + q_5 + q_6 + q_7}$$

Table 8 shows the optimal inventory policies determined by the inventory models that consider and ignore purchase dependence. Based on the assumption that purchase dependence exists, Table 8 also shows the increased average costs per unit time of the inventory model that ignores purchase dependence.

For the three dissimilarities, we plot the cost increment against the total average backordering rate in Figure 5. From Figure 5, we can notice that the cost increment increases monotonically as dissimilarity decreases. This notice can deduce the following interpretation. A decrease in dissimilarity means that the item set

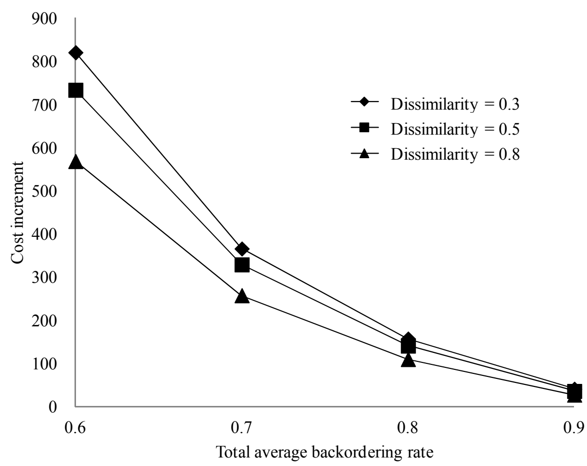

Figure 5. Comparison of the cost increments.

Table 8. Comparison of the optimal inventory policies

	Total average	Dissimilarity = 0.3					Dissimilarity = 0.5					Dissimilarity = 0.8				
	backordering rate	T	F_1	F_2	F_3	Γ	T	F_1	F_2	F_3	Γ	T	F_1	F_2	F_3	Γ
Case of considering purchanse dependence	0.6	0.21	0.65	0.65	0.65	$30,997.59	0.21	0.64	0.64	0.64	$30,941.21	0.22	0.63	0.63	0.63	$30,854.38
	0.7	0.28	0.45	0.45	0.45	$27,890.41	0.28	0.45	0.45	0.45	$27,805.59	0.28	0.44	0.44	0.44	$27,677.64
	0.8	0.30	0.36	0.36	0.36	$23,996.19	0.30	0.35	0.35	0.35	$23,898.76	0.30	0.35	0.35	0.35	$23,752.17
	0.9	0.31	0.29	0.29	0.29	$19,652.35	0.31	0.28	0.28	0.28	$19,545.72	0.31	0.28	0.28	0.28	$19,385.39
Case of ignoring purchanse dependence	0.6	0.21	0.62	0.64	0.85	$31,817.88	0.21	0.61	0.63	0.84	$31,674.44	0.21	0.60	0.62	0.82	$31,422.60
	0.7	0.27	0.43	0.44	0.56	$28,255.91	0.27	0.42	0.44	0.55	$28,134.27	0.28	0.42	0.43	0.55	$27,934.89
	0.8	0.30	0.34	0.35	0.42	$24,153.13	0.30	0.34	0.35	0.41	$24,039.29	0.30	0.34	0.34	0.41	$23,861.60
	0.9	0.31	0.28	0.28	0.31	$19,692.94	0.31	0.28	0.28	0.34	$19,581.28	0.31	0.27	0.28	0.31	$19,412.18

has a higher correlation in terms of order demand, which implies a higher degree of purchase dependence. Thus, the impact of purchase dependence can gain greater significance as the item set becomes more closely correlated in terms of order demand. Consequently, we can deduce that the consideration of purchase dependence in inventory management is important.

8. CONCLUDING REMARKS

This paper considered purchase dependence, a phenomenon that occurs when the purchase of one item is dependent on the availability of other items demanded in the same order. Although purchase dependence is an important factor in designing inventory replenishment policies, it has remained largely unaddressed. Park and Seo (2013) proposed the first approximate continuous and periodic review models that consider purchase dependence when unmet demand orders are lost entirely. However, this paper developed the EOQ model to address situations in which unmet demand orders are partially lost and partially backordered, when purchase dependence exists.

The computational analyses compared the performance of the inventory models that consider and ignore purchase dependence by changing the backordering rate and the degree of purchase dependence. The results demonstrated that the inventory model that ignores purchase dependence incurs more average cost per unit time than the inventory model that considers purchase dependence; in addition, the impact of purchase dependence can become more significant as the correlation of the item set increases with regard to order demand. As a result, this paper argued for the consideration of purchase dependence in inventory management.

The EOQ model proposed in this paper assumes that a common order cycle is used for all items. However, as asserted by Zhang *et al*. (2011), the assumption of identical order cycles may be too stringent in practice. From a practical point of view, for example, expensive items (e.g. A class items) are often more important to inventory management and may be replenished more frequently to maintain a lower inventory holding cost. Thus, future studies are required that extend the proposed EOQ model to include various order cycles for multiple items.

It is also worthwhile to mention the loss of goodwill when purchase dependence exists. Since the purchase of an item can be influenced by the availability of other items, it would be more reasonable to assume that goodwill loss costs would depend on the order that is backordered or left unsold due to the unavailability of a particular item, rather than simply the sum of the goodwill loss of its component items. Future studies are required in order to estimate individual items' goodwill loss costs and the differences in goodwill loss costs that depend on whether the item is ordered individually or as part of a package of items.

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Appendix 1

*G*s in Eq. (10) are as follows.

$$G_{01} = C_{o1} + C_{o2}$$

$$G_{11} = \frac{C_{h1}D_1 + C_{h2}D_2 + \beta_1^{(2)}C_{b1}D_1 - \alpha_2^{(2)}C_{h2}D_2}{2}$$

$$G_{12} = \frac{\beta_1^{(2)}C_{b1}D_1}{2}$$

$$G_{13} = C_{l1}D_1(1-\beta_1^{(2)}) + C_{l2}D_2(1-\alpha_2^{(2)})$$

$$G_{21} = \frac{(\beta_1 - \beta_1^{(2)})C_{b1}D_1 + \beta_2 C_{b2}D_2 + \alpha_2^{(2)}C_{h2}D_2}{2}$$

$$G_{22} = \frac{(\beta_1 - \beta_1^{(2)})C_{b1}D_1 + \beta_2 C_{b2}D_2}{2}$$

$$G_{23} = C_{l1}D_1(\beta_1^{(2)} - \beta_1) + C_{l2}D_2(\alpha_2^{(2)} - \beta_2)$$

$$G_{02} = \frac{\beta_1 C_{b1}D_1 + \beta_2 C_{b2}D_2}{2}$$

$$G_{03} = C_{l1}D_1(1-\beta_1) + C_{l2}D_2)(1-\beta_2)$$

Appendix 2

In order to solve Eq. (12), there are 8 cases to check:

Case 1: $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ gives

$$T = \sqrt{\frac{A}{B}} \quad A = G_{01} - \frac{G_{13}^2}{4G_{11}} - \frac{G_{23}^2}{4G_{21}}, \quad B = G_{02} - \frac{G_{12}^2}{G_{11}} - \frac{G_{22}^2}{G_{21}}$$

$$F_1 = \frac{2G_{12}T + G_{13}}{2G_{11}T}, \quad F_2 = \frac{2G_{22}T + G_{23}}{2G_{21}T}$$

Case 2: $\lambda_1 = 0, \lambda_2 = 0, F_2 = 1$ gives

$$T = \sqrt{\frac{A}{B}}, \quad A = G_{01} - \frac{G_{13}^2}{4G_{11}}, \quad B = G_{02} - \frac{G_{12}^2}{G_{11}} + G_{21} - 2G_{22}$$

$$F_1 = \frac{2G_{12}T + G_{13}}{2G_{11}T}, \quad \lambda_3 = -2G_{21}T + 2G_{22}T + G_{23} = 0$$

Case 3: $\lambda_1 = 0,\ F_1 = F_2 = F,\ \lambda_3 = 0$ gives

$$T = \sqrt{\frac{A}{B}}, \quad A = G_{01} - \frac{(G_{13}+G_{23})^2}{4(G_{11}+G_{21})}, \quad B = G_{02} - \frac{(G_{12}+G_{22})^2}{G_{11}+G_{21}}$$

$$F = \frac{2(G_{12}+G_{22})T + (G_{13}+G_{23})}{2(G_{11}+G_{21})T},$$

$$\lambda_2 = -2G_{11}TF + 2G_{12}T + G_{13}$$

Case 4: $\lambda_1 = 0,\ F_1 = F_2,\ F_2 = 1$ gives

$$T = \sqrt{\frac{A}{B}}, A = G_{01}, \quad B = G_{02} + (G_{11}+G_{21}) - 2(G_{12}+G_{22})$$

$$\lambda_2 = -2G_{11}TF + 2G_{12}T + G_{13},$$

$$\lambda_3 = -2G_{21}T + 2G_{22}T + G_{23} + \lambda_2$$

Case 5: $F_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0$ gives

$$T = \sqrt{\frac{A}{B}}, \quad A = G_{01} - \frac{G_{23}^2}{4G_{21}}, \quad B = G_{02} - \frac{G_{22}^2}{G_{21}}$$

$$F_2 = \frac{2G_{22}T + G_{23}}{2G_{21}T}, \quad \lambda_1 = 2G_{12}T + G_{13}$$

Case 6: $F_1 = 0,\ \lambda_2 = 0,\ F_2 = 1$ gives

$$T = \sqrt{\frac{A}{B}}, \quad A = G_{01}, \quad B = G_{02} + G_{21} - 2G_{22}$$

$$\lambda_1 = 2G_{12}T + G_{13}, \quad \lambda_3 = -2G_{21}T + 2G_{22}T + G_{23}$$

Case 7: $F_1 = 0, F_1 = F_2, \lambda_3 = 0$ gives

$$T = \sqrt{\frac{A}{B}},\ A = G_{01},\ B = G_{02}$$

$$\lambda_1 = 2G_{12}T + G_{13} - \lambda_2,\ \lambda_2 = -2G_{22}T - G_{23}$$

Case 8: $F_1 = 0, F_1 = F_2, F_3 = 1$ is infeasible.

Appendix 3

Gs in Eq. (13) are as follows.

$$G_{01} = \sum_{i=1}^{k} C_{oi}$$

$$G_{11} = \frac{1}{2}\left(C_{h1}D_1 + \sum_{i=2}^{k} (1-\alpha_i^{(k)})C_{hi}D_i + \beta_1^{(k)}C_{b1}D_1 \right)$$

$$G_{j1} = \frac{1}{2}\left(\begin{array}{l} \sum_{i=1}^{j-1} (\beta_i^{(k-j+1)} - \beta_i^{(k-j+2)})C_{bi}D_i + \beta_j^{(k-j+1)}C_{bj}D_j \\ +\alpha_j^{(k-j+2)}C_{hj}D_j + \sum_{i=j+1}^{k} (\alpha_i^{(k-j+2)} - \alpha_i^{(k-j+1)})C_{hi}D_i \end{array} \right),$$
$$2 \le j < k$$

$$G_{k1} = \frac{1}{2}\left(\sum_{i=1}^{k-1} (\beta_i - \beta_i^{(2)})C_{bi}D_i + \beta_k C_{bk}D_k + \alpha_k^{(2)}C_{hk}D_k \right)$$

$$G_{12} = \frac{1}{2}\left(\beta_1^{(k)}C_{b1}D_1 \right)$$

$$G_{j2} = \frac{1}{2}\left(\sum_{i=1}^{j-1} (\beta_i^{(k-j+1)} - \beta_i^{(k-j+2)})C_{bi}D_i + \beta_j^{(k-j+1)}C_{bj}D_j \right),$$
$$2 \le j < k$$

$$G_{k2} = \frac{1}{2}\left(\sum_{i=1}^{k-1} (\beta_i - \beta_i^{(2)})C_{bi}D_i + \beta_k C_{bk}D_k \right)$$

$$G_{13} = (1-\beta_1^{(k)})C_{l1}D_1 + \sum_{i=2}^{k} (1-\alpha_i^{(k)})C_{li}D_i$$

$$G_{j3} = \left(\begin{array}{l} \sum_{i=1}^{j-1} (\beta_i^{(k-j+2)} - \beta_i^{(k-j+1)})C_{li}D_i + (\alpha_j^{(k-j+2)} - \beta_j^{(k-j+1)})C_{lj}D_j \\ + \sum_{i=j+1}^{k} (\alpha_i^{(k-j+2)} - \alpha_i^{(k-j+1)})C_{li}D_i \end{array} \right),$$
$$2 \le j < k$$

$$G_{k3} = \sum_{i=1}^{k-1} (\beta_i^{(2)} - \beta_i)C_{li}D_i + (\alpha_k^{(2)} - \beta_k)C_{lk}D_k$$

$$G_{02} = \frac{1}{2}\left(\sum_{i=1}^{k} \beta_i C_{bi}D_i \right)$$

$$G_{03} = \sum_{i=1}^{k} C_{li}D_i(1-\beta_i)$$