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SELF SIMILAR TILES ARISING FROM THE UNITARY NUMBER SYSTEMS IN EUCLIDEAN RINGS OF IMAGINARY QUADRATIC INTEGERS

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Abstract. In [7], it was shown that Euclidean rings R of imaginary quadratic integers admit unitary number systems. In this paper, as an application of the result, we obtain all self similar tiles arising from the unitary number systems of R.

1. Introduction

Let R be a ring of imaginary quadratic integers, which consists of all algebraic integers in the field $\mathbb{Q}\left(\sqrt{d}\right)$ for some square-free negative integer d, and let b be an element of R whose norm $N(b)$ is greater than one. Then a complete representatives D of $R/(b)$ is called a *digit set* for a *base* b, where (b) denotes an ideal generated by b. Then a *unitary number system* (b, D) in R is defined so that all non-zero elements of digits in D consist of units of R.

From [7] we see that among the nine principal ideal domains of imaginary quadratic integers, only the five rings

$$\mathbb{Z}\left[\sqrt{-1}\right],\ \mathbb{Z}\left[\sqrt{-2}\right],\ \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right],\ \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right],\ \mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}\right],$$

equipped with Euclidean functions defined by norms admit unitary number systems. As proposed in [7], the concept of a unitary number system can be replacement of a 'universal side divisor' by Motzkin. Furthermore unitary number systems are already available in various applications of complex based number systems. In particular, Khmelnik [3] and Penny [6] independently showed that a number system with a base $b = -1 + \sqrt{-1}$ and a digit set $D = \{0, 1\}$ in $\mathbb{Z}\left[\sqrt{-1}\right]$ yields so called *the twin dragon* (see Figure 2 in Appendix). Knuth [5] proposed another binary number system with a base $b = \sqrt{-2}$ and a digit

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set $D = \{0, 1\}$. For applications in data processing Khmelnik [4] introduced a rather mysterious binary number system $\left(b = \frac{1+\sqrt{-7}}{2},\ D = \{0, 1\}\right)$ which yields so called the *the tamed dragon* (see Figure 9 in Appendix).

We classify all unitary number systems in Euclidean rings of imaginary quadratic integers, up to equivalence of associated self similar tiles adopted from [1], as follows:

Theorem 1.1. *The list of all unitary number systems* (b, D) *up to equivalence is given as follows:*

(1) *for binary number systems,* (i) $b = 1 + \sqrt{-1}$, $D = \{0, 1\}$ *in* $\mathbb{Z}\left[\sqrt{-1}\right]$, (ii) $b = \sqrt{-2}$, $D = \{0, 1\}$ *in* $\mathbb{Z}\left[\sqrt{-1}\right]$, (iii) $b = \frac{1+\sqrt{-7}}{2}$, $D = \{0, 1\}$ *in* $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$;

(2) *for ternary number systems,* (i) $b = 1 + \sqrt{-2}$, $D = \{0, \pm 1\}$ *in* $\mathbb{Z}\left[\sqrt{-2}\right]$, (ii) $b = 2\omega$, $D = \{0, 1, \omega\}$ *in* $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$, (iii) $b = 2\omega$, $D = \{0, \pm 1\}$ *in* $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$, (iv) $b = \frac{1+\sqrt{-11}}{2}$, $D = \{0, \pm 1\}$ *in* $\mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}\right]$;

(3) *for* 4*-ary number systems,* (i) $b = 2\omega$, $D = \{0, 1, \omega, \omega^2\}$ *in* $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$, (ii) $b = 2\omega$, $D = \{0, 1, -\omega, \omega^2\}$ *in* $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$;

(4) *for* 5*-ary number systems,* (i) $b = 2 + \sqrt{-1}$, $D = \{0, \pm 1, \pm\sqrt{-1}\}$ *in* $\mathbb{Z}\left[\sqrt{-1}\right]$;

(5) *for* 7*-ary number systems,* (i) $b = 2 + \omega$, $D = \{0, \pm 1, \pm\omega, \pm\omega^2\}$ *in* $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$*, where* $\omega = \frac{1+\sqrt{-3}}{2}$.

We will provide a proof of Theorem 1.1 in Section 2, based on a crucial observation that $R/(b)$ is a finite field for the base b of each unitary number system.

All self similar tiles associated with the above theorem turn out to be already known in fractal geometry of complex based number systems.

2. Proof of Theorem 1.1

In this section we exhibit all unitary number systems available in the five Euclidean rings R of imaginary quadratic integers; the observation will provide a proof of Theorem 1.1.

Let R be a ring of imaginary quadratic integers. For every $r \in R$, let N denote the norm defined by $N(r) = |r|^2 = r\bar{r}$ where $\bar{r}$ is the complex conjugate of r.

Denote by $R^\times$ the multiplicative group of units of R, and $R_0^\times = R^\times \cup \{0\}$.

We here list some basic facts, which are necessary for our observation; the results are well known or can be found in [7].

Lemma 2.1. *Let* R *be a ring of imaginary quadratic integers. The group* $R^\times$ *of units of* R *is listed as follow:*

(i) $R^{\times} = \{\pm 1, \pm i\}$ *for* $R = \mathbb{Z}[i]$, *where* $i = \sqrt{-1}$;
(ii) $R^{\times} = \{\pm 1, \pm\omega \cdot \pm \omega^2\}$ *for* $R = \mathbb{Z}[\omega]$, *where* $\omega = \frac{1+\sqrt{-3}}{2}$;
(iii) *except for the above two cases,* $R^{\times} = \{\pm 1\}$.

Lemma 2.2. *Let R be a ring of imaginary quadratic integers. If $b \in R \setminus R_0^{\times}$, then the number of cosets modulo (b) equals to $N(b)$.*

Lemma 2.3. *Let R be a ring of imaginary quadratic integers. If b is a base of a unitary number system in R, then $R/(b)$ is a finite field, whose order is one of* $2, 3, 4, 5$ *and* 7.

Adopted from [1], we here have the following definition of equivalence of self similar tiles:

Let $T = T(b, D)$, $T' = T'(b', D')$ be the self similar tiles arising from number systems (b, D), (b', D') respectively. Then we say that T and T' are *equivalent* if and only if there exists a Euclidean similarity $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that for each $d \in D$ there exists uniquely $d' \in D'$ such that $\phi(d + T) = d' + T'$.

Then the following facts are well known in fractal geometry of complex based number systems.

Lemma 2.4. *Let (b, D) be a unitary number system in a Euclidean ring R of imaginary quadratic integers. Then for each $u \in R^{\times}$ we have :*

(i) (ub, D) *is a unitary numbers system such that* $T(ub, D)$ *is equivalent to* $T(b, D)$.

(ii) (b, uD)*) is a unitary numbers system such that* $T(b, uD)$ *is equivalent to* $T(b, D)$.

(iii) *If the digit set D preserves the complex conjugation (namely $\bar{d} \in D$ for each $d \in D$), then $\bar{d} \in D$ is a unitary numbers system such that $T(\bar{d}, D)$ is equivalent to $T(b, D)$.*

Consequently, once we get a unitary number system from a solution of equation $N(b) = k$, it is easy to see that all of its other solutions yield unitary number systems equivalent to it. We here determine the unitary number systems up to equivalence in the five Euclidean rings of imaginary quadratic integers.

I. Unitary number systems in $R = \mathbb{Z}[i]$

Since $R^{\times} = \{\pm 1, \pm i\}$ and $N(b) = p^2 + q^2$ for $b = p + qi \in R$, available cardinalities of unitary digit sets are $2, 4$ and 5.

(i) *number systems with the maximal digit set* $D = R_0^{\times} = \{0, \pm 1, \pm i\}$
Equation $N(b) = p^2 + q^2 = 5$ has a solution $(p, q) = (2, 1); b = 2 + i$. As shown in figure 1(a), $R_0^{\times}$ represent the set $R/(b)$ in the fundamental domain determined by b. Each circled element in the fundamental domain represents the unit modulo $b = 2 + i$, for instance, $2i$ represented by 1 with a circle in the fundamental domain is congruent to 1 modulo $b = 2 + i$. Figure 1(b) shows the fractional set of the number system $b = 2 + i$, $D = R_0^{\times} = \{0, \pm 1, \pm i\}$.

The other solutions of equation $N(b) = p^2 + q^2 = 5$ yield unitary number systems equivalent to the one in the above.

(ii) *binary number systems*
Equation $N(b) = p^2 + q^2 = 2$ has a solution $b = 1 + i$ for $(p, q) = (1, 1)$. Up to equivalency, a base $b = 1 + i$ and a digit set $D = \{0, 1\}$ form a unitary number system whose fractional set is known as the twin dragon in Figure 2.

Remark. In $\mathbb{Z}[i]$ there are no 4-ary number systems which are unitary. This is because any base of 4-ary number system is equivalent to a base $b = 2i$ for which $\mathbb{Z}[i]/(b)$ cannot be a field. By Lemma 2.3, $b = 2i$ cannot be a base of a unitary number system.

II. Unitary number systems in $R = \mathbb{Z}\left[\sqrt{-2}\right]$

Since $R^\times = \{\pm 1\}$ and $N(b) = p^2 + 2q^2$ for $b = p + q\sqrt{-2} \in R$, available cardinalities of unitary digit sets are 2 or 3.

(i) *ternary (maximal) number systems*
Up to equivalency, equation $N(b) = p^2 + 2q^2 = 3$ yields a base $b = 1 + \sqrt{-2}$ for $(p, q) = (1, 1)$ which has the fractional set with $b = 1+\sqrt{-2}$, $D = R_0^\times = \{0, \pm 1\}$ in Figure 3.

(ii) *binary number systems*
Up to equivalency, equation $N(b) = p^2 + 2q^2 = 2$ yields a base $b = \sqrt{-2}$ for $(p, q) = (0, 1)$ which has the fractional set with $b = \sqrt{-2}$, $D = R_0^\times = \{0, 1\}$ in Figure 4.

III. Unitary number systems in $R = \mathbb{Z}[\omega]$, $\omega = \frac{1+\sqrt{-3}}{2}$

Since $R^\times = \{\pm 1, \pm\omega, \pm \omega^2\}$ and $N(b) = p^2 + pq + q^2$ for $b = p + q\omega \in R$, available cardinalities of unitary digit sets are $3, 4$ or 7.

(i) *number systems with the maximal digit set* $D = R_0^\times = \{0, \pm 1, \pm\omega, \pm \omega^2\}$
Up to equivalency, equation $N(b) = p^2 + pq + q^2 = 7$ has a solution $(p, q) = (2, 1); b = 2 + \omega$. Figure 5(a) shows $R_0^\times$ in the fundamental domain which represents $R/(b)$. Figure 5(b) shows the fractional set of the number system, the boundary curve of which is known as the gosper curve.

(ii) 4-*ary number systems*
Up to equivalency, the equation $N(b) = p^2 + pq + q^2 = 4$ has a solution $(p, q) = (0, 2); b = 2\omega$. Given base $b = 2\omega$, we have two choices of unitary digit sets the associated number systems of which are considered as the same: (a) $D = \{0, 1, \omega, \omega^2\}$ and (b) $D = \{0, 1, -\omega, \omega^2\}$; the fractional sets of the number systems are shown in Figure 6(b) and Figure 7(b), respectively.

Remark. Note that Figure 7(b) is introduced in [2] as the fractional set of a number system $(b = -2, D = \{0, 1, -\omega, \omega^2\})$ which is equivalent to $(b = 2\omega, D = \{0, 1, -\omega, \omega^2\})$.

(iii) *ternary number systems*
Up to equivalency, the equation $N(b) = p^2 + pq + q^2 = 3$ has a solution $(p, q) = (1, 1); b = 1 + \omega$. Given base $b = 1 + \omega$, we have two choices of unitary digit

sets: (a) $D = \{0, 1, \omega\}$,, and (b) $D = \{0, \pm 1\}$; the fractional sets of the number systems are shown in Figure 8(a) and Figure 8(b), respectively.

IV. Unitary number systems in $R = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$

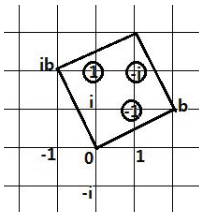
Since $R^\times = \{\pm 1\}$ and $N(b) = p^2 + pq + 2q^2$ for $b = p + q(\frac{1+\sqrt{-7}}{2} \in R$, 2 is only available cardinality of a unitary digit set. Up to equivalency, the equation $N(b) = p^2 + pq + 2q^2 = 2$ yields a base $b = \frac{1+\sqrt{-7}}{2}$ for $(p, q) = (0, 1)$, which has the fractional set with $b = \frac{1+\sqrt{-7}}{2}$, $D = \{0, 1\}$, shown in Figure 9.

V. Unitary number systems in $R = \mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}\right]$

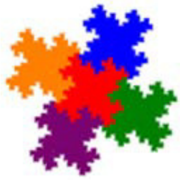
Since $R^\times = \{\pm 1\}$ and $N(b) = p^2 + pq + 3q^2$ for $b = p + q(\frac{1+\sqrt{-11}}{2} \in R$, 3 is only available cardinality of a unitary digit set. Up to equivalency, the equation $N(b) = p^2 + pq + 2q^2 = 3$ yields a base $b = \frac{1+\sqrt{-11}}{2}$ for $(p, q) = (0, 1)$ which has the fractional set with $b = \frac{1+\sqrt{-11}}{2}$, $D = \{0, \pm 1\}$, shown in Figure 10.

The proof of Theorem 1.1 is now complete with the above observations from I to V.

Appendix: List of Figures



(a)



(b)

FIGURE 1

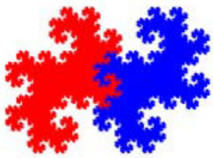


FIGURE 2

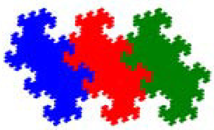
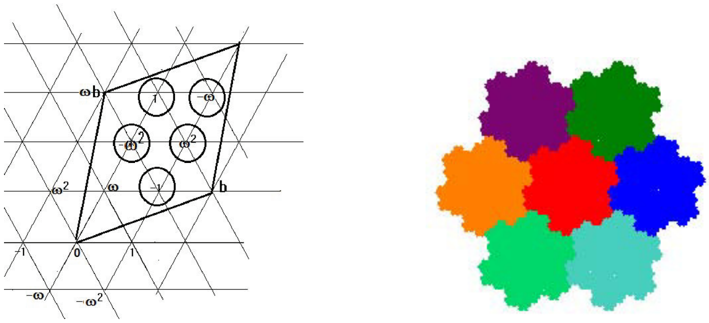


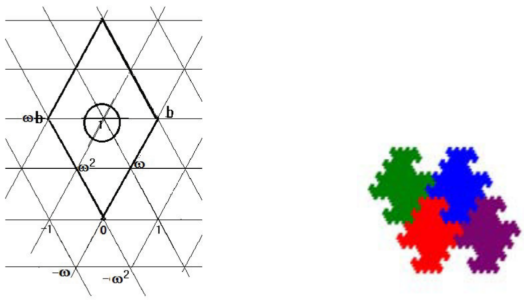
FIGURE 3

FIGURE 4



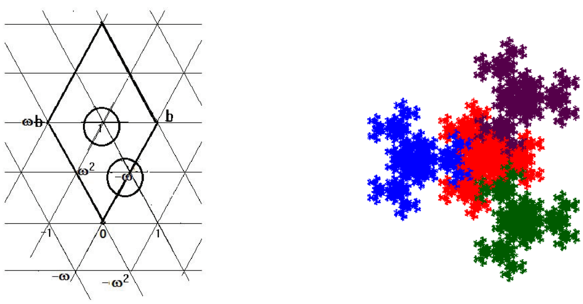
(a) (b)

FIGURE 5



(a) (b)

FIGURE 6



(a) (b)

FIGURE 7

(a) (b)

FIGURE 8

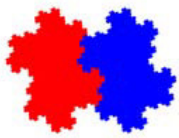


FIGURE 9

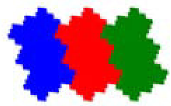


FIGURE 10

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