East Asian Math. J.
Vol. 31 (2015), No. 5, pp. 717–725
http://dx.doi.org/10.7858/eamj.2015.052

THE CORES OF PAIRED-DOMINATION GAMES

Hye Kyung Kim

Abstract. Velzen introduced the rigid and relaxed dominating set games and showed that the rigid game being balanced is equivalent to the relaxed game being balanced in 2004. After then various variants of dominating set games were introduced and it was shown that for each variant, a rigid game being balanced is equivalent to a relaxed game being balanced. It is natural to ask if for any other variant of dominating set game, the balancedness of a rigid game and the balancedness of a relaxed game are equivalent. In this paper, we show that the answer for the question is negative by considering the rigid and relaxed paired-domination games, which is considered as a variant of dominating set games. We characterize the cores of both games and show that the rigid game being balanced is not equivalent to the relaxed game being balanced. In addition, we study the cores of paired-dominations games on paths and cycles.

1. Introduction

Throughout this paper, we assume that a graph means a simple graph with at least two vertices. An edge of a graph with endpoints u and v is denoted by uv. A weighted graph is a graph G with a vertex weight function $\omega : V(G) \to \mathbb{R}_+ \setminus \{0\}$ ($\mathbb{R}_+$ is the set of nonnegative real numbers), and for $D \subseteq V(G)$, we let $\omega(D) = \sum_{v \in D} \omega(v)$. For a graph G and a set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. For a vertex v of a graph G, the set of vertices that are adjacent to v is denoted by $N_G(v)$.

A *dominating set* of a graph G is a vertex set $D \subset V(G)$ such that for each vertex $v \in V(G) \setminus D$, there exists a vertex $w \in D$ which is adjacent to v. A *perfect matching* of a graph G is an edge set $M \subset E(G)$ such that the set of all endpoints of M is equal to $V(G)$ and any two edges in M does not share an endpoint. For a graph G, a set $D \subset V(G)$ is called a *paired-domination set* of G if D is a dominating set of G and $G[D]$ has a perfect matching.

Received September 3, 2015; Accepted September 16, 2015.
2010 *Mathematics Subject Classification.* 05C69.
Key words and phrases. paired-dominating set games, cores, games on graphs.
This work was supported by research grants from the Catholic University of Daegu in 2014.

Let G be a weighted graph with no isolated vertex a vertex weight function ω. The *paired-domination number* of G is defined by

$$\gamma_p(G) = \min\{\omega(D) \mid D \text{ is a paired-domination set of } G\}.$$

The paired-domination problem of G is to find the paired-domination number of G.

The paired-domination problem of a weighted graph is one of domination problems on graphs, which was introduced by Haynes and Slater [4]. The paired-domination problem has some practical applications: we assume that each vertex of a graph is the location of villages and the weight of a vertex is the cost for placing a guard on the village corresponding to the vertex. One might want to place guards on some villages in which guards' location must be selected as a pair of adjacent vertices so that each guard is assigned on other. From the authority's point of view, one wishes to minimize the total cost for placing guards, and finding the minimum cost is the paired-domination problem. Then it is natural to ask how to distribute the total cost of placing guards to the villages, and this question is closely related to a solution of a cooperative game.

A *cooperative game* (or game for short) is an ordered pair (V, c) of a player set V and a characteristic function $c : 2^V \to \mathbb{R} \cup \{\infty\}$ with $c(\emptyset) = 0$. For $S \subset V$ and $z \in \mathbb{R}^V$, we denote $z(S) = \sum_{v \in S} z_v$ [1]. For a game (V, c), a subset X of $\mathbb{R}^V$ in which each element $x \in X$ satisfies the property that $x(V) = c(V)$ is called a *solution* of (V, c). For a game (V, c), $c(S)$ represents the cost which the players in S achieve together for each $S \subseteq V$, and a solution of (V, c) shows how to distribute the total cost $c(V)$ among all the players. As an important solution, the *core* $\mathcal{C}(V, c)$ of a game (V, c) is defined by

$$\mathcal{C}(V, c) = \{z \in \mathbb{R}^V \mid z(V) = c(V) \text{ and } \forall S \subseteq V, z(S) \leq c(S)\}.$$

We say a game is *balanced* if its core is nonempty. In this paper, we introduced two kinds of paired-domination games, a rigid paired-domination game and a relaxed paired-domination game, which are cooperative games arising from paired-domination problems on graph structures, and focus on their cores.

Cooperative games that arise from domination problem on graphs were studied by van Velzen [10] to model the cost allocation problem arising from domination problems on graphs (see [1, 3] for more information on cooperative games caused from graph structures). He introduced rigid dominating set games and relaxed dominating set games, and characterized their cores and then balancedness of two games are equivalent. After then, variants of dominating set games were introduced and studied (see [2, 5, 6, 7, 8, 9]), and it is shown that the result on equivalence of balancedness of rigid and relaxed dominating set games are well extended to those variants. It is natural to ask if there is a variant

[1]We denote by $\mathbb{R}^V$ the $|V|$-dimensional vector space over $\mathbb{R}$ in which the entries of a vector are indexed by the elements of V. For a vector $x \in \mathbb{R}^V$, x_v denotes the value of the entry of x indexed by $v \in V$.

of dominating set game which does not satisfy the equivalence of balancedness. We derive an answer for the question by introducing two kinds of paired-domination games, a rigid and a relaxed paired-domination games, and showing that a relaxed paired-domination game being balanced does not imply a rigid paired-domination game being balanced.

This paper is organized as follows. Section 2 introduces the rigid paired-domination game and the relaxed paired-domination game associated with a weighted graph and gives characterization results on the cores of paired-domination games. Then we show that a rigid paired-domination game being balanced is not equivalent to a relaxed paired-domination game being balanced. Section 3 discusses on the balancedness of the rigid and relaxed paired-domination games associated with a path and a cycle. Section 4 gives concluding remarks.

2. Characterization of the cores of paired-domination games

Let G be a weighted graph with no isolated vertices whose vertex weight function is ω. For $S \subseteq V(G)$, $\omega(S) = \sum_{v \in S} \omega(v)$. For $S \subseteq V(G)$, a set $D \subset S$ is called a *rigid paired-domination set for* S if $G[S]$ has no isolated vertex and D is a paired-domination set. In addition, for any $S \subseteq V(G)$, a set $D \subset V(G)$ is a *relaxed paired-domination set for* S, if for each vertex $v \in S \setminus D$, there exists a vertex $w \in D$ which is adjacent to v, and $G[D]$ has a perfect matching. Since a graph G has no isolated vertex, it is true that any $S \subset V(G)$ has a relaxed paired-domination set.

Now we introduce two games associated with a weighted graph in the following.

Definition. Given a weighted graph G with no isolated vertex whose vertex weight function is ω, (V, c) is called the *rigid paired-domination game associated with* G if $V = V(G)$ and a function $c : 2^{V(G)} \to \mathbb{R} \cup \{\infty\}$ is defined as follows:
i) $c(\emptyset) = 0$;
ii) for $\emptyset \subsetneq S \subseteq V(G)$, if $G[S]$ has an isolated vertex then $c(S) = \infty$, and if $G[S]$ has no isolated vertex then

$$c(S) = \min\{\omega(D) \mid D \text{ is a rigid paired-domination set for } S\} = \gamma_p(G[S]).$$

Definition. Given a weighted graph G with no isolated vertex whose vertex weight function is ω, $(V, \widetilde{c})$ is called the *relaxed paired-domination game associated with* G if $V = V(G)$ and a function $\widetilde{c} : 2^{V(G)} \to \mathbb{R} \cup \{\infty\}$ is defined as follows:
i) $\widetilde{c}(\emptyset) = 0$;
ii) for $\emptyset \subsetneq S \subseteq V(G)$,

$$\widetilde{c}(S) = \min\{\omega(D) \mid D \text{ is a relaxed paired-domination set for } S\}.$$

Whenever we consider a paired-domination game associated with a graph G, we always assume that G has no isolated vertices.

In the following, we investigate the cores of rigid paired-domination games and relaxed paired-domination games. Let G be a graph. For an edge uv of G, a subset of $N_G(u) \cup N_G(v)$ containing $\{u, v\}$ is called a *uv-star*. A subset of $V(G)$ is called a *star* if it is a uv-star for some edge uv. We denote by $\mathcal{T}_{uv}(G)$ the set of all uv-stars of G.

We will present characterization results of the cores of both rigid and relaxed paired-domination games. Theorem 2.1 and Theorem 2.2 are characterizations for the cores of the rigid and relaxed paired-domination games, respectively. First, we see a characterization result of the cores for rigid paired-domination games.

Theorem 2.1. *Let (V, c) be the rigid paired-domination game associated with a weighted graph G whose vertex weight function is ω. Then $z \in \mathcal{C}(V, c)$ if and only if the following hold:*

(a) $z(V) = c(V)$;
(b) *for any edge uv, for any uv-star T of G, $z(T) \le \omega(u) + \omega(v)$.*

Proof. The 'only if' part is easy. Let z be an element of $\mathcal{C}(V, c)$. Then by the definition of the core, (a) follows immediately. To show (b), take an uv-star $T \in \mathcal{T}_{uv}(G)$ for some edge uv. Then $z(T) \le c(T)$ by the definition of the core. Then $\{u, v\}$ is a rigid paired-domination set for T. Since $c(T)$ is the minimum weight of rigid paired-domination sets for T, $c(T) \le \omega(u) + \omega(v)$. Therefore, $z(T) \le \omega(u) + \omega(v)$ and so (b) holds.

Now we will show the 'if' part. Suppose that z satisfies (a) and (b). It is sufficient to show that $z(S) \le c(S)$ for any $S \subset V$. Take $S \subset V$. If $G[S]$ has an isolated vertex, $c(S) = \infty$ and so $z(S) \le \infty = c(S)$ holds. Now we consider the case where $G[S]$ has no isolated vertex. Then there is a rigid paired-domination set D for S such that $c(S) = w(D)$. Let $D = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$ where $\{u_i v_i \mid 1 \le i \le k\}$ is a perfect matching of $G[D]$. Then there exists a function $f : (S \setminus D) \to D$ such that $f(v) \in N_{G[S]}(v)$ for any $v \in S \setminus D$. For each $1 \le i \le k$, let

$$T_i = \{u_i, v_i\} \cup f^{-1}(\{u_i, v_i\}).$$

Then T_i is $u_i v_i$-star. Also, $\{T_1, T_2, \ldots, T_k\}$ is a partition of S, and so $z(S) = \sum_{i=1}^{k} z(T_i)$. By the assumption (b), we have $z(T_i) \le \omega(u_i) + \omega(v_i)$. Therefore,

$$z(S) \le \sum_{i=1}^{k} (\omega(u_i) + \omega(v_i)) = \sum_{v \in D} \omega(v) = \omega(D) = c(S).$$

Hence $z(S) \le c(S)$. □

Now we give a characterization of a core element of a relaxed paired-domination game. For an edge uv of a graph G, the uv-star $N_G(u) \cup N_G(v)$ of G is called the *maximal uv-star* of G.

Theorem 2.2. *Let $(V, \widetilde{c})$ be the relaxed paired-domination game associated with a weighted graph G whose vertex weight function is ω. Then $z \in \mathcal{C}(V, \widetilde{c})$ if and only if the following hold:*

(a) $z(V) = \widetilde{c}(V)$;
(b) *for each* $v \in V(G)$, $z_v \geq 0$;
(c) *for any edge* uv, $z(T^*_{uv}) \leq \omega(u) + \omega(v)$ *where* T^*_{uv} *is the maximal* uv*-star of* G.

Proof. We will show the 'only if' part first. Let z be an element of $\mathcal{C}(V, \widetilde{c})$. By the definition of $\mathcal{C}(V, c)$ (a) holds. Since $\widetilde{c}(S) \leq \widetilde{c}(T)$ for every $S \subseteq T \subseteq V$, $z_v = z(V) - z(V \setminus \{v\}) \geq \widetilde{c}(V) - \widetilde{c}(V \setminus \{v\}) \geq 0$ for every $v \in V$. From this observation, (b) holds immediately. To show (c), take an edge uv. Let T^*_{uv} be the maximal uv-star of G. Then $\{u, v\}$ is a relaxed paired-domination set for T^*_{uv}. Therefore $\widetilde{c}(T^*_{uv}) \leq \omega(u) + \omega(v)$, and so (c) holds.

To show the 'if' part, suppose that z satisfies (a), (b) and (c). It is sufficient to show that $z(S) \leq \widetilde{c}(S)$ for any $S \subset V$. Take a subset $S \subset V$. Then there is a relaxed paired-domination set D for S such that $\widetilde{c}(S) = w(D)$. Let $D = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$ where $\{u_i v_i \mid 1 \leq i \leq k\}$ is a perfect matching of $G[D]$. For each $1 \leq i \leq k$, let

$$T_i = N_G(u_i) \cup N_G(v_i).$$

Then for each $1 \leq i \leq k$, T_i is the maximal $u_i v_i$-star of G and $\bigcup_{i=1}^k T_i \supseteq S$. By the assumption that $z_v > 0$ for all $v \in V$,

$$z(S) \leq z(\bigcup_{i=1}^{k} T_i) \leq \sum_{i=1}^{k} z(T_i).$$

Since $z(T_i) \leq \omega(u_i) + \omega(v_i)$ by (c),

$$\sum_{i=1}^{k} z(T_i) \leq \sum_{i=1}^{k} (\omega(u_i) + \omega(v_i)) = \sum_{v \in D} \omega(v) = \omega(D) = \widetilde{c}(S).$$

Hence $z(S) \leq \widetilde{c}(S)$. □

Theorem 2.2 says that determining if given vector is in the core or not of a relaxed paired-domination game can be done in polynomial time. From Theorem 2.1 and Theorem 2.2, it can be easily checked that the core of a relaxed paired-domination game coincides the nonnegative vectors in the core of a rigid paired-domination game.

Corollary 2.3. *Let (V, c) be the rigid paired-domination game and $(V, \widetilde{c})$ be the relaxed paired-domination game, both of which associate with a weighted graph G whose vertex weight function is ω. Then $\mathcal{C}(V, c) \cap \mathbb{R}^V_+ = \mathcal{C}(V, \widetilde{c})$.*

For dominating set games [10], the balancedness of a rigid dominating set game is equivalent to the balancedness of a relaxed dominating set game. For all variants known until now, that is, integer dominating set games [5], fractional

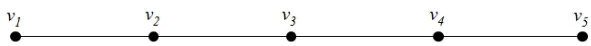

FIGURE 1. A graph G whose vertex weight function ω is defined by $\omega(v_1) = \omega(v_2) = \omega(v_4) = \omega(v_5) = 2$, $\omega(v_3) = 1$

dominating set games [6], edge covering games [7], and integer edge covering games [9], the balancedness of a rigid one is equivalent to balancedness of corresponding relaxed one. Thus it can be natural to ask if for any other variant of dominating set game, the balancedness of a rigid game and the balancedness of a relaxed game are equivalent. However, Example 1 shows that it the answer for the question is negative in the case of paired-domination games, that is, the rigid paired-domination game being balanced is not equivalent to the relaxed paired-domination game being balanced.

Example 1. See Figure 1, which shows a graph G such that $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$. Let a vertex weight function ω of G be defined by $\omega(v_1) = \omega(v_2) = \omega(v_4) = \omega(v_5) = 2$ and $\omega(v_3) = 1$. Let (V, c) be the rigid paired-domination game and $(V, \widetilde{c})$ be the relaxed paired-domination game, both of which associate with a weighted graph G We will see that $\mathcal{C}(V, c) \neq \emptyset$ and $\mathcal{C}(V, \widetilde{c}) = \emptyset$. Then $c(V) = 7$ and $(4, 0, -1, 0, 4) \in \mathcal{C}(V, c)$. Therefore, $\mathcal{C}(V, c) \neq \emptyset$. Suppose that $\mathcal{C}(V, \widetilde{c}) \neq \emptyset$. Take $z \in \mathcal{C}(V, \widetilde{c})$. Then $\widetilde{c}(V) = 7$ and $z_{v_1} + z_{v_2} + z_{v_3} + z_{v_4} + z_{v_5} = 7$. Since the maximal v_2v_3-star satisfies the condition (c) of Theorem 2.2, $z_{v_1} + z_{v_2} + z_{v_3} + z_{v_4} \leq \omega(v_2) + \omega(v_3) = 3$. Therefore $z_{v_5} \geq 4$. Similarly, $z_{v_2} + z_{v_3} + z_{v_4} + z_{v_5} \leq 3$ and so $z_{v_1} \geq 4$. Then $z_{v_1} + z_{v_5} \geq 8$ and $z_{v_1} + z_{v_2} + z_{v_3} + z_{v_4} + z_{v_5} = 7$. It follows that $z_{v_2} + z_{v_3} + z_{v_4} \leq -1$, which implies that there exists a vertex v_i such that $z_{v_i} < 0$. Therefore z does not satisfies the condition (b) of Theorem 2.2, a contradiction. Thus $\mathcal{C}(V, \widetilde{c}) = \emptyset$.

By Corollary 2.3 and Example 1, the following holds:

Theorem 2.4. *Let (V, c) be the rigid paired-domination game and $(V, \widetilde{c})$ be the relaxed paired-domination game, both of which associate with a weighted graph G whose vertex weight function is ω. Then (V, c) being balanced implies $(V, \widetilde{c})$ being balanced, and the converse is not true.*

3. The paired-domination games on paths and cycles

In this section, we always assume that any vertex weight function of a graph is the constant function 1. Then the paired-domination number of a graph is equal to the minimum size of a paired-domination set. We denote by P_n and C_n a path with n vertices and a cycle with n vertices, respectively. It is well known that for an integer n greater than 2, $\gamma_p(P_n) = \gamma_p(C_n) = 2 \times \lceil \frac{n}{4} \rceil$ (see [4]).

We will show that both the rigid and relaxed paired-domination games associate with a path are always balanced, whereas two games associate with a cycle are not. In addition, we give the sufficient and necessary condition for the balancedness of both games associate with a cycle.

Theorem 3.1. *The rigid paired-domination game* (V, c) *and the relaxed paired-domination game* $(V, \widetilde{c})$*, both of which associate with a path* P_n $(n \geq 2)$*, are always balanced.*

Proof. If $n = 2$, then $V(P_n)$ is a paired-domination set and any vector $(x, 2-x)$ belongs to $\mathcal{C}(V, c)$. Suppose that $n \geq 3$. Let the vertices of P_n be v_1, v_2, $\ldots$, v_n such that $v_i v_{i+1} \in E(P_n)$ for each $i = 1, 2, \ldots, n-1$. Let z be the vector in $\mathbb{R}^V$ defined by

$$z_v = \begin{cases} 2 & \text{if } v = v_{4k+1} \text{ for some integer } k \\ 0 & \text{otherwise} \end{cases} .$$

Then $\sum_{v \in V} z_v = 2 \times \lceil \frac{n}{4} \rceil$. Since $\gamma_p(P_n) = 2 \times \lceil \frac{n}{4} \rceil$, $\sum_{v \in V} z_v = \gamma_p(P_n) = \widetilde{c}(V)$. In addition, $z_v \geq 0$ for any vertex v. For an edge uv, the maximal uv-star T has at most 4 consecutive vertices and so $\sum_{v \in V(T)} z_v \leq 2$. By Theorem 2.2, z is an element of $\mathcal{C}(V, \widetilde{c})$. By Corollary 2.3, $z \in \mathcal{C}(V, c)$. Hence both (V, c) and $(V, \widetilde{c})$ are balanced. $\square$

Theorem 3.2. *Let* (V, c) *and* $(V, \widetilde{c})$ *be the rigid paired-domination game and the relaxed paired-domination game, both of which associate with a cycle* C_n $(n \geq 4)$*. Then the following are equivalent:*

(i) (V, c) *is balanced;*
(ii) $(V, \widetilde{c})$ *is balanced;*
(iii) n *is a multiple of* 4.

Proof. Let the vertices of C_n be v_1, v_2, $\ldots$, v_n such that $v_i v_{i+1} \in E(C_n)$ for each $i = 1, 2, \ldots, n$, and we assume that the subscripts are reduced to modulo n. By Corollary 2.3, it is sufficient to show that (iii) implies (ii) and (i) implies (iii).

First, we show that (iii) implies (ii). Suppose that $n = 4q$ for some integer q. Let z be the vector in $\mathbb{R}^V$ defined by

$$z_v = \begin{cases} 2 & \text{if } v = v_{4k+1} \text{ for some integer } k \\ 0 & \text{otherwise} \end{cases} .$$

Then $\sum_{v \in V} z_v = 2q = 2 \times \lceil \frac{n}{4} \rceil = c(V)$. In addition, $z_v \geq 0$ for each vertex v. For an edge uv and for a uv-star T of G, T contains at most 4 consecutive vertices and the sum of 4 consecutive vertices is always 2. Therefore $\sum_{v \in V(T)} z_v \leq 2$. By Theorem 2.2, z is an element of $\mathcal{C}(V, \widetilde{c})$. Thus $(V, \widetilde{c})$ is balanced.

To show that (i) implies (iii) by contradiction, suppose that $\mathcal{C}(V, c) \neq \emptyset$ and n is not a multiple of 4. Then there exists an element $z \in (V, c)$. For convenience, we denote z_{v_i} by z_i. By division algoritheorem, there exist two integers q and r such that $n = 4q + r$ and $0 < r \leq 3$. By the definition of the core, $\sum_{j=1}^{n} z_j = \gamma_p(C_n) = 2q + 2$. For each vertex v_i, the subgraph obtained by deleting the r consecutive vertices v_i, $\ldots$, v_{i+r-1} is a path P_{4q} of $4q$ vertices and so $\sum_{j=1}^{n} z_j - \sum_{j=1}^{r} z_{i+j-1} \leq c(P_{4q})$ by the definition of the core. Since $c(P_{4q}) = 2q$, it holds that

$$(\forall i \in \{1, 2, \ldots, n\}) \quad \sum_{j=1}^{r} z_{i+j-1} \geq 2. \tag{1}$$

If $r = 1$, then $z_i \geq 2$ for each vertex v_i by (1), and then $2q + 2 = \sum_{i=1}^{n} z_i \geq 2n = 8q + 2r$, a contradiction. If $r = 2$, then $z_i + z_{i+1} \geq 2$ for each vertex v_i by (1), and then

$$2q + 2 = \sum_{i=1}^{n} z_i = \sum_{i=1}^{2q+1} (z_{2i-1} + z_{2i}) \geq (2q + 1) \times 2 = 4q + 2,$$

a contradiction. Suppose that $r = 3$. Take a vertex v_i. Then $z_i + z_{i+1} + z_{i+2} \geq 2$ by (1). By the condition (b) of Theorem 2.1,

$$\begin{aligned} &z_i + z_{i+1} + z_{i+2} \leq 2; \\ &z_{i+1} + z_{i+2} \leq 2. \end{aligned}$$

Then $z_i + z_{i+1} + z_{i+2} = 2$ and so $z_i \geq 0$. Therefore $z_i \geq 0$ for each vertex v_i.

Let m and s be nonnegative integers such that $q = 3m + s$ and $0 \leq s \leq 2$. Then $2q + 2 = 6m + 2s + 2$ and $n = 4q + 3 = 4(3m + s) + 3 = 3(4m + 1 + s) + s$. Therefore,

$$\begin{aligned} 6m + 2s + 2 = 2q + 2 &= \sum_{i=1}^{n} z_i \\ &\geq \sum_{i=1}^{4m+1+s} (z_{3i-2} + z_{3i-1} + z_{3i}) \\ &= 2 \times (4m + 1 + s) = 8m + 2s + 2. \end{aligned}$$

It implies that $m = 0$, and so $n = 4q + 3 = 4s + 3$. Since $n \geq 4$, $s > 0$. In addition, $m = 0$ implies that $\sum_{i=1}^{n} z_i = \sum_{i=1}^{4m+1+s} (z_{3i-2} + z_{3i-1} + z_{3i})$. Then $\sum_{i=1}^{s} z_{n-s+i} = 0$ (since $s > 0$, the summation is well-defined). Since $z_i \geq 0$ for each vertex v_i, we conclude that $z_n = 0$. By relabeling the vertices, we can show that $z_i = 0$ for any vertex v_i, a contradiction. Hence (i) implies (iii). □

4. Concluding Remarks

In this paper, we introduce the rigid and relaxed paired-domination games and investigate their cores. In addition, we study the cores of paired-domination games on paths and cycles if their vertex weight functions are the constant function 1. We present several research questions related to this topic.

- Find an algorithm for finding a core element of a rigid (or relaxed) paired-domination game if it is balanced.

- Comparing Example 1 and Theorem 3.1, it follows that the relaxed paired-domination game on a weighted path is not always balanced. We may ask if the rigid paired-domination game on a weighted path is always balanced.
- Characterize the core of the rigid (or relaxed) paired-domination game associated with an interval graph or a circular arc graph.

References

[1] J. M. Bilbao, *Cooperative games on combinatorial structures*, Kluwer Academic Publishers, Boston (2000).

[2] T. Bietenhader and Y. Okamoto: Core stability of minimum coloring games, *Mathematics of Operations Research* **31** (2006) 418-431.

[3] X. Deng, T. Ibaraki and H. Nagamochi: Algoritheoremic aspects of the core of combinatorial optimization games, *Mathematics of Operations Research* **24** (1999) 751-766.

[4] T. W. Haynes and P. J. Slater: Paired-domination in graphs, *Networks* **32** (1998) 199–206.

[5] H. Kim and Q. Fang: Balancedness of integer domination set games, *Journal of Korean Mathematics Society* **43** (2006) 297-309.

[6] H. Kim and Q. Fang: Balancedness and concavity of fractional domination games, *Bulletin of the Korean Mathematical Society* **43** (2006) 265-275.

[7] Y. Liu and Q. Fang: Balancedness of edge covering games, *Applied Mathematics Letters* **20** (2007) 1064-1069.

[8] Y. Okamoto: Fair cost allocations under conflicts -a game-theoretic point of view-, *Discrete Optimization* **5** (2008) 1-18.

[9] B. Park, S.-R. Kim and H. Kim: On the cores of games arising from integer edge covering functions of graphs, *preprint*.

[10] B. V. Velzen: Dominating set games, *Operations Research Letters* **32** (2004) 565-573.

Department of Mathematics Education, Catholic University of Daegu
Kyeongsan 712-702, Republic of Korea
E-mail address: hkkim@cu.ac.kr