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A FIXED POINT APPROACH TO THE ORTHOGONAL STABILITY OF MIXED TYPE FUNCTIONAL EQUATIONS

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Abstract. In this paper, we investigate the following orthogonally additive-quadratic functional equation

$$f(2x+y)-f(x+2y)-f(x+y)-f(y-x)-f(x)+f(y)+f(2y)=0.$$

and prove the generalized Hyers-Ulam stability for it in orthogonality spaces by using the fixed point method.

1. Introduction

Assume that X is a real inner product space and $f : X \longrightarrow \mathbb{R}$ is a solution of the orthogonally Cauchy functional equation $f(x+y) = f(x)+f(y)$, $< x, y >= 0$. By the Pythagorean theorem, $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus, orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

The orthogonally Cauchy functional equation

$$f(x+y) = f(x) + f(y), \quad x \perp y$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [6]. Rätz [15] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [16] investigated the problem in a rather more general framework.

Definition 1. [16] Let X be a real vector space with $\dim X \geq 2$ and $\perp$ a binary relation on X with the following properties:

(O1) totality of $\perp$ for zero: $x \perp 0$ and $0 \perp x$ for all $x \in X$;
(O2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;
(O3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

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(O4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and a non-negative real number k, then there exists an $y \in P$ such that $x \perp y$ and $x + y \perp kx - y$.
The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space having a normed structure.

Remark 1. (i) The trivial orthogonality on a vector space X defined by (O1) and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, < \cdot, \cdot >)$ given by $x \perp y$ if and only if $< x, y >= 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, \| \cdot \|)$ defined by $x \perp y$ if and only if $\|x + ky\| \geq \|x\|$ for all $k \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Then clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

In 1940, S. M. Ulam proposed the following stability problem (cf. [18]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [7] gave a partial solution of Ulam's problem for the case of approximate additive mappings. In 1978, Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy difference. The result of Rassias has provided a lot of influence in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam stability of functional equations. Ger and Sikorska [5] investigated the orthogonal stability of the Cauchy functional equation

$$f(x + y) = f(x) + f(y), \ \ x \perp y \tag{1}$$

and Vajzović [19] investigated the orthogonally additive-quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \ \ x \perp y \tag{2}$$

when X is a Hilbert space, Y is a scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, many mathematicians have investigated the orthogonal stability of functional equations ([3], [4], [10], [11], [17], [9], [12], and [13]).

In this paper, we deal with the following functional equation

$$f(2x+y) - f(x+2y) - f(x+y) - f(y-x) - f(x) + f(y) + f(2y) = 0, \ \ x \perp y. \tag{3}$$

It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of (3).

1.1. Orthogonal stability for (3)

Let $(X, \perp)$ be an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ be a Banach space. For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x,y) = f(2x+y) - f(x+2y) - f(x+y) - f(y-x) - f(x) + f(y) + f(2y)$$

for all $x, y \in X$ and let

$$f_o(x) = \frac{f(x) - f(-x)}{2}, \quad f_e(x) = \frac{f(x) + f(-x)}{2}.$$

In this section, we prove the generalized Hyers-Ulam stability for the orthogonal additive-quadratic functional equation (3) by using the fixed point method.

Lemma 1.1. *Suppose that $f : X \longrightarrow Y$ is a solution of the functional equation (3). Then f is an orthogonally additive-quadratic mapping.*

Proof. Clearly, $f(0) = 0$. Letting $y = 0$ in (3), we have

$$f(2x) = 3f(x) + f(-x) \tag{4}$$

for all $x \in X$. By (3) and (4), we have

$$f_o(2x+y) - f_o(x+2y) - f_o(x+y) + f_o(x-y) - f_o(x) + 3f_o(y) = 0 \tag{5}$$

for all $x, y \in X$ with $x \perp y$. Since $\perp$ is symmetric, interchanging x and y in (5), we have

$$f_o(x+2y) - f_o(2x+y) - f_o(x+y) - f_o(x-y) - f_o(y) + 3f_o(x) = 0 \tag{6}$$

for all $x, y \in X$ with $x \perp y$. By (5) and (6), f is an orthogonally additive mapping.

By (3) and (4), we have

$$f_e(2x+y) - f_e(x+2y) - f_e(x+y) - f_e(x-y) - f_e(x) + 5f_e(y) = 0 \tag{7}$$

for all $x, y \in X$ with $x \perp y$. Since $\perp$ is symmetric, interchanging x and y in (7), we have

$$f_e(x+2y) - f_e(2x+y) - f_e(x+y) - f_e(x-y) - f_e(y) + 5f_e(x) = 0 \tag{8}$$

for all $x, y \in X$ with $x \perp y$. By (7) and (8), f_e is an orthogonally quadratic mapping. Hence $f = f_o + f_e$ is an orthogonally additive-quadratic mapping. □

In 1996, Isac and Rassias [8] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 1.2. [1], [2] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ and

(4) $d(y, y^*) \leq \dfrac{1}{1-L}\ d(y, Jy)$ *for all* $y \in Y$.

Theorem 1.3. *Assume that* $\phi : X^2 \longrightarrow [0, \infty)$ *is a function such that*

$$\phi(x, y) \leq \frac{L}{4}\phi(2x, 2y) \tag{9}$$

for all $x, y \in X$ *with* $x \perp y$ *and some* L *with* $0 < L < 1$. *Let* $f : X \longrightarrow Y$ *be a mapping such that* $f(0) = 0$ *and*

$$\|Df(x, y)\| \leq \phi(x, y) \tag{10}$$

for all $x, y \in X$ *with* $x \perp y$. *Then there exists a unique orthogonally additive-quadratic mapping* $F : X \longrightarrow Y$ *such that*

$$\|f(x) - F(x)\| \leq \frac{(4-3L)L}{4(1-L)(2-L)}\psi(x, 0) \tag{11}$$

for all $x \in X$, *where* $\psi(x, y) = \frac{1}{2}[\phi(x, y) + \phi(-x, -y)]$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and define the generalized metric d on S by

$$d(g, h) = \inf\{c \in [0, \infty) \mid \|g(x) - h(x)\| \leq c\ \psi(x, 0), \forall x \in X\}.$$

Then (S, d) is a complete metric space([9]). Define a mapping $T_o : S \longrightarrow S$ by $T_o g(x) = 2g(\frac{x}{2})$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (9), we have

$$\|T_o g(x) - T_o h(x)\| = 2\|g(\frac{x}{2}) - h(\frac{x}{2})\| \leq c\frac{L}{2}\psi(x, 0)$$

for all $x \in X$. Hence we have $d(T_o g, T_o h) \leq \frac{L}{2} d(g, h)$ for all $g, h \in S$ and so T_o is a strictly contractive mapping. By (10) and (O3), we get

$$\|Df_o(x, y)\| \leq \psi(x, y) \tag{12}$$

for all $x, y \in X$ with $x \perp y$. Putting $y = 0$ in (12), we get

$$\|f_o(2x) - 2f_o(x)\| \leq \psi(x, 0)$$

for all $x \in X$ and hence

$$\|f_o(x) - 2f_o(\frac{x}{2})\| \leq \frac{L}{4}\psi(x, 0)$$

for all $x \in X$ and hence $d(f_o, T_o f_o) \leq \frac{L}{4} < \infty$. By Theorem 1.2, there exists a mapping $A : X \longrightarrow Y$ which is a fixed point of T_o such that $d(T_o^n f_o, A) \to 0$ as $n \to \infty$ and

$$\|A(x) - f_o(x)\| \leq \frac{L}{2(2-L)}\psi(x, 0) \tag{13}$$

for all $x \in X$. Replacing x, y by $\frac{x}{2^n}$, $\frac{y}{2^n}$ in (12), respectively, and multiplying (12) by 2^n, by (O3), we have

$$\|2^n Df_o(\frac{x}{2^n}, \frac{y}{2^n})\| \leq \frac{L^n}{2^n}\psi(x, y)$$

for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DA(x, y) = 0 \tag{14}$$

for all $x, y \in X$ with $x \perp y$.

Define a mapping $T_e : S \longrightarrow S$ by $T_e g(x) = 4g(\frac{x}{2})$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (9), we have

$$\|T_e g(x) - T_e h(x)\| = 4\|g(\frac{x}{2}) - h(\frac{x}{2})\| \leq cL\psi(x, 0)$$

for all $x \in X$. Hence we have $d(T_e g, T_e h) \leq Ld(g, h)$ for all $g, h \in S$ and so T_e is a strictly contractive mapping. By (10) and (O3), we get

$$\|Df_e(x, y)\| \leq \psi(x, y) \tag{15}$$

for all $x, y \in X$ with $x \perp y$. Putting $y = 0$ in (15), we get

$$\|f_e(2x) - 4f_e(x)\| \leq \psi(x, 0)$$

for all $x \in X$ and hence

$$\|f_e(x) - 4f_e(\frac{x}{2})\| \leq \frac{L}{4}\psi(x, 0)$$

for all $x \in X$ and hence $d(f_e, T_e f_e) \leq \frac{L}{4} < \infty$. By Theorem 1.2, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of T_e such that $d(T_e^n f_e, Q) \to 0$ as $n \to \infty$ and

$$\|Q(x) - f_e(x)\| \leq \frac{L}{4(1 - L)}\psi(x, 0) \tag{16}$$

for all $x \in X$. Replacing x, y by $\frac{x}{2^n}$, $\frac{y}{2^n}$ in (15), respectively, and multiplying (15) by 2^n, by (O3), we have

$$\|4^n Df_e(\frac{x}{2^n}, \frac{y}{2^n})\| \leq L^n\psi(x, y)$$

for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DQ(x, y) = 0 \tag{17}$$

for all $x, y \in X$ with $x \perp y$.

Let $F = A + Q$. Then we can easily show that A is odd and Q is even. Hence by (14) and (17), F is a solution of (3). By Lemma 1.1, F is an orthogonally additive-quadratic mapping. Since $\|F - f\| \leq \|A - f_o\| + \|Q - f_e\|$ and hence we have (11).

Now, we will show the uniqueness of F. Let $G : X \longrightarrow Y$ be another orthogonally additive-quadratic mapping with (11). By (11), we get

$$\|G_o(x) - F_o(x))\| \leq \frac{1}{2}\|G(x) - F(x)\| + \frac{1}{2}\|G(-x) - F(-x)\| \leq \frac{(4 - 3L)L}{4(1 - L)(2 - L)}\psi(x, 0)$$

for all $x \in V$. Since F_o and G_o are fixed points of T_o, we have

$$\|G_o(x) - F_o(x)\| \le \|T_o^n G_o(x) - T_o^n F_o(x))\| \le \frac{4-3L}{4(1-L)(2-L)} L^{n+1}\psi(x,0)$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Hence $F_o = G_o$ and similarly, we have $F_e = G_e$. Thus $F = G$. □

Related with Theorem 1.3, we can also have the following theorem. And the proof is similar to that of Theorem 1.3.

Theorem 1.4. *Assume that $\phi : X^2 \longrightarrow [0,\infty)$ is a function such that*

$$\phi(2x, 2y) \le 2L\phi(x,y) \tag{18}$$

for all $x, y \in X$ with $x \perp y$ and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping satisfying $f(0) = 0$ and (10). Then there exists a unique orthogonally additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$\|f(x) - F(x)\| \le \frac{4-3L}{4(1-L)(2-L)}\psi(x,0) \tag{19}$$

for all $x \in X$, where $\psi(x,y) = \frac{1}{2}[\phi(x,y) + \phi(-x,-y)]$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and define the generalized metric d on S by

$$d(g,h) = \inf\{c \in [0,\infty) \mid \|g(x) - h(x)\| \le c\ \psi(x,0), \forall x \in X\}.$$

Then (S,d) is a complete metric space([9]). Define a mapping $T_o : S \longrightarrow S$ by $T_o g(x) = \frac{1}{2} g(2x)$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (18), we have

$$\|T_o g(x) - T_o h(x)\| = \frac{1}{2}\|g(2x) - h(2x)\| \le cL\psi(x,0)$$

for all $x \in X$. Hence we have $d(T_o g, T_o h) \le Ld(g,h)$ for all $g, h \in S$ and so T_o is a strictly contractive mapping. By (10) and (O3), we get

$$\|Df_o(x,y)\| \le \psi(x,y) \tag{20}$$

for all $x, y \in X$ with $x \perp y$. Putting $y = 0$ in (20), we get

$$\|f_o(2x) - 2f_o(x)\| \le \psi(x,0)$$

for all $x \in X$ and hence

$$\|f_o(x) - \frac{1}{2}f_o(2x)\| \le \frac{1}{2}\psi(x,0)$$

for all $x \in X$ and hence $d(f_o, T_o f_o) \le \frac{1}{2} < \infty$. By Theorem 1.2, there exists a mapping $A : X \longrightarrow Y$ which is a fixed point of T_o such that $d(T_o^n f_o, A) \to 0$ as $n \to \infty$ and

$$\|A(x) - f_o(x)\| \le \frac{1}{2(1-L)}\psi(x,0) \tag{21}$$

for all $x \in X$. Replacing x, y by $2^n x$, $2^n y$ in (20), respectively, and multiplying (12) by $\frac{1}{2^n}$, by (O3), we have

$$\|\frac{1}{2^n} Df_o(2^n x, 2^n y)\| \leq L^n \psi(x, y)$$

for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DA(x, y) = 0 \tag{22}$$

for all $x, y \in X$ with $x \perp y$.

Define a mapping $T_e : S \longrightarrow S$ by $T_e g(x) = \frac{1}{4} g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (9), we have

$$\|T_e g(x) - T_e h(x)\| = \frac{1}{4}\|g(2x) - h(2x)\| \leq c\frac{L}{2}\psi(x, 0)$$

for all $x \in X$. Hence we have $d(T_e g, T_e h) \leq \frac{L}{2} d(g, h)$ for all $g, h \in S$ and so T_e is a strictly contractive mapping. By (10) and (O3), we get

$$\|Df_e(x, y)\| \leq \psi(x, y) \tag{23}$$

for all $x, y \in X$ with $x \perp y$. Putting $y = 0$ in (23), we get

$$\|f_e(2x) - 4f_e(x)\| \leq \psi(x, 0)$$

for all $x \in X$ and hence

$$\|f_e(x) - \frac{1}{4} f_e(2x)\| \leq \frac{1}{4}\psi(x, 0)$$

for all $x \in X$ and hence $d(f_e, T_e f_e) \leq \frac{1}{4} < \infty$. By Theorem 1.2, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of T_e such that $d(T_e^n f_e, Q) \to 0$ as $n \to \infty$ and

$$\|Q(x) - f_e(x)\| \leq \frac{1}{2(2 - L)}\psi(x, 0) \tag{24}$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 1.3. □

As an example of $\phi(x, y)$ in Theorem 1.3 and Theorem 1.4, we can take $\phi(x, y) = \epsilon(\|x\|_X^p \|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p})$ for some positive real numbers ϵ and p. Then we can formulate the following corollary :

Corollary 1.5. *Let $(X, \perp)$ be an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ a Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\|Df(x, y)\| \leq \epsilon(\|x\|_X^p \|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p}). \tag{25}$$

for all $x, y \in X$ with $x \perp y$ and a fixed positive number p with $0 < p < \frac{1}{2}$ or $1 < p$. Then there exists a unique orthogonally additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \begin{cases} \frac{2^{2p+2} - 12}{(2^{2p} - 4)(2^{2p+1} - 4)}\epsilon\|x\|_X^{2p}, & \text{if } 1 < p \\ \\ \frac{8 - 3\cdot 2^{2p}}{2(2 - 2^{2p})(4 - 2^{2p})}\epsilon\|x\|_X^{2p}, & \text{if } 0 < p < \frac{1}{2}, \end{cases}$$

for all $x \in X$.

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