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ON OPTIMALITY AND DUALITY FOR GENERALIZED FRACTIONAL ROBUST OPTIMIZATION PROBLEMS

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Abstract. In this paper, we consider a generalized fractional robust optimization problem (FP). Establishing a nonfractional optimization problem (NFP) equivalent to (FP), we establish necessary optimality conditions and duality results.

1. Introduction

In this paper, we consider the following generalized fractional robust optimization problem (FP):

$$\text{(FP)} \qquad \text{Minimize} \quad \frac{\max\limits_{u \in U} f(x,u)}{\min\limits_{u \in U} g(x,u)}$$
$$\text{subject to} \quad h_i(x, v_i) \leq 0, \ \forall v_i \in V_i, \ i = 1, \cdots, m,$$

where u, v_i are uncertain parameters and $u \in U$, $v_i \in V_i$, $i = 1, \cdots, m$ for some convex compact sets $U \in \mathbb{R}^p$, $V_i \subset \mathbb{R}^q$, $i = 1, \cdots, m$, respectively and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \cdots, m$ are continuously differentiable functions. Assume that $f(x,u) \geq 0$ and $g(x,u) > 0$ for any $u \in U$.

Let $F := \{x \in \mathbb{R}^n : h_i(x, v_i) \leq 0, \ \forall v_i \in V_i, \ i = 1, \cdots, m\}$ be the robust feasible set of (FP). Then we say that x^* is a robust solution of (FP) if $x^* \in F$ and $\frac{\max_{u \in U} f(x,u)}{\min_{u \in U} g(x,u)} \geqq \frac{\max_{u \in U} f(x^*,u)}{\min_{u \in U} g(x^*,u)}$ for any $x \in F$. We denote $\nabla_1 g$ the derivative of g with respect to the first variable.

Consider the following nonfractional robust optimization problem:

$$\begin{aligned} \text{(NFP)} \qquad & \text{Minimize} && p \\ & \text{subject to} && f(x,u) - pg(x,u) \leq 0, \quad \forall u \in U, \\ &&& h_i(x, v_i) \leq 0, \quad \forall v_i \in V_i, \ i = 1, \cdots, m. \end{aligned}$$

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Following the approaches in [10], we can establish an equivalent relationship between (FP) and (NFP) as follows:

Proposition 1.1. *Let* $\bar{x} \in F$.

(1) If $\bar{x}$ *is a robust solution of (FP), then* $(\bar{x}, \bar{p})$ *is a robust solution of (NFP), where* $\bar{p} = \frac{\max_{u \in U} f(\bar{x},u)}{\min_{u \in U} g(\bar{x},u)}$.

(2) If $(\bar{x}, \bar{p})$ *is a robust solution of (NFP) where* $\bar{p} = \frac{\max_{u \in U} f(\bar{x},u)}{\min_{u \in U} g(\bar{x},u)}$, *then* $\bar{x}$ *is a robust solution of (FP).*

Proof. (1) Suppose that $(\bar{x}, \bar{p})$ is not a robust solution of (NFP). Then there exists $(\widetilde{x}, \widetilde{p})$ such that

$$\begin{aligned} &\widetilde{p} < \bar{p} \\ &f(\widetilde{x}, u) - \widetilde{p} g(\widetilde{x}, u) \leqq 0 \quad \forall u \in U \quad \text{ and} \\ &h_i(\widetilde{x}, v_i) \leqq 0 \ \ \forall v_i \in V_i, \ \ i = 1, \cdots, m. \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{\max_{u \in U} f(\widetilde{x}, u)}{\min_{u \in U} g(\widetilde{x}, u)} \leqq \widetilde{p} < \bar{p} = \frac{\max_{u \in U} f(\bar{x}, u)}{\min_{u \in U} g(\bar{x}, u)} \quad \text{ and} \\ &h_i(\widetilde{x}, v_i) \leqq 0 \ \ \forall v_i \in V_i, \ \ i = 1, \cdots, m. \end{aligned}$$

So, $\bar{x} \in F$, but $\bar{x}$ is not a robust solution of (FP).

(2) Suppose that $\bar{x}$ is not a robust solution of (FP). Then there exists $\widehat{x} \in F$ such that

$$\frac{\max_{u \in U} f(\widehat{x}, u)}{\min_{u \in U} g(\widehat{x}, u)} < \frac{\max_{u \in U} f(\bar{x}, u)}{\min_{u \in U} g(\bar{x}, u)} = \bar{p}.$$

Let $\widehat{p} = \frac{\max_{u \in U} f(\widehat{x},u)}{\min_{u \in U} g(\widehat{x},u)}$. Then $f(\widehat{x}, u) - \widehat{p} g(\widehat{x}, u) \leqq 0 \ \ \forall u \in U$. So, $(\bar{x}, \bar{p})$ is not a robust solution of (NFP). □

Many authors have introduced robust optimization problems and have obtained their optimality theorems and duality theorems ([1] − [9]).

Recently, Kim [7] considered the following fractional robust optimization problem (P):

$$\text{(P)} \quad \inf_{x \in \mathbb{R}^n} \left\{ \frac{f(x)}{g(x)} : h_j(x, v_j) \leq 0, \ \forall v_j \in V_j, \ i = 1, \cdots, m \right\},$$

where v_j are uncertain parameters and $v_j \in V_j, \ i = 1, \cdots, m$ for some convex compact sets $V_j \subset \mathbb{R}^q, \ j = 1, \cdots, m$ and $f : \mathbb{R}^n \to \mathbb{R}, \ g : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, \ j = 1, \cdots, m$ are continuously differentiable functions.

In this paper, we extend the generalized fractional robust optimization problem (FP) and prove necessary optimality theorems for (FP). Establishing a nonfractional optimization problem (NFP) equivalent to (FP), we formulate a Mond-Weir type dual problem for (FP) and obtain duality theorems for (FP).

2. Optimality theorems and duality theorems

In this section, we give necessary optimality conditions for the fractional robust optimization problem (FP).

Let $\bar{x} \in F$ and $\bar{p} = \frac{\max_{u \in U} f(\bar{x},u)}{\min_{u \in U} g(\bar{x},u)}$ and let us decompose $J := \{1, \cdots, m\}$ into two index sets $J = J_1(\bar{x}) \cup J_2(\bar{x})$ where $J_1(\bar{x}) = \{j \in J \mid \exists v_i \in V_i \text{ s.t. } h_i(\bar{x}, v_i) = 0\}$ and $J_2(\bar{x}) = J \setminus J_1(\bar{x})$. Let $U^0 = \{u \in U \mid f(\bar{x},u) - \bar{p}g(\bar{x},u) = 0\}$ and $V_i^0 = \{v_i \in V_i \mid \; h_i(\bar{x}, v_i) = 0\}$ for $i \in J_1(\bar{x})$.

Now we say that an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds at $(\bar{x}, \bar{p})$ for (NFP) if there exists $d \in \mathbb{R}^n$ such that for any $i \in J_1(\bar{x})$ and any $u^0 \in U^0$ and $v_i^0 \in V_i^0$,

$$\begin{aligned}&\left[\nabla_1 f(\bar{x}, u^0) - \bar{p}\nabla_1 g(\bar{x}, u^0)\right]^T d < 0 \quad \text{ and}\\ &\nabla_1 h_i(\bar{x}, v_i^0)^T d < 0.\end{aligned}$$

Now we present a necessary optimality theorem for a solution of (FP). For the proof of the following theorem, we follow the approaches for Theorem 3.1 in [5].

Theorem 2.1. *Let $\bar{x} \in F$ be a robust solution of (FP). Suppose that $f(\bar{x}, \cdot), -g(\bar{x}, \cdot)$ are concave on U and $h_i(\bar{x}, \cdot)$ is concave on V_i, $i = 1, \cdots, m$. Suppose that the Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds at $(\bar{x}, \bar{p})$ for (NFP). Then there exist $\bar{\mu}_i \geq 0$, $i = 1, \cdots, m$, $\bar{u} \in U$, $\bar{v}_i \in V_i$, $j = 1, \cdots, m$ such that*

$$\begin{aligned}&\nabla_1 f(\bar{x}, \bar{u}) - \bar{p}\nabla_1 g(\bar{x}, \bar{u}) + \sum_{i=1}^{m} \bar{\mu}_i \nabla_1 h_i(\bar{x}, \bar{v}_i) = 0,\\ &f(\bar{x}, \bar{u}) - \bar{p}g(\bar{x}, \bar{u}) = 0,\\ &\bar{\mu}_i h_i(\bar{x}, \bar{v}_i) = 0, \; i = 1, \cdots, m.\end{aligned}$$

Proof. Suppose that $\bar{x}$ is a robust solution of (FP). By Proposition 1.1, $(\bar{x}, \bar{p})$ is a robust solution of (NFP), where $\bar{p} = \frac{\max_{u \in U} f(\bar{x},u)}{\min_{u \in U} g(\bar{x},u)}$. By Theorem 3.1 in [5], there exist $\lambda \geqq 0$, $\mu_i \geqq 0$, $i = 1, \cdots, m$, $\bar{u} \in U$ and $\bar{v}_i \in V_i$, $i = 1, \cdots, m$,

$$\begin{aligned}&\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \nabla_1 f(\bar{x}, \bar{u}) - \bar{p}\nabla_1 g(\bar{x}, \bar{u}) \\ -g(\bar{x}, \bar{u}) \end{pmatrix} + \sum_{i=1}^{m} \mu_i \begin{pmatrix} \nabla_1 h_i(\bar{x}, \bar{v}_i) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\\ &\lambda\left[f(\bar{x}, \bar{u}) - \bar{p}g(\bar{x}, \bar{u})\right] = 0 \quad \text{and}\\ &\mu_i h_i(\bar{x}, \bar{v}_i) = 0, \; i = 1, \cdots, m.\end{aligned}$$

Hence $\lambda > 0$ and so, letting $\bar{\mu}_i = \frac{\mu_i}{\lambda}$, $i = 1, \cdots, m$, we get the conclusion. □

Using the equivalent relationship in Proposition 1.1, we formulate a Mond-Weir type robust dual problem (FD) for (FP).

$$
\begin{aligned}
\text{(FD)} \qquad & \text{Maximize} \quad p \\
& \text{subject to} \quad \nabla_1 f(x,u) - p\nabla_1 g(x,u) + \sum_{i=1}^{m} \mu_i \nabla_1 h_i(x,v_i) = 0, \quad (1)\\
& \qquad\qquad\quad f(x,u) - pg(x,u) \geqq 0,\\
& \qquad\qquad\quad \sum_{i=1}^{m} \mu_i h_i(x,v_i) \geqq 0,\\
& \qquad\qquad\quad u \in U,\ v_i \in V_i,\ \mu_i \geqq 0,\ i = 1,\cdots,m.
\end{aligned}
$$

Let $V = V_1 \times \cdots \times V_m$.

Theorem 2.2. *(Weak Duality) Let $x \in F$ be feasible for (FP) and $(\bar{x},\bar{u},\bar{v},\bar{\mu},\bar{p}) \in \mathbb{R}^n \times U \times V \times \mathbb{R}^m \times \mathbb{R}$ be feasible for (FD). Suppose that $f(\cdot,\bar{u}) - \bar{p}g(\cdot,\bar{u})$ is convex at $\bar{x}$ and $h_i(\cdot,\bar{v}_i),\ i = 1,\cdots,m$ are convex at $\bar{x}$, then*

$$\frac{\max_{u\in U} f(x,u)}{\min_{u\in U} g(x,u)} \geqq \bar{p}.$$

Proof. Let $x \in F$ be feasible for (FP) and $(\bar{x},\bar{u},\bar{v},\bar{\mu},\bar{p}) \in \mathbb{R}^n \times U \times V \times \mathbb{R}^m \times \mathbb{R}$ be feasible for (FD). Now suppose, contrary to the result. Then we have

$$\frac{\max_{u\in U} f(x,u)}{\min_{u\in U} g(x,u)} < \bar{p}, \text{ that is, } \max_{u\in U} f(x,u) - \bar{p}\min_{u\in U} g(x,u) < 0.$$

Since $f(\bar{x},\bar{u}) - \bar{p}g(\bar{x},\bar{u}) \geq 0$, $\max_{u\in U} f(\bar{x},u) - \bar{p}\min_{u\in U} g(\bar{x},u) = 0$, we have

$$
\begin{aligned}
f(x,\bar{u}) - \bar{p}g(x,\bar{u}) &\leqq \max_{u\in U} f(x,u) - \bar{p}\min_{u\in U} g(x,u)\\
&< \max_{u\in U} f(\bar{x},u) - \bar{p}\min_{u\in U} g(\bar{x},u)\\
&\leq f(\bar{x},\bar{u}) - \bar{p}\min_{u\in U} g(\bar{x},\bar{u}).
\end{aligned}
$$

By the convexity of $f(\cdot,\bar{u}) - \bar{p}g(\cdot,\bar{u})$ at $\bar{x}$,

$$[\nabla_1 f(\bar{x},\bar{u}) - \bar{p}\nabla_1 g(\bar{x},\bar{u})]^T (x - \bar{x}) < 0. \qquad (2)$$

Since $\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x},\bar{v}_j) \geqq \sum_{j=1}^{m} \bar{\mu}_j h_j(x,\bar{v}_j)$, by the convexity $h_i(\cdot,\bar{v}_i)$ at $\bar{x}$,

$$\left[\sum_{j=1}^{m} \bar{\mu}_i \nabla_1 h_i(\bar{x},\bar{v}_i)\right]^T (x - \bar{x}) \leqq 0. \qquad (3)$$

From (2) and (3),

$$\left[\nabla_1 f(\bar{x},\bar{u}) - \bar{p}\nabla_1 g(\bar{x},\bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 h_i(\bar{x},\bar{v}_i)\right]^T (x - \bar{x}) < 0,$$

which contradicts (1). □

Theorem 2.3. *(Strong Duality) Let $\bar{x}$ be a robust solution of (FP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds at $\bar{x}$. Then, there exist $(\bar{u}, \bar{v}, \bar{\mu})$ such that $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{p})$ is feasible for (FD) and the objective values of (FP) and (FD) are equal. If $f(\cdot, \bar{u}) - \bar{p}g(\cdot, \bar{u})$ is convex at $\bar{x}$, $h_i(\cdot, \bar{v}_i),\ i = 1, \cdots, m$ are convex at $\bar{x}$, then $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{p})$ is a solution of (FD).*

Proof. By Theorem 2.1, there exist $\bar{\mu}_j \geq 0,\ j = 1, \cdots, m$, $\bar{v}_j \in V_j,\ j = 1, \cdots, m$ such that

$$\begin{aligned} &\nabla_1 f(\bar{x}, \bar{u}) - \bar{p}\nabla_1 g(\bar{x}, u) + \sum_{i=1}^{m} \mu_i \nabla_1 h_i(\bar{x}, \bar{v}_i) = 0, \\ &f(\bar{x}, \bar{u}) - \bar{p}g(\bar{x}, \bar{u}) = 0, \\ &\bar{\mu}_i h_i(\bar{x}, \bar{v}_i) = 0,\ i = 1, \cdots, m. \end{aligned}$$

Thus $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{p})$ is a feasible for (FD). By Theorem 2.2, $\frac{\max_{u\in U} f(\bar{x},u)}{\min_{u\in U} g(\bar{x},u)} \geq \widetilde{p}$, for any feasible solution $(\widetilde{x}, \widetilde{u}, \widetilde{v}, \widetilde{\mu}, \widetilde{p})$ for (FD). Let $\frac{\max_{u\in U} f(\bar{x},u)}{\min_{u\in U} g(\bar{x},u)} = \bar{p}$, $\bar{p} \geq \widetilde{p}$. Hence $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{p})$ is a solution of (FD). □

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