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RELATIONS OF L-REGULAR FUNCTIONS ON QUATERNIONS IN CLIFFORD ANALYSIS

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Abstract. In this paper, we provide some properties of several left regular functions in Clifford analysis. We find the corresponding Cauchy-Riemann system and conjugate harmonic functions of the harmonic functions, for each left regular function in the sense of several complex variables. And we investigate certain properties of generalized quaternions in Clifford analysis.

1. Introduction

In 1971, Naser [7] has given some properties with respect to regular (hyperholomorphic) functions over the quaternion field $\mathcal{T}$. Naser [7] obtained some theorems in the using quaternionic differential operator $D^* = \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}}$, where e_2 is a base of $\mathcal{T}$. Also Naser [7] obtained properties of regular functions in Clifford analysis and conjugate harmonic functions of quaternion variables. In 2010, Koriyama et al. [6] have given regularities of quaternionic functions and corresponding Cauchy-Riemann system for each differential operator. In 2013, Kim et al. [4, 5] have given some properties of regular functions valued ternary number and reduced quaternion. Jung and Shon [1] investigated properties of hyperholomorphic function on dual ternary numbers and Jung et al. [2] investigated structures of hyperholomorphic function on dual quaternion numbers. And Kim and Shon [3] have given some properties of regular functions for dual split quaternions in Clifford analysis. In this paper, we investigate corresponding Cauchy-Riemann system for each quaternionic differential operator and properties of regular functions for generalized quaternions.

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2. Notations on quaternion

The quaternion field $\mathcal{T}$ over the real field $\mathbb{R}$ is generated by a basis $\{e_0, e_1, e_2, e_3\}$,

$$\mathcal{T} = \{z \mid z = \sum_{j=0}^{3} e_j x_j,\ x_j \in \mathbb{R}\ (j = 0,\ 1,\ 2,\ 3)\} \approx \mathbb{C}^2 \approx \mathbb{R}^4. \tag{1}$$

The quaternion $z = \sum_{j=0}^{3} e_j x_j$ is an element of four dimensional skew field of real numbers such that all base elements e_0, e_1, e_2 and e_3 satisfy the followings:

$$e_0 = id,\ e_1^2 = e_2^2 = e_3^2 = -1\ (e_1 = \sqrt{-1}),$$

$$e_1e_2 = -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = e_1,\ e_3e_1 = -e_1e_3 = e_2. \tag{2}$$

A quaternion z is denoted by

$$z = z_1 + z_2e_2$$

by putting $z_1 = x_0 + e_1x_1$ and $z_2 = x_2 + e_1x_3$. The quaternionic conjugate z^* and the absolute value $|z|$ of z are defined by

$$z^* = x_0 - \sum_{j=0}^{3} e_j x_j = \overline{z_1} - z_2e_2,$$

$$|z|^2 = zz^* = \sum_{j=0}^{3} x_j^2.$$

And every non-zero quaternion z of $\mathcal{T}$ has a unique inverse $z^{-1} = z^*/|z|^2$. Let Ω be a bounded open set in $\mathbb{C}^2$ and a function $f : \Omega \to \mathcal{T}$ is expressed by

$$\begin{aligned} f(z) &= \sum_{j=1}^{3} e_j u_j(x_0, x_1, x_2, x_3) \\ &= f_1(z_1, z_2) + f_2(z_1, z_2)e_2, \end{aligned}$$

where u_j $(j = 0,\ 1,\ 2,\ 3)$ are real valued functions and $f_1(z_1, z_2),\ f_2(z_1, z_2)$ are complex valued functions of two complex variables.

We use the following quaternionic differential operators

$$\begin{aligned} D &:= \frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}(\frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}), \\ D^* &= \frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3}), \end{aligned}$$

where $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z_j}}$ $(j = 1,\ 2)$ are usual complex differential operators. Since $\mathcal{T}$ given by (1) and (2) is a non-commutative field, we have

$$\frac{\partial}{\partial z_1}e_2 = e_2\frac{\partial}{\partial \overline{z_1}},\ \ \frac{\partial}{\partial \overline{z_1}}e_2 = e_2\frac{\partial}{\partial z_1}.$$

Let Ω be a bounded open set in $\mathbb{C}^2$. A function $f(z) = f_1(z) + f_2(z)e_2$ is said to be a *L-regular function* in Ω, if (i) $f \in C^1(\Omega)$ and (ii) $D^*f = 0$ in Ω.

The above equation (ii) is equivalent to the following system:

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial z_2}. \tag{3}$$

This system (3) is called a corresponding Cauchy-Riemann system in $\mathcal{T}$.

We consider the following quaternionic differential operators:

$$\begin{aligned}
D_1 &:= \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2}, \quad D_1^* = \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial z_2}, \\
D_2 &:= \frac{\partial}{\partial \overline{z_1}} - e_2 \frac{\partial}{\partial z_2}, \quad D_2^* = \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2}, \\
D_3 &:= \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial \overline{z_2}}, \quad D_3^* = \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}}, \\
D_4 &:= \frac{\partial}{\partial \overline{z_1}} - e_2 \frac{\partial}{\partial \overline{z_2}}, \quad D_4^* = \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial \overline{z_2}}.
\end{aligned} \tag{4}$$

Definition 1. Let Ω be a bounded open set in $\mathbb{C}^2$. A function f is said to be a L_j*-regular function* in Ω for $j = 1, \ 2, \ 3, \ 4$, if
(a) $f \in C^1(\Omega)$,
(b) $D_j^* f = 0$ in $\Omega \quad (j = 1, \ 2, \ 3, \ 4)$.

The L-regular function in Ω is regarded as the L_3-regular function in Ω. We denote that $f \in L_j(\Omega)$ if the function is a L_j-regular function in Ω where $j = 1, \ 2, \ 3, \ 4$.

Remark 1. In the condition (b) of Definition 1, D_j^* $(j = 1, \ 2, \ 3, \ 4)$ operate to f by multiplication of the quaternion,

$$D_1^* f = (\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial \overline{z_2}}) + (\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial \overline{z_2}}) e_2 = 0, \ D_2^* f = (\frac{\partial f_1}{\partial z_1} - \frac{\partial \overline{f_2}}{\partial \overline{z_2}}) + (\frac{\partial f_2}{\partial z_1} + \frac{\partial \overline{f_1}}{\partial \overline{z_2}}) e_2 = 0,$$

$$D_3^* f = (\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial z_2}) + (\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2}) e_2 = 0, \ D_4^* f = (\frac{\partial f_1}{\partial z_1} - \frac{\partial \overline{f_2}}{\partial z_2}) + (\frac{\partial f_2}{\partial z_1} + \frac{\partial \overline{f_1}}{\partial z_2}) e_2 = 0.$$

Remark 2. Each corresponding Cauchy-Riemann system satisfying the above equations in $\mathcal{T}$ are

$$\begin{aligned}
&\text{if } \ j = 1, \ \frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}}, \ \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial \overline{z_2}}, \\
&\text{if } \ j = 2, \ \frac{\partial f_1}{\partial z_1} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}}, \ \frac{\partial f_2}{\partial z_1} = -\frac{\partial \overline{f_1}}{\partial \overline{z_2}}, \\
&\text{if } \ j = 3, \ \frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial z_2}, \\
&\text{if } \ j = 4, \ \frac{\partial f_1}{\partial z_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial z_1} = -\frac{\partial \overline{f_1}}{\partial z_2}.
\end{aligned} \tag{5}$$

Theorem 2.1. *If f is a L_1-regular function on Ω, then the system is a generalized Cauchy-Riemann system,*

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}},\ \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial \overline{z_2}}.$$

Proof. Let f be a function defined by

$$\begin{aligned} f(z) &= f_1(z_1, z_2) + f_2(z_1, z_2)e_2 \\ &= u_0(x_0, x_1, x_2, x_3) + e_1 u_1(x_0, x_1, x_2, x_3) \\ &\quad +(u_2(x_0, x_1, x_2, x_3) + e_1 u_3(x_0, x_1, x_2, x_3))e_2, \end{aligned}$$

where u_j are real valued functions for $j = 0,\ 1,\ 2,\ 3$ on Ω. Suppose that

$$D_1^* f = (\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial \overline{z_2}}) + (\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial \overline{z_2}})e_2 = 0.$$

Hence $D_1^* f = 0$ is equivalent to the system:

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}},\ \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial \overline{z_2}}.$$

Now, we prove that each u_j is harmonic function for $j = 0,\ 1,\ 2,\ 3$. Since

$$\begin{aligned} \frac{\partial f_1}{\partial \overline{z_1}} &= (\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1})(u_0 + e_1 u_1) \\ &= \frac{\partial u_0}{\partial x_0} + e_1 \frac{\partial u_1}{\partial x_0} + e_1 \frac{\partial u_0}{\partial x_1} - \frac{\partial u_1}{\partial x_1} = (\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1}) + e_1(\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1}), \\ \frac{\partial \overline{f_2}}{\partial \overline{z_2}} &= (\frac{\partial}{\partial x_2} + e_1 \frac{\partial}{\partial x_3})(u_2 - e_1 u_3) \\ &= \frac{\partial u_2}{\partial x_2} - e_1 \frac{\partial u_3}{\partial x_2} + e_1 \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_3} = (\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}) + e_1(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}), \\ \frac{\partial f_2}{\partial \overline{z_1}} &= (\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1})(u_2 + e_1 u_3) \\ &= \frac{\partial u_2}{\partial x_0} + e_1 \frac{\partial u_3}{\partial x_0} + e_1 \frac{\partial u_2}{\partial x_1} - \frac{\partial u_3}{\partial x_1} = (\frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1}) + e_1(\frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1}), \\ \frac{\partial \overline{f_1}}{\partial \overline{z_2}} &= (\frac{\partial}{\partial x_2} + e_1 \frac{\partial}{\partial x_3})(u_0 - e_1 u_1) \\ &= \frac{\partial u_0}{\partial x_2} - e_1 \frac{\partial u_1}{\partial x_2} + e_1 \frac{\partial u_0}{\partial x_3} + \frac{\partial u_1}{\partial x_3} = (\frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3}) + e_1(\frac{\partial u_0}{\partial x_3} - \frac{\partial u_1}{\partial x_2}), \end{aligned}$$

we have the following equations

$$\begin{aligned} &\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3},\ \frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} = \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}, \\ &\frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial x_0} = \frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3},\ \frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_0}{\partial x_3}. \end{aligned} \tag{6}$$

For u_0, from (6) we have the following equations:

$$\frac{\partial^2 u_0}{\partial x_0 \partial x_0} - \frac{\partial^2 u_1}{\partial x_1 \partial x_0} = \frac{\partial^2 u_2}{\partial x_2 \partial x_0} + \frac{\partial^2 u_3}{\partial x_3 \partial x_0}, \tag{7}$$

$$\frac{\partial^2 u_1}{\partial x_0 \partial x_1} + \frac{\partial^2 u_0}{\partial x_1 \partial x_1} = \frac{\partial^2 u_2}{\partial x_3 \partial x_1} - \frac{\partial^2 u_3}{\partial x_2 \partial x_1}, \tag{8}$$

$$\frac{\partial^2 u_3}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_0 \partial x_2} = \frac{\partial^2 u_0}{\partial x_2 \partial x_2} + \frac{\partial^2 u_1}{\partial x_3 \partial x_2}, \tag{9}$$

$$\frac{\partial^2 u_3}{\partial x_0 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3} = \frac{\partial^2 u_1}{\partial x_2 \partial x_3} - \frac{\partial^2 u_0}{\partial x_3 \partial x_3}. \tag{10}$$

By adding (7) + (8) − (9) + (10), we have

$$\frac{\partial^2 u_0}{\partial x_0 \partial x_0} + \frac{\partial^2 u_0}{\partial x_1 \partial x_1} = -\frac{\partial^2 u_0}{\partial x_2 \partial x_2} - \frac{\partial^2 u_0}{\partial x_3 \partial x_3}.$$

Thus,

$$\triangle\, u_0 = 0,$$

where $\triangle$ is a Laplacian operator in $\mathbb{R}^4$. Hence, u_0 is a harmonic function on Ω. Similarly, we can prove that u_j $(j = 1,\ 2,\ 3)$ are harmonic in Ω. □

We use the following theorem proved by Naser [7].

Theorem 2.2. ([7]) *For any complex harmonic function $f_1(z)$ in a domain of holomorphy $\Omega \subset \mathbb{C}^2$, we can find a function $f_2(z)$ so that $f(z) = f_1(z) + f_2(z)e_2$ is a regular function in Ω.*

We consider the complex harmonic function defined by $f_1 = \overline{z_1}/|z|^4$ in a domain of holomorphy $\Omega \subset \mathbb{C}^2$. Then we can find $f_2 = -\overline{z_2}/|z|^4$ as conjugate harmonic function of $f_1(z)$ in Ω and show $f = f_1 + f_2 e_2$ is a L_3-regular function. Similarly, for a quaternionic function in a domain of holomorphy, if we know complex valued harmonic function f_1, then by the Theorem 2.2, we can find conjugate harmonic function f_2 of f_1 such that f is L_j-regular for $j = 1,\ 2,\ 4$.

Example 2.3. Let Ω be a bounded open set in $\mathbb{C}^2$ and $f \in L_j(\Omega)$, where $j = 1,\ 2,\ 4$.
(a) The function $f(z) = (\overline{z_1} - z_2 e_2)/|z|^4$ is a L_1-regular function.
(b) The function $f(z) = (z_1 - z_2 e_2)/|z|^4$ is a L_2-regular function.
(c) The function $f(z) = (z_1 - \overline{z_2} e_2)/|z|^4$ is a L_4-regular function.

Example 2.4. Let Ω be a bounded open set in $\mathbb{C}^2$ and $f \in L_j(\Omega)$, where $j = 1,\ 2,\ 3,\ 4$.
(a) The function $f(z) = e^{z_1} \cos \overline{z_2} + (e^{z_1} \sin \overline{z_2})e_2$ is L_1-regular.
(b) The function $f(z) = e^{\overline{z_1}} \cos \overline{z_2} + (e^{\overline{z_1}} \sin \overline{z_2})e_2$ is L_2-regular.
(c) The function $f(z) = e^{z_1} \cos z_2 + (e^{z_1} \sin z_2)e_2$ is L_3-regular.
(d) The function $f(z) = e^{\overline{z_1}} \cos z_2 + (e^{\overline{z_1}} \sin z_2)e_2$ is L_4-regular.

Example 2.5. The function $f(z) = z^2$ is not L_j-regular, where $j = 1,\ 2,\ 3,\ 4$. In fact, $z^2 = (z_1 + z_2e_2)^2 = (z_1z_1 - z_2\overline{z_2}) + (z_1z_2 + z_2\overline{z_1})e_2$ and the function f does not satisfy any corresponding Cauchy-Riemann system.

3. Properties of regular functions on $\mathcal{T}$

Naser [7] proved the following corresponding Cauchy theorem.

Theorem 3.1. ([7]) *Let Ω be a open set in $\mathbb{C}^2$. If a function $f = f_1 + f_2e_2$ is L_3-regular in Ω, then for a domain $D \subset \Omega$ with smooth boundary ∂D,*

$$\int_{\partial D} \kappa f = 0,$$

where $\kappa = dz_1 \wedge dz_2 \wedge d\overline{z_2} - dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}e_2$.

The quaternion form κ given by Theorem 3.1 is called a kernel for the corresponding Cauchy theorem. We find kernels κ_j for corresponding Cauchy theorems for all differential operators given by (4) as follows:

$$\kappa_1 = dz_1 \wedge dz_2 \wedge d\overline{z_2} - dz_1 \wedge dz_2 \wedge d\overline{z_1}e_2,$$
$$\kappa_2 = dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} - dz_1 \wedge dz_2 \wedge d\overline{z_1}e_2,$$
$$\kappa_3 = dz_1 \wedge dz_2 \wedge d\overline{z_2} - dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}e_2,$$
$$\kappa_4 = dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} - dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}e_2.$$

Theorem 3.2. *Let κ_j $(j = 1, 2, 3, 4)$ be kernels for the corresponding Cauchy theorem for each differential operator and Ω be a bounded open set in $\mathbb{C}^2$. If a function f is L_j-regular in Ω, then for a domain $D \subset \Omega$ with smooth boundary ∂D,*

$$\int_{\partial D} \kappa_j f = 0 \ \ (j = 1,\ 2,\ 3,\ 4), \tag{11}$$

where $\kappa_j f$ is the quaternion product of form (11) *upon the function f.*

Proof. In the case of $j = 1$, by the rule of the quaternionic multiplication, we have

$$\kappa_1 f = f_1dz_1\wedge dz_2\wedge d\overline{z_2}+\overline{f_2}dz_1\wedge dz_2\wedge d\overline{z_1}+(f_2dz_1\wedge dz_2\wedge d\overline{z_2}-\overline{f_1}dz_1\wedge dz_2\wedge d\overline{z_1})e_2.$$

Thus,

$$\begin{aligned} d(\kappa_1 f) \quad = \quad & (\partial f_1/\partial\overline{z_1} - \partial\overline{f_2}/\partial\overline{z_2})dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} \\ & +(\partial f_2/\partial\overline{z_1} + \partial\overline{f_1}/\partial\overline{z_2})dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}e_2. \end{aligned}$$

Since f is L_1-regular, from the corresponding Cauchy-Riemann system (5), we have $d(\kappa_1 f) = 0$. By Stokes' theorem, we obtain

$$\int_{\partial D} \kappa_1 f = \int_D d(\kappa_1 f) = 0.$$

Similarly, in the cases of $j = 2,\ 3,\ 4$, we can have the results. □

4. Generalized quaternion

Now we define a generalized quaternion $z = x_0 + e_1x_1 + e_2x_2 + e_3x_3$. For non-zero Euclidean numbers α and β, we put e_1, e_2 and e_3 such that

$$e_1^2 = -\alpha,\ e_2^2 = -\beta,\ e_3^2 = -\alpha\beta.$$

These base e_1, e_2 and e_3 satisfy followings:

$$e_1e_2 = -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = \beta e_1,\ e_3e_1 = -e_1e_3 = \alpha e_2.$$

Then the quaternion z can be denoted by $z_1 + z_2e_2$, where $z_1 = x_0 + e_1x_1$ and $z_2 = x_2 + e_1x_3$ as before. The quaternionic conjugate z^* also can be defined by $z^* = x_0 - e_1x_1 - e_2x_2 - e_3x_3 = \overline{z_1} - z_2e_2$. The absolute value $|z|$ is defined by

$$|z|^2 = zz^* = x_0^2 + \alpha x_1^2 + \beta x_2^2 + \alpha\beta x_3^2.$$

Every non-zero generalized quaternion z has a unique inverse $z^{-1} = z^*/|z|^2$.

We use the following generalized quaternionic differential operators:

$$D := \frac{1}{2}(\frac{\partial}{\partial x_0} - \alpha^{-1}e_1\frac{\partial}{\partial x_1} - \beta^{-1}e_2\frac{\partial}{\partial x_2} + (\alpha^{-1}\beta^{-1})e_3\frac{\partial}{\partial x_3}) = \frac{\partial}{\partial z_1} - \beta^{-1}e_2\frac{\partial}{\partial \overline{z_2}},$$
$$D^* = \frac{1}{2}(\frac{\partial}{\partial x_0} + \alpha^{-1}e_1\frac{\partial}{\partial x_1} + \beta^{-1}e_2\frac{\partial}{\partial x_2} - (\alpha^{-1}\beta^{-1})e_3\frac{\partial}{\partial x_3}) = \frac{\partial}{\partial \overline{z_1}} + \beta^{-1}e_2\frac{\partial}{\partial \overline{z_2}},$$

where $\frac{\partial}{\partial z_1} = \frac{1}{2}(\frac{\partial}{\partial x_0} - \alpha^{-1}e_1\frac{\partial}{\partial x_1})$ and $\frac{\partial}{\partial z_2} = \frac{1}{2}(\frac{\partial}{\partial x_2} - \alpha^{-1}e_1\frac{\partial}{\partial x_3})$. Let Ω be a bounded open set in $\mathbb{C}^2$. A function $f(z) = f_1(z) + f_2(z)e_2$ is said to be a *L-regular function* in Ω, if (i) $f \in C^1(\Omega)$ and (ii) $D^*f = 0$ in Ω.

The above equation (ii) is equivalent to the following system:

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial z_2},\ \frac{\partial f_2}{\partial \overline{z_1}} = -\beta^{-1}\frac{\partial \overline{f_1}}{\partial z_2}. \tag{12}$$

This system (12) is called a corresponding Cauchy-Riemann system in generalized quaternionic field. We consider the following quaternionic differential operators:

$$D_1 := \frac{\partial}{\partial z_1} - \beta^{-1}e_2\frac{\partial}{\partial z_2},\ \ D_1^* = \frac{\partial}{\partial \overline{z_1}} + \beta^{-1}e_2\frac{\partial}{\partial z_2},$$
$$D_2 := \frac{\partial}{\partial \bar{z_1}} - \beta^{-1}e_2\frac{\partial}{\partial z_2},\ \ D_2^* = \frac{\partial}{\partial z_1} + \beta^{-1}e_2\frac{\partial}{\partial z_2},$$
$$D_3 := \frac{\partial}{\partial z_1} - \beta^{-1}e_2\frac{\partial}{\partial \overline{z_2}},\ \ D_3^* = \frac{\partial}{\partial \overline{z_1}} + \beta^{-1}e_2\frac{\partial}{\partial \overline{z_2}},$$
$$D_4 := \frac{\partial}{\partial \overline{z_1}} - \beta^{-1}e_2\frac{\partial}{\partial \overline{z_2}},\ \ D_4^* = \frac{\partial}{\partial z_1} + \beta^{-1}e_2\frac{\partial}{\partial \overline{z_2}},$$

where $\frac{\partial}{\partial z_1} = \frac{1}{2}(\frac{\partial}{\partial x_0} - \alpha^{-1}e_1\frac{\partial}{\partial x_1})$ and $\frac{\partial}{\partial z_2} = \frac{1}{2}(\frac{\partial}{\partial x_2} - \alpha^{-1}e_1\frac{\partial}{\partial x_3})$.

Definition 2. Let Ω be a bounded open set in $\mathbb{C}^2$. A function f is said to be a *L_j-regular function* in Ω for $j = 1,\ 2,\ 3,\ 4$, if
(a) $f \in C^1(\Omega)$,
(b) $D_j^*f = 0$ in Ω (j=1, 2, 3, 4).

As before, the $L-$regular function in Ω is regarded as the L_3-regular function in generalized quaternionic field.

Remark 3. For each differential operator D_j^* $(j = 1,\ 2,\ 3,\ 4)$, corresponding Cauchy-Riemann system is equivalent to the followings:

$$\begin{aligned} &\text{if } j=1, \quad \frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}}, \ \frac{\partial f_2}{\partial \overline{z_1}} = -\beta^{-1}\frac{\partial \overline{f_1}}{\partial \overline{z_2}}, \\ &\text{if } j=2, \quad \frac{\partial f_1}{\partial z_1} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}}, \ \frac{\partial f_2}{\partial z_1} = -\beta^{-1}\frac{\partial \overline{f_1}}{\partial \overline{z_2}}, \\ &\text{if } j=3, \quad \frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial \overline{z_1}} = -\beta^{-1}\frac{\partial \overline{f_1}}{\partial z_2}, \\ &\text{if } j=4, \quad \frac{\partial f_1}{\partial z_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial z_1} = -\beta^{-1}\frac{\partial \overline{f_1}}{\partial z_2} \end{aligned} \tag{13}$$

for $\beta \in \mathbb{R}$.

Now we find kernels for the corresponding Cauchy theorem and prove the corresponding Cauchy theorem for generalized quaternion.

Theorem 4.1. *Let κ_j $(j = 1, 2, 3, 4)$ be kernels for the corresponding Cauchy theorem for each differential operator and Ω be a bounded open set in $\mathbb{C}^2$. If a function f is L_j-regular in Ω, then for a domain $D \subset \Omega$ with smooth boundary ∂D,*

$$\int_{\partial D} \kappa_j f = 0 \ \ (j = 1, 2, 3, 4),$$

where

$$\begin{aligned} \kappa_1 &= dz_1 \wedge dz_2 \wedge d\overline{z_2} - \beta^{-1} dz_1 \wedge dz_2 \wedge d\overline{z_1} e_2, \\ \kappa_2 &= dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} - \beta^{-1} dz_1 \wedge dz_2 \wedge d\overline{z_1} e_2, \\ \kappa_3 &= dz_1 \wedge dz_2 \wedge d\overline{z_2} - \beta^{-1} dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} e_2, \\ \kappa_4 &= dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} - \beta^{-1} dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} e_2 \end{aligned}$$

for $\beta \in \mathbb{R}$.

Proof. In the case of $j = 1$, by the rule of the generalized quaternionic multiplication, we have

$$\begin{aligned} \kappa_1 f &= (\kappa_1 = dz_1 \wedge dz_2 \wedge d\overline{z_2} - \beta^{-1} dz_1 \wedge dz_2 \wedge d\overline{z_1} e_2)(f_1 + f_2 e_2) \\ &= f_1 dz_1 \wedge dz_2 \wedge d\overline{z_2} + \overline{f_2} dz_1 \wedge dz_2 \wedge d\overline{z_1} \\ &\quad +(f_2 dz_1 \wedge dz_2 \wedge d\overline{z_2} - \beta^{-1}\overline{f_1} dz_1 \wedge dz_2 \wedge d\overline{z_1}) e_2 \end{aligned}$$

for $\beta \in \mathbb{R}$. Thus,

$$\begin{aligned} d(\kappa_1 f) &= (\partial f_1/\partial \overline{z_1} - \partial \overline{f_2}/\partial \overline{z_2}) dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} \\ &\quad +(\partial f_2/\partial \overline{z_1} + \beta^{-1}\partial \overline{f_1}/\partial \overline{z_2}) dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} e_2. \end{aligned}$$

Since f is L_1-regular, from the corresponding Cauchy-Riemann system (13), we have $d(\kappa_1 f) = 0$. By Stokes' theorem, we obtain

$$\int_{\partial D} \kappa_1 f = \int_D d(\kappa_1 f) = 0.$$

Similarly, in cases of $j = 2,\ 3,\ 4$, we can have the results. □

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