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DYNAMICS OF A DISCRETE RATIO-DEPENDENT PREDATOR-PREY SYSTEM INCORPORATING HARVESTING

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ABSTRACT. In this paper, we consider a discrete ratio-dependent predator-prey system with harvesting effect. In order to investigate dynamical behaviors of this system, first we find out all fixed points of the system and then classify their stabilities by using their Jacobian matrices and local stability method. Next, we display some numerical examples to substantiate theoretical results and finally, we show numerically, by means of a bifurcation diagram, that various dynamical behaviors including cycles, periodic doubling bifurcation and chaotic bands can be existed.

1. Introduction

In population dynamics, a number of continuous-time systems of predator-prey type have been investigated by many researchers for several decades (see [2, 6, 8, 9, 10, 11, 13, 15, 16, 17, 18, 19]). However, sometimes it is useful to consider discrete-time systems described by difference equations, discrete dynamical systems or iterative maps. For instance, many species of insect have no overlap between successive generations and thus their population evolves in discrete-time steps ([15, 20]). Such a population dynamics can be described by a sequence $\{x_n\}$, for example, the well-known logistic difference equation is modeled as

$$x_{n+1} = rx_n(1 - x_n), \tag{1}$$

where x_n denotes the population of a single species in the n-th generation and r is the intrinsic growth rate.

On the other hand, in recent years, many researchers have studied dynamical behaviors of predator-prey systems with nonzero harvesting rates of either species or both species simultaneously(see [3, 7, 12, 15, 21, 22, 23]). In general, human needs are not immutable for a long time. Actually, human needs could be changed according to the amount of biological resources. It is an undoubted fact that nonconstant harvesting method considering prey or predator populations is more efficient and economic than

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constant harvesting method. For the reason, it is needed to take into account harvesting rate according to the species population. Thus, based on the above discussion, in the paper, we consider the following ratio dependent predator-prey system with linear harvesting rates ([3, 21]).

$$\begin{cases} \dfrac{dx}{dt} = rx(1-x) - \dfrac{axy}{x+y} - hx, \\ \dfrac{dx}{dt} = -dy + \dfrac{bxy}{x+y} - ky, \end{cases} \tag{2}$$

where $x(t), y(t)$ denote prey and predator densities respectively, a, b and d stand for the capturing rate, the conversion efficiency of prey into predators and the predator's death rate, respectively and h, k are nonnegative harvesting constants. System (2) has been studied by the authors in [21] for the stability of non-hyperbolic equilibria and for bifurcation phenomena including Hopf bifurcation. In [3], the author has investigated the spatiotemporal complexity of system (2) with respective to h.

Another possible way to understand the complex dynamics of predator-prey system is using discrete systems [1, 4, 5, 6, 14, 20, 24]. In the present work, we take into account a discrete ratio-dependent prey-predator system with linear harvesting rate as follows:

$$\begin{cases} x_{n+1} = rx_n(1-x_n) - \dfrac{ax_ny_n}{x_n+y_n} - hx_n, \\ y_{n+1} = -dy_n + \dfrac{bx_ny_n}{x_n+y_n} - ky_n. \end{cases} \tag{3}$$

However, since $-dy_n - ky_n = -(d+k)y_n$, the dynamics of system (3) are the same as the following system:

$$T : \begin{cases} x_{n+1} = rx_n(1-x_n) - \dfrac{ax_ny_n}{x_n+y_n} - hx_n, \\ y_{n+1} = -dy_n + \dfrac{bx_ny_n}{x_n+y_n}. \end{cases} \tag{4}$$

With an initial condition (x_0, y_0), the iteration of system (4) uniquely determines a trajectory of the states of population output in the following form

$$(x_n, y_n) = T^n(x_0, y_0), \tag{5}$$

where $n = 0, 1, 2, \cdots$.

The main purpose of this paper is to investigate the dynamical behaviors of system (4) near fixed points of system (4) and to illustrate some numerical simulation to substantiate theoretical results.

2. Stability of fixed points of system (4)

From biological points of view, we will focus on the dynamical behaviors of system (4) in the closed first quadrant $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$. First, we determine the existence of fixed points of system (4) by considering the following nonlinear system;

$$\begin{cases} x = rx(1-x) - \dfrac{axy}{x+y} - hx, \\ y = -dy + \dfrac{bxy}{x+y}. \end{cases} \tag{6}$$

Then it follows from simple algebraic calculation that there exist at most three fixed points of system (4) as follows:

(i) $E_0(0,0)$ is the origin,

(ii) $E_1(\frac{r-h-1}{r}, 0)$ is the axial fixed point in the absence of predator ($y = 0$) and

(iii) $E_2(x^*, y^*)$ is the positive fixed point of system (4), where

$$x^* = \frac{a-b-ab+ad-bh+br}{br}, y^* = \frac{b-d-1}{1+d}x^*. \tag{7}$$

From now no, for the existence of the positive fixed point E_2, we assume that the following conditions

$$\begin{aligned} a(d+1) + br &> b(a+h+1) \\ b &> d+1 \end{aligned} \tag{8}$$

hold.

In order to investigate the local dynamical behaviors of system (4) around each of the above fixed points, The Jacobian matrix of system (4) at the given state variable must be considered, which is given by

$$J(x,y) = \begin{pmatrix} r - 2rx - \dfrac{ay^2}{(x+y)^2} - h & -\dfrac{ax^2}{(x+y)^2} \\ \dfrac{by^2}{(x+y)^2} & -d + \dfrac{bx^2}{(x+y)^2} \end{pmatrix}. \tag{9}$$

The characteristic equation of the Jacobian matrix $J(x,y)$ can be obtained as

$$\lambda^2 - B(x,y)\lambda + C(x,y) = 0, \tag{10}$$

where $B(x,y)$ is the trace and $C(x,y)$ is the determinant of the Jacobian matrix $J(x,y)$ which is defined as

$$\begin{aligned} B(x,y) &= r - h - d - 2rx - \frac{ay^2}{(x+y)^2} + \frac{bx^2}{(x+y)^2}, \\ C(x,y) &= \frac{bx^2(r-h-2rx) + d(ay^2 + h(x+y)^2 + r(2x-1)(x+y)^2)}{(x+y)^2}. \end{aligned} \tag{11}$$

In order to study the stability of the fixed points of system (4), we give the following lemma, which can be easily proved by the relations between roots and coefficients of the quadratic equation[1, 15].

Lemma 2.1. *Let $F(\lambda) = \lambda^2 - B\lambda + C$. Suppose that $F(1) > 0$, λ_1 and λ_2 are the two roos of $F(\lambda) = 0$. Then*

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;

(iv) $\lambda_1 = -1$ *and* $\lambda_2 \neq 1$ *if and only if* $F(-1) = 0$ *and* $B \neq 0, 2$*;*

(v) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $B^2 - 4C < 0$ *and* $C = 1$.

Let λ_1 and λ_2 be the two roots of equation (10), which are called eigenvalues of the fixed point (x, y). We recall some definitions of topological types for a fixed point (x, y). A fixed point (x, y) is called a *sink* if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink is locally asymptotically stable. (x, y) is called a *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source is locally unstable. (x, y) is called a *saddle* if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). And (x, y) is called *non-hyperbolic* if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Theorem 2.2. *For the fixed point* E_0*, we have the following topological types:*

(i) E_0 *is a sink if* $|d| < 1$ *and* $|r - h| < 1$*;*

(ii) E_0 *is a source if* $|d| > 1$ *and* $|r - h| > 1$*;*

(iii) E_0 *is a saddle if either* $|d| < 1$ *and* $|r - h| > 1$ *or* $|d| > 1$ *and* $|r - h| < 1$*;*

(iv) E_0 *is non-hyperbolic if* $d = 1$ *or* $0 < r = h \pm 1$.

Proof. It is easy to get that the Jacobian matrix J at E_0 is given by

$$J(E_0) = \begin{pmatrix} r - h & 0 \\ 0 & -d \end{pmatrix}. \tag{12}$$

Hence the eigenvalues of the matrix are $\lambda_1 = r - h$ and $\lambda_2 = d$. Therefore, it is clear that (i)-(iv) hold. □

Theorem 2.3. *If* $r > h + 1$ *holds, then there exists the fixed point* $E_1(\frac{r-h-1}{r}, 0)$*. Moreover,*

(i) E_1 *is a sink if* $|b - d| < 1$ *and* $0 < r < h + 3$*;*

(ii) E_1 *is a source if* $|b - d| > 1$ *and* $r > h + 3$*;*

(iii) E_1 *is non-hyperbolic if* $|b - d| = 1$ *or* $r = h + 3$*;*

(iv) E_1 *is a saddle for the other values of parameters except those values in (i)-(iii).*

Proof. The Jacobian matrix J of E_1 can be obtained as

$$J(E_1) = \begin{pmatrix} h - r + 2 & -a \\ 0 & b - d \end{pmatrix}. \tag{13}$$

Thus elementary calculation yields that the eigenvalues of the matrix are $\lambda_1 = h - r + 2$ and $\lambda_2 = b - d$. Since $r > h + 1$, $\lambda_1 < 1$. Thus one can have the results (i)-(iv). □

Now, we shall discuss the stability of the interior fixed point $E_2(x^*, y^*)$. To do this we must think over the Jacobian matrix $J(x^*, y^*)$ given by

$$J(x^*, y^*) = \begin{pmatrix} r - 2rx^* - \dfrac{a(y^*)^2}{(x + y^*)^2} - h & -\dfrac{a(x^*)^2}{(x^* + y^*)^2} \\ \dfrac{b(y^*)^2}{(x^* + y^*)^2} & -d + \dfrac{b(x^*)^2}{(x + y^*)^2} \end{pmatrix}. \tag{14}$$

As one sees, it is difficult to get eigenvalues of the matrix $J(x^*, y^*)$ directly to analyze the stability of system (4) at the fixed point $E_2(x^*, y^*)$. Thus we will use the Lemma (2.1) to obtain the following proposition.

Theorem 2.4. *Under the condition* (8)*, there exists the fixed point* E_2 *and*
(i) E_2 *is a sink if* $\Delta_1 < h < \Delta_2$*;*
(ii) E_2 *is a source if* $\max\{\Delta_1, \Delta_2\} < h$*;*
(iii) E_2 *is a saddle if* $h < \Delta_1$*;*
(iv) E_2 *is non-hyperbolic if* $h = \Delta_1$ *and* $B \neq 0, 2$*,*
where

$$\begin{aligned} \Delta_1 &= r - 3 + \frac{a(b^2(-1+d) - 2b(1+d)^2 + (1+d)^2(3+d))}{b((1+d)^2 - b(d-1))}, \\ \Delta_2 &= r - 2 + \frac{a(b^2 d - 2b(1+d)^2 + (1+d)^2(2+d))}{b(d^2 + (2-b)d + 1)}. \end{aligned} \tag{15}$$

Proof. First we calculate the parameters $B(x^*, y^*)$ and $C(x^*, y^*)$ of $F(\lambda) = \lambda^2 - B\lambda + C$ in Lemma 2.1 as follows:

$$\begin{aligned} B(x^*, y^*) &= a + h - r + 2 - d + \frac{(b-a)(d+1)^2}{b^2} \\ C(x^*, y^*) &= d(r - a - h - 2) + \frac{(a(2b - d - 2) + b(h - r + 2))(d+1)^2}{b^2}. \end{aligned} \tag{16}$$

From this, we can obtain $F(1) = \dfrac{(d - b + 1)(d + 1)(a(b - d - 1) + b(h - r + 1))}{b^2}$. It follows from equation (8) that $F(1) > 0$, which satisfies the assumption of Lemma 2.1. Also, from (16) we can have that $F(-1) =$

$$\frac{a(b^2(-d+1) + 2b(d+1)^2 - (d+1)^2(d+3)) + b(b(d-1) - (d+1)^2)(r - h - 3)}{b^2}.$$

Note that $F(-1) > 0$ and $C < 1$ hold if and only if $\Delta_1 < h$ and $h < \Delta_2$ are satisfied, respectively. Thus from Lemma 2.1 we can get the results of this theorem. $\square$

3. Numerical Examples

In this section, to provide some numerical evidence for analytic results, especially Theorem 2.4, obtained in section 2, the phase portraits and bifurcation diagrams will be demonstrated.

In order to implement numerical examples, we regard the harvesting rate h of the prey as a variable. Let the other parameters be as follows:

$$a = 3.6, b = 1.5, d = 0.2 \text{ and } r = 3.2. \tag{17}$$

If $h = 0.9$ is taken, then one can get the positive fixed point $E_2(x^*, y^*) = (0.1813, 0.0453)$. Elementary calculation yields that the values $\Delta_1 = -1.1745$ and $\Delta_2 = 1.0379$ in Theorem 2.4 are obtained. Thus it follows from Theorem 2.4 that the fixed point E_2 is stable. In fact, the eigenvalues of the matrix $J(x^*, y^*)$ are $\lambda_{1,2} = 0.8780 \pm$

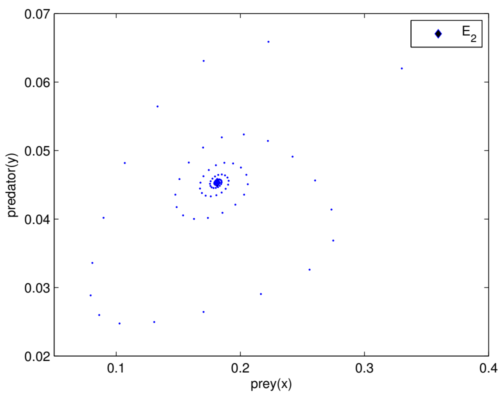

FIGURE 1. The fixed point $E_2(0.1813, 0.0453)$ of system (4) is stable when $a = 3.6, b = 1.5, d = 0.2, r = 3.2$ and $h = 0.9$.

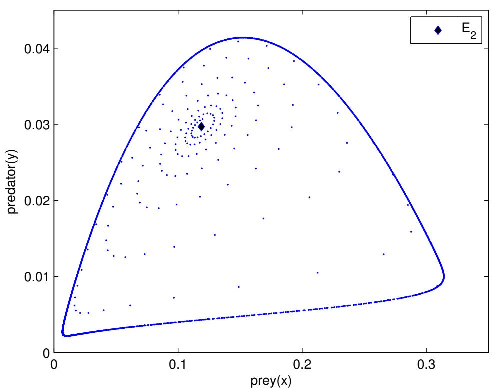

FIGURE 2. A limit cycle could be observed numerically in system (4) when $a = 3.6, b = 1.5, d = 0.2, r = 3.2$ and $h = 1.1$ and $E_2 = (0.1188, 0.0297)$.

$i0.3526$ and then $|\lambda_{1,2}| = 0.9462$. Figure 1 displays that the fixed point E_2 is a stable attractor at $h = 0.9$.

It deserves to note that a limit cycle could be observed numerically when the value h varies increasing. To show such phenomena, let us take $h = 1.1$. One can know from Theorem 2.4 that the fixed point $E_2(0.1188, 0.0297)$ becomes a source since $h > \Delta_1, \Delta_2$. In fact, if one takes an initial values $(x_0, y_0) = (0.1198, 0.0307)$ one can observe that the trajectory $T^n(x_0, y_0)$ goes away from the fixed point E_2 as n goes the infinity as shown in Figure 2.

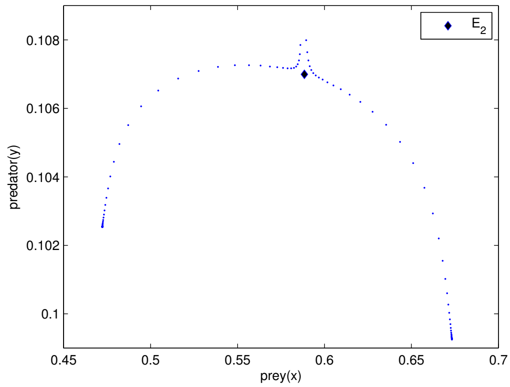

FIGURE 3. The fixed point $E_2(0.5885, 0.1070)$ of system (4) is a saddle point when $a = 1.6, b = 1.3, d = 0.1, r = 4$ and $h = 0.4$.

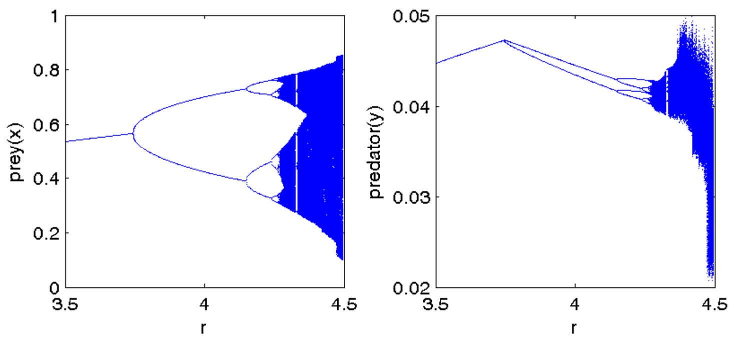

FIGURE 4. Bifurcation diagram of system (4) when $a = 1.6, b = 1.3, d = 0.2, h = 0.5$ and $3.5 \leq r \leq 4.5$.

In other words, this figures shows that a stable limit cycle could exist, which is a numerical evidence of the existence of Hopf bifurcation.

In order to illustrate a saddle phenomenon, we need to take another parameters set such as

$$a = 1.6, b = 1.3, d = 0.1, r = 4 \text{ and } h = 0.4. \tag{18}$$

Then from Theorem 2.4, one can figure out that the fixed point $E_2(0.5885, 0.1070)$ is a saddle point since $\Delta_1 = 0.5263, \Delta_2 = 2.7068$, which is shown in Figure 3.

It is difficult to search out all dynamics of system (4) theoretically. However, using bifurcation diagram, we can probe into numerically what kinds of dynamical behaviors happen to system (4). In this context, we illustrate a typical bifurcation diagram in Figure 4 for system (4) with respect to r when $a = 1.6, b = 1.3, d = 0.2, h = 0.5$ and

$3.5 \leq r \leq 4.5$. The resulting bifurcation diagram clearly shows that system (4) has rich dynamics including cycles, periodic doubling bifurcation, chaotic bands, periodic windows etc.

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