Korean J. Math. **23** (2015), No. 3, pp. 491–502
http://dx.doi.org/10.11568/kjm.2015.23.3.491

ON MARTINGALE PROPERTY OF THE STOCHASTIC INTEGRAL EQUATIONS

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Abstract. A martingale is a mathematical model for a fair wager and the modern theory of martingales plays a very important and useful role in the study of the stochastic fields. This paper is devoted to investigate a martingale and a non-martingale on the several stochastic integral or differential equations. Specially, we show that whether the stochastic integral equation involving a standard Wiener process with the associated filtration is or not a martingale.

1. Introduction

A differential equation in which one or more of the terms has a random component is called a stochastic differential equation. The stochastic differential equations are frequently used to model diverse applied fields. Generally speaking, the stochastic differential equations have continuous paths with both random and non-random components and to drive the random component of the model they usually incorporate a Wiener process. Before we discuss the models in depth, we first look at the definition and example of a Wiener process.

Definition 1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability measure space. Then a stochastic process $\{W_t : t \geqslant 0\}$ is called a standard Wiener process

Received September 6, 2015. Revised September 13, 2015. Accepted September 14, 2015.
2010 Mathematics Subject Classification: 28C20, 60H15, 60H20, 60G44, 65C30.
Key words and phrases: Martingale, Wiener Process, Stochastic Differential Equation, Stochastic Integral Equation.

if $W_0 = 0$ and $\{W_t : t \geqslant 0\}$ has continuous sample paths, a stationary increment and an independent increment.

A standard Wiener process is a standardized version of a Wiener process, which need not begin at $W_0 = 0$ and may have a non-zero drift term and a variance term not necessarily equal to t. The general definition of Wiener process is as follows.

Definition 1.2. A process $\{\widetilde{W}_t : t \geqslant 0\}$ is called a Wiener process if $\{\widetilde{W}_t : t \geqslant 0\}$ can be written as

$$\widetilde{W}_t = \xi + \mu t + \sigma W_t$$

where $\xi, \mu, \sigma \in \mathbb{R}$, $(\sigma > 0)$ and W_t is a standard Wiener process.

The Wiener process in fact is a stochastic process sharing the similar behaviour as Brownian motion([1],[6]). The Wiener process has a strong Markov property, which is an important result in establishing many other properties of Wiener processes such as martingales([7],[8],[10]). In general, the strong Markov property implies the Markov property but not vice versa([3],[4]). Once we have established the Markov property, we can use them to show that a Wiener process is a martingale. The following example shows this process. Consider the following symmetric random walk model

$$M_k = \sum_{i=1}^{k} Z_i \text{ starting } M_0\,,$$

where $P(Z_i = -1) = \frac{1}{2}$, $P(Z_i = 1) = \frac{1}{2}$ on a probability measure space $(\Omega, \mathcal{F}, P)$. Using the independent increment property, $E(M_k - M_j) = 0$ for $j < k$, $(j, k \in \mathbb{Z}^+)$,

$$\begin{aligned} E(M_k|\mathcal{F}_j) &= E(M_k - M_j + M_j\,|\mathcal{F}_j) \\ &= E(M_k - M_j\,|\mathcal{F}_j) + E(M_j\,|\mathcal{F}_j) \\ &= E(M_k - M_j) + M_j \\ &= M_j \end{aligned}$$

and

$$|M_k| \;=\; \left|\sum_{i=1}^{k} Z_i\right| \;\leqslant\; \sum_{i=1}^{k} |Z_i| \;=\; k \;<\; +\infty\,.$$

Moreover, M_k is clearly $\mathcal{F}_k$-adapted, it follows that M_k is a martingale. By the Donsker's theorem([8]), if

$$W_t^{(n)} = \frac{1}{\sqrt{n}} M_{[nt]} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} Z_i$$

is a symmetric random walk for a fixed time t, then we obtain the Wiener process where

$$\lim_{n\to\infty} W_t^{(n)} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} Z_i \xrightarrow{d} N(0,1)\,.$$

This example shows that the symmetric random walk satisfies the martingale property as a Wiener process and then the limiting distribution of the symmetric random walk is a standard normal distribution $N(0, 1)$, and also is giving explain the sequence of all process referred to in this paper.

As already stated in the abstract, this paper deals with martingale property of some Wiener process. The tool of the proof is the quadratic variation method and if the case can not use a quadratic variation, then we prove the martingale property using the stochastic integration method. In general, the quadratic variation of any stochastic process is not a easy thing to prove whether that satisfies the martingale property. Finally, we introduce an example using the results of theorem.

2. Martingale Property

Let $(\Omega, \mathcal{F}, P)$ be a probability measure space and $\{W_t : t \geqslant 0\}$ be a standard Wiener process. Then the following terms such as

$$W_t,\ W_t^2 - t,\ \exp\Big(\lambda W_t - \frac{1}{2}\lambda^2 t\Big)\ (\lambda \in \mathbb{R})$$

satisfy the martingale property([2],[6]). Using this results, we obtain the martingale property for the hyperbolic processes in the next theorem.

Theorem 2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability measure space and let $\{W_t : t \geqslant 0\}$ be a standard Wiener process. Then the hyperbolic processes*

$$X_t = \exp\Big(-\frac{1}{2}\lambda^2 t\Big)\cosh(\lambda W_t),\ \ Y_t = \exp\Big(-\frac{1}{2}\lambda^2 t\Big)\sinh(\lambda W_t),\ (\lambda \in \mathbb{R})$$

are a martingale, respectively.

Proof. We prove only the first formula. Then X_t can be expresses as

$$\begin{aligned} X_t &= \exp\Big(-\frac{1}{2}\lambda^2 t\Big)\cosh(\lambda W_t) \\ &= \frac{1}{2}\exp\Big(\lambda W_t - \frac{1}{2}\lambda^2 t\Big) + \frac{1}{2}\exp\Big(-\lambda W_t - \frac{1}{2}\lambda^2 t\Big),\ (\lambda \in \mathbb{R}) \\ &= A_t + B_t \end{aligned}$$

Since A_t and B_t are martingales, we have $E(A_t + B_t|\mathcal{F}_s) = A_s + B_s$ for $s < t$ and also $E(|A_t + B_t|) < +\infty$. And since X_t is a function of W_t, X_t is $\mathcal{F}_t$-adapted. As a result, X_t is a margingale. $\square$

An one-dimensional stochastic differential equation can be described as

$$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dW_t$$

where W_t is a standard Wiener process, $\mu(X_t, t)$ is the drift term and $\sigma(X_t, t)$ is the volatility term([5]). From the initial condition $X_0 = x_0$, it can be written as the form of

$$X_t = X_0 + \int_0^t \mu(X_s, s)\,ds + \int_0^t \sigma(X_s, s)\,dW_s\,,$$

where

$$\int_0^t (|\,\mu(X_s, s)\,| + |\,\sigma(X_s, s)\,|^2)\,ds < +\infty$$

and the solution of the stochastic integral equation is called an Itô's diffusion([5]). We consider the stochastic integral with respect to a Wiener process and write

$$I_t = \int_0^t f(W_s, s)\,dW_s$$

where the integrand $f(W_t, t)$ is $\mathcal{F}_t$-measurable and

$$E\Big(\int_0^t |f(W_s, s)|^2\,ds\Big) < +\infty \text{ for all } t \geqslant 0.$$

Let $t > 0$ be any constant. Suppose that $f(W_{t_i}, t_i)$ is a constant on the subinterval $[t_i, t_{i+1})$, where $t_i = \frac{t}{n}i$, $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ for $n \in \mathbb{N}$. Then we call such a process f a simple process. For a simple process, the stochastic integral I_t can be defined as

$$I_t = \int_0^t f(W_s, s)\,dW_s = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})\,.$$

Now we prove a martingale case and a non-martingale case on the several stochastic integral equations involing the Wiener process. One of the most useful property of a Wiener process is the quadratic variation. First, we give a related lemma that we need in the sequal.

LEMMA 2.1. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability measure space and let* $\{W_t : t \geqslant 0\}$ *be a standard Wiener process. Then the finite quadratic variation of* $\{W_t : t \geqslant 0\}$ *is given by*

$$\langle W, W\rangle_t \;=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} \big(W_{t_{i+1}} - W_{t_i}\big)^2 \;=\; t\,,$$

where $t_i = \frac{t}{n} i$, $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$ *for* $n \in \mathbb{N}$.

Proof. Since the quadratic variation is a sum of random variables, we need to show that its expected value is t and its variance converges to zero as $n \to \infty$. Let

$$\Delta W_{t_i} \;=\; W_{t_{i+1}} - W_{t_i} \;\sim\; N(0, \frac{t}{n})\,,$$

where $E(\Delta W_{t_i}^2) = \frac{t}{n}$, we have

$$E\bigg(\lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta W_{t_i}^2\bigg) \;=\; \lim_{n\to\infty} \sum_{i=0}^{\infty} E(\Delta W_{t_i}^2) \;=\; t\,.$$

Because $\Delta W_{t_i}^2/(t/n) \sim \chi^2(1)$, we have $E(\Delta W_{t_i}^4) = \frac{3t^2}{n^2}$, it follows that

$$\begin{aligned}
\mathrm{Var}\bigg(\lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta W_{t_i}^2\bigg) \;&=\; E\bigg\{\bigg(\lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta W_{t_i}^2 - t\bigg)^2\bigg\} \\
&=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} E\big\{\big(\Delta W_{t_i}^2 - \frac{t}{n}\big)^2\big\} \\
&=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} \big(3\frac{t^2}{n^2} - 2\frac{t^2}{n^2} + \frac{t^2}{n^2}\big) \\
&=\; 0\,.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \big(W_{t_{i+1}} - W_{t_i}\big)^2 \;=\; \int_0^t dW_s\, dW_s \;=\; t$$

holds. □

Using the result of Lemma 2.1, Theorem 2.2 satisfies the martigale property.

Theorem 2.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability measure space and let $\{W_t : t \geqslant 0\}$ be a standard Wiener process. Then*

$$I_t \;=\; \int_0^t W_s \, dW_s \;=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}) \,,$$

where $t_i = \frac{t}{n} i$, $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$ for $n \in \mathbb{N}$, is a martingale.

Proof. Since the quadratic variation of W_t is

$$\langle W, W \rangle_t \;=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} \big(W_{t_{i+1}} - W_{t_i}\big)^2 \;=\; t$$

and

$$\begin{aligned} I_t \;&=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}) \\ &=\; \lim_{n\to\infty} \sum_{i=0}^{n-1} \Big[\frac{1}{2}(W_{t_{i+1}}^2 - W_{t_i}^2) - \frac{1}{2}(W_{t_{i+1}} - W_{t_i})^2\Big] \\ &=\; \frac{1}{2}\,(W_t^2 - t)\,. \end{aligned}$$

Therefore I_t is a martingale. □

Remark. Another proof of Theorem 2.2 is as follow.

$$\begin{aligned} dX_t \;&=\; \frac{\partial X_t}{\partial t}\, dt + \frac{\partial X_t}{\partial W_t}\, dW_t + \frac{1}{2}\frac{\partial^2 X_t}{\partial W_t^2}\, dW_t^2 + \cdots \\ &=\; W_t \, dW_t + \frac{1}{2}\, dt\,. \end{aligned}$$

Taking integrals,

$$\int_0^t dX_s \;=\; \int_0^t W_s \, dW_s + \frac{1}{2}\int_0^t ds$$

we obtain

$$\int_0^t W_s \, dW_s \;=\; \frac{1}{2}\,(W_t^2 - t)\,.$$

Next theorem is not a martingale case.

THEOREM 2.3. *Let $(\Omega, \mathcal{F}, P)$ be a probability measure space and let $\{W_t : t \geqslant 0\}$ be a standard Wiener process with the associated filtration $\mathcal{F}_t$. Then the quadratic variation of W_t is defined by*

$$\langle W, W\rangle_t \;=\; \lim_{n\to\infty}\sum_{i=0}^{n-1}\big(W_{t_{i+1}} - W_{t_i}\big)^2 \;=\; t$$

and the stochastic integral of $W_t * dW_t$

$$J_t \;=\; \int_0^t W_s * dW_s \;=\; \lim_{n\to\infty}\sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i})\,,$$

where $t_i = \frac{t}{n}i$, $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$ for $n \in \mathbb{N}$, is not a martingale.

Proof. By LEMMA 2.1, the quadratic variation of W_t is

$$\langle W, W\rangle_t \;=\; \lim_{n\to\infty}\sum_{i=0}^{n-1}\big(W_{t_{i+1}} - W_{t_i}\big)^2 \;=\; t\,,$$

we have

$$\begin{aligned} J_t &= \lim_{n\to\infty}\sum_{i=0}^{n-1} W_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\Big[\frac{1}{2}(W_{t_{i+1}}^2 - W_{t_i}^2) + \frac{1}{2}(W_{t_{i+1}} - W_{t_i})^2\Big] \\ &= \frac{1}{2}\,(W_t^2 + t)\,. \end{aligned}$$

Under the filtration $\mathcal{F}_u$ $(u < t)$,

$$\begin{aligned} E(J_t \,|\, \mathcal{F}_u) &= \frac{1}{2}E\big(W_t^2 + t \,|\, \mathcal{F}_u\big) \\ &= \frac{1}{2}E\big(W_t^2 \,|\, \mathcal{F}_u\big) + \frac{1}{2}\,t \\ &= \frac{1}{2}\,(W_u^2 - u)\,. \end{aligned}$$

Hence J_t is not a martingale. □

The next theorem shows the martingale property as using the integrand of the stochastic integral is a simple function but the quadratic variation method is not.

THEOREM 2.4. *Let $(\Omega, \mathcal{F}, P)$ be a probability measure space and let $\{W_t : t \geqslant 0\}$ be a standard Wiener process with the associated filtration $\mathcal{F}_t$. The stochastic integral I_t with respect to the standard Wiener process defined by*

$$I_t = \int_0^t f(W_s, s)\, dW_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}),$$

where f is a simple function and $|f(W_{t_i}, t_i)| < +\infty$, $t_i = \frac{t}{n} i$, $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$ for $n \in \mathbb{N}$, is a martingale.

Proof. Since I_t is a function of W_t, it is $\mathcal{F}_t$-adapted. Under the filtration $\mathcal{F}_u$ $(u < t)$, we have

$$\begin{aligned} I_t &= \int_0^t f(W_s, s)\, dW_s \\ &= \int_0^u f(W_s, s)\, dW_s + \int_u^t f(W_s, s)\, dW_s \\ &= \lim_{n\to\infty} \sum_{i=0}^{k-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) \\ &\qquad + \lim_{n\to\infty} \sum_{i=k}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}), \end{aligned}$$

where $k < n-1$ and

$$E\Big(\int_0^u f(W_s, s)\, dW_s \,\Big|\, \mathcal{F}_u\Big) = \int_0^u f(W_s, s)\, dW_s .$$

Furthermore, we have

$$\begin{aligned} E(I_t|\mathcal{F}_u) &= E\Big[\lim_{n\to\infty} \sum_{i=0}^{k-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})\big|\mathcal{F}_u\Big] \\ &\qquad + E\Big[\lim_{n\to\infty} \sum_{i=k}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})\big|\mathcal{F}_u\Big] \\ &= \int_0^u f(W_s, s)\, dW_s + \lim_{n\to\infty} \sum_{i=k}^{n-1} E\big[f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})\big|\mathcal{F}_u\big] \end{aligned}$$

Hence it follows that

$$\begin{aligned} E(I_t|\mathcal{F}_u) &= \int_0^u f(W_s,s)\,dW_s \\ &\quad + \lim_{n\to\infty}\sum_{i=k}^{n-1} E\Big[E\big[f(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i})\big|\mathcal{F}_{t_i}\big]\Big|\mathcal{F}_u\Big] \\ &= \int_0^u f(W_s,s)\,dW_s + \lim_{n\to\infty}\sum_{i=k}^{n-1} E\big[f(W_{t_i},t_i)(W_{t_i}-W_{t_i})\big|\mathcal{F}_u\big] \\ &= \int_0^u f(W_s,s)\,dW_s \;=\; I_u\,. \end{aligned}$$

Finally, since

$$\begin{aligned} |\,I_t\,| &\leqslant \lim_{n\to\infty}\sum_{i=0}^{n-1}\big|f(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i})\big| \\ &\leqslant \lim_{n\to\infty}\Big\{\Big(\max_{0\leqslant m\leqslant n-1}|W_{t_{m+1}}-W_{t_m}|\Big)\sum_{i=0}^{n-1}|f(W_{t_i},t_i)|\Big\}\,, \end{aligned}$$

we have $E(|I_t|) < +\infty$. Therefore I_t is a martingale. □

As a result, if we use the martingale property of the quadratic variation in Theorem 2.2 or the stochastic integration in Theorem 2.4, then it follows that the limiting distribution of a standard Wiener process or a standard Wiener process with the associated filtration $\mathcal{F}_t$, $(t \geqslant 0)$ is a standard normal distribution $N(0,1)$.

Example. Let $(\Omega,\mathcal{F},P)$ be a probability space and let $\{M_t : t \geqslant 0\}$ be a martingale with respect to the filtration $\mathcal{F}_t$, $(t \geqslant 0)$, where $M_0 = 0$, M_t has continuous sample paths whose quadratic variation

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\big(M_{t_{i+1}}-M_{t_i}\big)^2 \;=\; t\,,$$

where $t_i = \frac{t}{n}i$, $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$ for $n \in \mathbb{N}$. Then M_t is a standard Wiener process and the limiting distribution of M_t is a standard normal distribution $N(0,1)$. We first show that M_t is a standard Wiener process. Since M_t is a martingale, fot $s < t$

$$\begin{aligned} E(M_t \,|\, \mathcal{F}_s) &= E(M_t - M_s \,|\, \mathcal{F}_s) + E(M_s \,|\, \mathcal{F}_s) \\ &= E(M_t - M_s \,|\, \mathcal{F}_s) + M_s \\ &= M_s \end{aligned}$$

and for $t > 0$ and $s > 0$, we have

$$E(M_{t+s} - M_t) \;=\; E(M_{t+s}) - E(M_t) \;=\; 0$$

and

$$\begin{aligned} \mathrm{Var}(M_{t+s} - M_t) &= \mathrm{Var}(M_{t+s}) + \mathrm{Var}(M_t) - 2\mathrm{Cov}(M_{t+s}, M_t) \\ &= 2t + s - 2E(M_t)\, E(M_{t+s} - M_t) - 2E(M_t^2) \\ &= s\,. \end{aligned}$$

Hence $M_{t+s} - M_t \sim N(0, s)$. Because $M_0 = 0$ and also M_t has continuous sample paths with independent and stationary increments, so M_t is a standard Wiener process. Next, let $f(M_t, t) = e^{\lambda M_t - \frac{1}{2}\lambda t}$ for any constant λ. Since $dM_t\, dM_t = dt$ and $(dt)^n = 0$ for $n \geqslant 2$, we have

$$\begin{aligned} df(M_t, t) &= \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial dM_t}\, dM_t + \frac{1}{2}\frac{\partial^2 f}{\partial M_t^2}\, (dM_t)^2 + \cdots \\ &= \Big(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial M_t^2}\Big)\, dt + \frac{\partial f}{\partial M_t}\, dM_t \\ &= \lambda f(M_t, t)\, dM_t\,. \end{aligned}$$

Taking integrals from 0 to t,

$$\begin{aligned} \int_0^t df(M_s, s) &= \lambda \int_0^t f(M_s, s)\, dM_s \\ f(M_t, t) - f(M_0, 0) &= \lambda \int_0^t f(M_s, s)\, dM_s \end{aligned}$$

and then taking expectations, we have

$$E\big(f(M_t, t)\big) \;=\; 1 + \lambda\, E\Big(\int_0^t f(M_s, s)\, dM_s\Big)\,.$$

By definition of the stochastic integral and since M_t is a martingale, it follows that

$$
\begin{aligned}
E\Big(\int_0^t f(M_s,s)\,dM_s\Big) &= \lim_{n\to\infty}\sum_{i=0}^{n-1} E\big\{f(M_{t_i},t_i)(M_{t_{i+1}}-M_{t_i})\big\}\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} E\big[E\big\{f(M_{t_i},t_i)(M_{t_{i+1}}-M_{t_i})|\mathcal{F}_{t_i}\big\}\big]\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} E\big\{f(M_{t_i},t_i)(M_{t_i}-M_{t_i})\big\}\\
&= 0\,.
\end{aligned}
$$

Hence

$$
E\big(f(M_t,t)\big) \;=\; 0 \quad \text{or} \quad E(e^{\lambda M_t}) \;=\; e^{\frac{1}{2}\lambda^2 t}\,,
$$

which is a moment generating function for the normal distribution with mean zero and variance t. Therefore, the limiting distribution of M_t is a standard normal distribution $N(0,1)$.

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