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STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR A MODIFIED SYSTEM OF VARIATIONAL INEQUALITIES AND A FINITE FAMILY OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, a new iterative scheme based on the extra-gradient-like method for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of solutions of modified variational inequalities in Banach spaces. A strong convergence theorem for this iterative scheme in Banach spaces is established. Our results extend recent results announced by many others.

1. Introduction

Let $(E, \|\cdot\|)$ be a Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T.

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Let $A, B : C \to E$ be two nonlinear mappings, I be the idnetity mapping. We consider the modified system of nonlinear variational inequalities for finding $(x^*, y^*) \in C \times C$ such that

$$(1.1)\quad \begin{cases} \langle x^* - (I - \lambda_1 A)(ax^* + (1-a)y^*), j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \lambda_2 B)x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases}$$

where $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$, J is the normalized duality mapping, $j \in J$.

In the case $a = 0$, problem (1.1) reduces to the following general system of nonlinear variational inequalities for finding $(x^*, y^*) \in C \times C$ such that

$$(1.2)\quad \begin{cases} \langle \lambda_1 Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \lambda_2 Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases}$$

which was considered by Wang and Yang [12], Yao et al. [13].

In particular, if $A = B$, then problem (1.2) reduces to the following system of variational inequalities for finding $(x^*, y^*) \in C \times C$ such that

$$(1.3)\quad \begin{cases} \langle \lambda_1 Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \lambda_2 Ax^* + y^* - x^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \end{cases}$$

which was studied by Qin et al. [6].

If $x^* = y^*$ in (1.3), then (1.3) reduces to

$$(1.4)\quad \langle Ax^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C,$$

which was considered by Aoyama et al. [1].

If $E = H$ is a real Hilbert space and $A, B : C \to H$ are nonlinear mappings, then (1.1) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$(1.5)\quad \begin{cases} \langle x^* - (I - \lambda_1 A)(ax^* + (1-a)y^*), x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \lambda_2 B)x^*, x - x^*\rangle \geq 0, & \forall x \in C. \end{cases}$$

Aoyama et al. [1] proved that an element $x^* \in C$ is a solution of the variational inequality (1.4) if and only if $x^* \in C$ is a fixed point of the mapping $Q_C(I - \lambda A)$, where $\lambda > 0$ is a constant and Q_C is a sunny nonexpansive retraction from E onto C.

Recently, Qin et al. [6] studied the problem of finding a common element in fixed point set of a nonexpansive mapping and solution set of a variational inequality for a inverse strongly accretive mapping. More precisely, they proved the following theorem.

THEOREM 1.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, C be a nonempty closed convex subset of E and Q_C be a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse strongly accretive mapping and $S : C \to C$ be a nonexpansive mapping with a fixed point. Assume that $\mathcal{F} = F(S) \cap F(D) \neq \phi$, where $Dx = Q_C[Q_C(x - \mu Ax) - \lambda A Q_C(x - \mu Ax)]$ for all $x \in C$. Let $\{x_n\}$ be a sequence generated in the following manner:*

(1.6)
$$\begin{cases} x_1 = u \in C, \\ y_n = Q_C(x_n - \mu A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta S x_n + (1-\delta) Q_C(y_n - \lambda A y_n)], \quad n \geq 1. \end{cases}$$

where $\delta \in (0,1)$, $\lambda, \mu \in (0, \frac{\alpha}{K^2})$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that

(a) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = Q_{\mathcal{F}} u$ and $(\overline{x}, \overline{y})$, where $\overline{y} = Q_C(\overline{x} - \mu A \overline{x})$, is a solution of the problem (1.3).

Motivated and inspired by the research work going on this field, in this paper, we consider the problem of convergence of an iterative algorithm for a modified system of nonlinear variational inequalities and a finite family of nonexpansive mappings. We prove the strong convergence of the purposed iterative scheme in uniformly convex and 2-uniformly smooth Banach spaces.

2. Preliminaries

Let C be a nonempty closed convex subset of a Banach space E with its dual space E^*. Let $\langle \cdot, \cdot \rangle$ denote the dual pair between E and E^*. Let

2^E denote the family of all the nonempty subsets of E. For $q > 1$,the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$ and J_q is single-valued if E^* is strictly convex or E is uniformly smooth. If $E = H$ is a Hilbert space, $J = I$, the identity mapping.

Let $B = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in B$,

$$\|x - y\| \geq \varepsilon \quad \text{implies} \quad \|\frac{x + y}{2}\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in B$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space E is called uniformly smooth if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$. E is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_E(t) \leq ct^q, \quad q > 1.$$

If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

Definition 2.1. Let $A : C \to E$ be a mapping. A is said to be
(i) accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0$$

for all $x, y \in C$.
(ii) ζ-inverse strongly accretive if there exist $j(x - y) \in J(x - y)$ and a constant $\zeta > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \zeta \|Ax - Ay\|^2$$

for all $x, y \in C$.

DEFINITION 2.2. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\eta_1, \cdots, \eta_N$ be real numbers such that $0 \leq \eta_i \leq 1$ for every $i = 1, \cdots, N$. Define a mapping $S : C \to C$ as follows:

$$U_1 = \eta_1 T_1 + (1 - \eta_1) I,$$

$$U_2 = \eta_2 T_2 U_1 + (1 - \eta_2) U_1,$$

$$U_3 = \eta_3 T_3 U_2 + (1 - \eta_3) U_2,$$

$$\vdots$$

$$U_{N-1} = \eta_{N-1} T_{N-1} U_{N-2} + (1 - \eta_{N-1}) U_{N-2},$$

$$S = U_N = \eta_N T_N U_{N-1} + (1 - \eta_N) U_{N-1}.$$

Such a mapping S is called the K-mapping generated by $T_1, \cdots, T_N$ and $\eta_1, \cdots, \eta_N$.

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if

$$Q[Q(x) + t(x - Q(x))] = Q(x),$$

whenever $Q(x) + t(x - Q(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Q(z) = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

In order to prove our main results in the next section, we also need the following lemmas.

LEMMA 2.1. ([10]) *Let E be a real 2-uniformly smooth Banach space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2, \quad \forall x, y \in E,$$

where K is the 2-uniformly smooth constant of E.

LEMMA 2.2. ([5]) *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \neq \phi$ and let $\eta_1, \cdots, \eta_N$ be real numbers such that $0 < \eta_i < 1$ for every $i = 1, \cdots, N-1$ and $0 < \eta_N \leq 1$.*

Let S be the K-mapping generated by $T_1 \cdots, T_N$ and $\eta_1, \cdots, \eta_N$. Then $F(S) = \cap_{i=1}^N F(T_i)$.

Remark 2.1. It is easy to see that the K-mapping is a nonexpansive mapping.

Lemma 2.3. ([9]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$ for all integer $n \ge 0$ and*

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.4. ([8]) *Let E be a uniformly smooth Banach space, C be a closed convex subset of E and $D : C \to C$ be a nonexpansive mapping with $F(D) \neq \phi$. For each fixed point $u \in C$ and every $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $x \mapsto tu + (1-t)Dx$ converges strongly as $t \to 0$ to a point of $F(D)$. Define $Q : C \to F(D)$ by $Q(u) = \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retraction from C onto $F(D)$, that is, Q satisfy the property:*

$$\langle u - Q(u), j(y - Q(u))\rangle \le 0, \quad \forall u \in C, y \in F(D).$$

Lemma 2.5. ([2]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{S_k\}$ be a sequence of nonexpansive mappings of C into E and $\{\beta_k\}$ be a sequence of positive real numbers such that $\sum_{k=1}^\infty \beta_k = 1$. If $\cap_{k=1}^\infty F(S_k) \neq \phi$, then the mapping $S = \sum_{k=1}^\infty \beta_k S_k$ is nonexpansive and $F(S) = \cap_{k=1}^\infty F(S_k)$.*

Lemma 2.6. ([11]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \beta_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions
(a) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(b) $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \le 0$ or $\sum_{n=1}^\infty |\beta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. ([7]) *Let C be a nonempty closed convex subset of a smooth Banach space E and let Q_C be a retraction from E onto C. Then the following are equivalent:*

(i) Q_C *is both sunny and nonexpansive;*
(ii) $\langle x - Q_C(x), j(y - Q_C(x))\rangle \leq 0$ *for all* $x \in E$ *and* $y \in C$.

LEMMA 2.8. ([3]) *In a Banach space* E*, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E,$$

where $j(x+y) \in J(x+y)$.

LEMMA 2.9. ([3]) *Let* C *be a nonempty closed convex subset of a smooth Banach space* E*. Let* $Q_C : E \to C$ *be a sunny nonexpansive retraction,* $A, B : C \to E$ *be mappings. For every* $\lambda_1, \lambda_2 > 0$ *and* $a \in [0,1]$*, the following statements are equivalent:*
(a) $(x^*, y^*) \in C \times C$ *is a solution of problem (1.1).*
(b) x^* *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = Q_C(I - \lambda_1 A)(ax + (1-a)Q_C(I - \lambda_2 B)x),$$

where $y^* = Q_C(I - \lambda_2 B)x^*$.

Proof. (a)⇒(b). Let $(x^*, y^*) \in C \times C$ be a solution of problem (1.1). For every $\lambda_1, \lambda_2 > 0$ and $a \in [0,1]$, we have

$$\begin{cases} \langle x^* - (I - \lambda_1 A)(ax^* + (1-a)y^*), j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \lambda_2 B)x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases}$$

From Lemma 2.7, we have

$$\begin{cases} x^* = Q_C(I - \lambda_1 A)(ax^* + (1-a)y^*), \\ y^* = Q_C(I - \lambda_2 B)x^*. \end{cases}$$

It implies that

$$\begin{aligned} x^* &= Q_C(I - \lambda_1 A)(ax^* + (1-a)Q_C(I - \lambda_2)x^*) \\ &= G(x^*). \end{aligned}$$

Hence, we have $x^* \in F(G)$, where $y^* = Q_C(I - \lambda_2 B)x^*$.

(b)⇒(a). Let $x^* \in F(G)$ and $y^* = Q_C(I - \lambda_2 B)x^*$. Then, we have

$$\begin{aligned} x^* &= G(x^*) \\ &= Q_C(I - \lambda_1 A)(ax^* + (1-a)Q_C(I - \lambda_2 B)x^*) \\ &= Q_C(I - \lambda_1 A)(ax^* + (1-a)y^*). \end{aligned}$$

From Lemma 2.7, we have

$$\begin{cases} \langle x^* - (I - \lambda_1 A)(ax^* + (1-a)y^*), j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle y^* - (I - \lambda_2 B)x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C. \end{cases}$$

Hence, we have $(x^*, y^*) \in C \times C$ is a solution of (1.1). □

3. Main results

Now we state and prove our main results.

THEOREM 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, C be a nonempty closed convex subset of E and Q_C be a sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be ζ_1, ζ_2-inverse strongly accretive mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = Q_C(I - \lambda_1 A)(ax + (1-a)Q_C(I - \lambda_2 B)x)$ for all $x \in C$, $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1)$. Let $S : C \to C$ be the K-mapping generated by $T_1, T_2, \cdots, T_N$ and $\eta_1, \eta_2, \cdots, \eta_N$, where $\eta_i \in (0, 1)$, for $i = 1, 2, \cdots, N-1$, and $\eta_N \in (0, 1]$ with $\mathcal{F} = \cap_{i=1}^N F(T_i) \cap F(G) \neq \phi$. Suppose that $\{x_n\}$ is the sequence generated by*

$$\begin{cases} x_1, u \in C, \\ y_n = Q_C(I - \lambda_2 B)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta S x_n + (1-\delta)Q_C(ax_n + (1-a)y_n \\ \qquad -\lambda_1 A(ax_n + (1-a)y_n))], \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\lambda_1 \in (0, \frac{\zeta_1}{K^2})$, $\lambda_2 \in (0, \frac{\zeta_2}{K^2})$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $x_0 = Q_{\mathcal{F}} u$ and (x_0, y_0) is a solution of (1.1), where $y_0 = Q_C(I - \lambda_2 B)x_0$.

Proof. First, we show that $Q_C(I - \lambda_1 A)$ and $Q_C(I - \lambda_2 B)$ are nonexpansive mappings for $\lambda_1 \in (0, \frac{\zeta_1}{K^2})$, $\lambda_2 \in (0, \frac{\zeta_2}{K^2})$. Let $x, y \in C$. Since A is an ζ_1-inverse strongly accretive mapping and $\lambda_1 < \frac{\zeta_1}{K^2}$, we have from

Lemma 2.1 that

$$\begin{aligned}
&\|(I-\lambda_1 A)x-(I-\lambda_2 A)y\|^2\\
&\le \|x-y\|^2-2\lambda_1\langle Ax-Ay, j(x-y)\rangle+2K^2\lambda_1^2\|Ax-Ay\|^2\\
&\le \|x-y\|^2-2\lambda_1\zeta_1\|Ax-Ay\|^2+2K^2\lambda_1^2\|Ax-Ay\|^2\\
&= \|x-y\|^2+2\lambda_1(\lambda_1K^2-\zeta_1)\|Ax-Ay\|^2\\
&\le \|x-y\|^2. \qquad (3.2)
\end{aligned}$$

Thus $(I-\lambda_1 A)$ is a nonexpansive mapping. So is $(I-\lambda_2 B)$. Hence $Q_C(I-\lambda_1 A)$, $Q_C(I-\lambda_2 B)$ are nonexpansive mappings. It is easy to see that the mapping G is a nonexpansive mapping. This show from Remark 2.1 that $\mathcal{F}=F(S)\cap F(G)$ is closed and convex. Let $x^*\in\mathcal{F}$. Then we have $x^*=Sx^*$ and

$$\begin{aligned}
x^*&=Gx^*\\
&=Q_c(I-\lambda_1 A)(ax^*+(1-a)Q_C(I-\lambda_2 B)x^*).
\end{aligned}$$

Putting $w_n=Q_C(I-\lambda_1 A)(ax_n+(1-a)y_n)$ and $y^*=Q_C(I-\lambda_2 B)x^*$, we can rewrite (3.1) by

$$x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n(\delta Sx_n+(1-\delta)w_n)$$

and $x^*=Q_C(I-\lambda_1 A)(ax^*+(1-a)y^*)$. Since $Q_C(I-\lambda_1 A)$ and $Q_C(I-\lambda_2 B)$ are nonexpansive, we have

(3.3)

$$\begin{aligned}
&\|w_n-x^*\|\\
&=\|Q_c(I-\lambda_1 A)(ax_n+(1-a)y_n)-Q_C(I-\lambda_1 A)(ax^*+(1-a)y^*)\|\\
&\le \|ax_n+(1-a)y_n-(ax^*+(1-a)y^*)\|\\
&\le a\|x_n-x^*\|+(1-a)\|y_n-y^*\|\\
&\le a\|x_n-x^*\|+(1-a)\|x_n-x^*\|\\
&=\|x_n-x^*\|.
\end{aligned}$$

It follows from the definition of x_n and (3.3) that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|\alpha_n u + \beta_n x_n + \gamma_n(\delta S x_n + (1-\delta) w_n) - x^*\| \\
&\le \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n[\delta \|S x_n - x^*\| + (1-\delta)\|w_n - x^*\|] \\
&\le \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n[\delta \|x_n - x^*\| + (1-\delta)\|x_n - x^*\|] \\
&= \alpha_n \|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\
&\le \max\{\|u - x^*\|, \|x_1 - x^*\|\}.
\end{aligned}$$

So, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{w_n\}$ and $\{Sx_n\}$ are also bounded. And we have

(3.4)
$$\begin{aligned}
&\|w_{n+1} - w_n\| \\
&= \|Q_C(I - \lambda_1 A)(a x_{n+1} + (1-a) y_{n+1}) - Q_C(I - \lambda_1 A)(a x_n + (1-a) y_n)\| \\
&\le a\|x_{n+1} - x_n\| + (1-a)\|y_{n+1} - y_n\| \\
&\le a\|x_{n+1} - x_n\| + (1-a)\|x_{n+1} - x_n\| \\
&= \|x_{n+1} - x_n\|.
\end{aligned}$$

Next, we will show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

Let

$$x_{n+1} = (1-\beta_n) z_n + \beta_n x_n, \quad \forall n \ge 1, \tag{3.6}$$

where $z_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ for each $n \geq 1$. Since $x_{n+1} - \beta_n x_n = \alpha_n u + \gamma_n[\delta S x_n + (1-\delta)w_n]$ and (3.6), we have

$$
\begin{aligned}
& z_{n+1} - z_n \\
&= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}u + \gamma_{n+1}[\delta S x_{n+1} + (1-\delta)w_{n+1}]}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n[\delta S x_n + (1-\delta)w_n]}{1-\beta_n} \\
&\quad - \frac{\gamma_{n+1}[\delta S x_n + (1-\delta)w_n]}{1-\beta_{n+1}} + \frac{\gamma_{n+1}[\delta S x_n + (1-\delta)w_n]}{1-\beta_{n+1}} \\
&= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}[\delta(S x_{n+1} - S x_n) + (1-\delta)(w_{n+1} - w_n)] \\
&\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)[\delta S x_n + (1-\delta)w_n] \\
&= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}[\delta(S x_{n+1} - S x_n) + (1-\delta)(w_{n+1} - w_n)] \\
&\quad + \left(\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)[\delta S x_n + (1-\delta)w_n].
\end{aligned}
$$

It follows from (3.4) that

$$
\begin{aligned}
& \|z_{n+1} - z_n\| \\
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \|u\| \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|\delta(S x_{n+1} - S x_n) + (1-\delta)(w_{n+1} - w_n)\| \\
&\quad + \left|\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right| \|\delta S x_n + (1-\delta)w_n\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| [\|u\| + \|Sx_n\| + \|w_n\|] \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} [\delta \|Sx_{n+1} - Sx_n\| + (1-\delta)\|w_{n+1} - w_n\|] \\
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| [\|u\| + \|Sx_n\| + \|w_n\|] \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} [\delta \|x_{n+1} - x_n\| + (1-\delta)\|x_{n+1} - x_n\|] \\
&= \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| [\|u\| + \|Sx_n\| + \|w_n\|] + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|x_{n+1} - x_n\| \\
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| [\|u\| + \|Sx_n\| + \|w_n\|] + \|x_{n+1} - x_n\|.
\end{aligned}
$$

From the conditions (ii) and (iii), we have

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 2.3 and (3.6), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $x_{n+1} - x_n = (1-\beta_n)(z_n - x_n)$, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

Next, we will show that

$$\limsup_{n\to\infty} \langle u - x_0, j(x_n - x_0)\rangle \leq 0,$$

where $x_0 = Q_{\mathcal{F}}u$. To show this inequality, define a mapping $D : C \to C$ by

$$
\begin{aligned}
Dx &= \delta Sx + (1-\delta) Q_C (I - \lambda_1 A)(ax + (1-a) Q_C (I - \lambda_2 B)x) \\
&= \delta Sx + (1-\delta) Gx, \quad \forall x \in C
\end{aligned}
$$

From Lemma 2.2 and 2.5, we have D is a nonexpansive mapping with

$$
\begin{aligned}
F(D) &= F(S) \cap F(G) \\
&= \cap_{i=1}^N F(T_i) \cap F(G) \\
&= \mathcal{F}.
\end{aligned} \tag{3.8}
$$

From the nonexpansiveness of the mapping D and the definition of x_n, we have

$$\begin{aligned}\|x_n - Dx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Dx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|u - Dx_n\| + \beta_n\|x_n - Dx_n\|.\end{aligned}$$

This implies that

$$(1-\beta_n)\|x_n - Dx_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|u - Dx_n\|.$$

From the conditions (ii), (iii) and (3.7), we have

$$\lim_{n\to\infty}\|x_n - Dx_n\| = 0. \tag{3.9}$$

Let x_t be the fixed point of the contraction $x \mapsto tu + (1-t)Dx$, where $t \in (0,1)$. That is,

$$x_t = tu + (1-t)Dx_t.$$

From the definition of x_t, we have

$$\begin{aligned}\|x_t - x_n\|^2 &= \|t(u - x_n) + (1-t)(Dx_t - x_n)\|^2 \\ &= (1-t)(\langle Dx_t - Dx_n, j(x_t - x_n)\rangle + \langle Dx_n - x_n, j(x_t - x_n)\rangle) \\ &\quad + t\langle u - x_t, j(x_t - x_n)\rangle + t\langle x_t - x_n, j(x_t - x_n)\rangle \\ &\le (1-t)(\|x_t - x_n\|^2 + \|Dx_n - x_n\|\|x_t - x_n\|) \\ &\quad + t\langle u - x_t, j(x_t - x_n)\rangle + t\|x_t - x_n\|^2 \\ &= \|x_t - x_n\|^2 + (1-t)\|Dx_n - x_n\|\|x_t - x_n\| \\ &\quad + t\langle u - x_t, j(x_t - x_n)\rangle. \end{aligned} \tag{3.10}$$

(3.10) implies that

$$\langle u - x_t, j(x_n - x_t)\rangle \le \frac{1-t}{t}\|Dx_n - x_n\|\|x_t - x_n\|. \tag{3.11}$$

From (3.9) and (3.11), we have

$$\limsup_{n\to\infty}\langle u - x_t, j(x_n - x_t)\rangle \le 0. \tag{3.12}$$

From Lemma 2.4 and (3.8), we see that $Q_{F(D)}u = \lim_{t\to 0} x_t$ and $F(D) = \mathcal{F}$. It follows that $\lim_{t\to 0} x_t = x_0 = Q_{\mathcal{F}}(u)$. Since

$$\begin{aligned}
&\langle u - x_0, j(x_n - x_0)\rangle \\
&\quad = \langle u - x_0, j(x_n - x_0)\rangle - \langle u - x_0, j(x_n - x_t)\rangle \\
&\qquad + \langle u - x_0, j(x_n - x_t)\rangle - \langle u - x_t, j(x_n - x_t)\rangle \\
&\qquad + \langle u - x_t, j(x_n - x_t)\rangle \\
&\quad = \langle u - x_0, j(x_n - x_0) - j(x_n - x_t)\rangle + \langle x_t - x_0, j(x_n - x_t\rangle \\
&\qquad + \langle u - x_t, j(x_n - x_t)\rangle \\
&\quad = \|u - x_0\|\|j(x_n - x_0) - j(x_n - x_t)\| + \|x_t - x_0\|x_n - x_t\| \\
&\qquad + \langle u - x_t, j(x_n - x_t)\rangle,
\end{aligned}$$

it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - x_0, j(x_n - x_0)\rangle \le{}& \limsup_{n\to\infty} \|u - x_0\|\|j(x_n - x_0) - j(x_n - x_t)\| \\
&+ \|x_t - x_0\| \limsup_{n\to\infty} \|x_n - x_t\| \\
&+ \limsup_{n\to\infty}\langle u - x_t, j(x_n - x_t)\rangle. \qquad (3.13)
\end{aligned}$$

Since j is norm-to-norm uniformly continuous on a bounded subset of E, (3.12) and (3.13), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - x_0, j(x_n - x_0)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle u - x_0, j(x_n - x_0)\rangle \\
&\le 0. \qquad (3.14)
\end{aligned}$$

Finally, we will show that the sequence $\{x_n\}$ converges strongly to $x_0 \in \mathcal{F}$. From the definition of x_n and Lemma 2.8, we have

$$\begin{aligned}
&\|x_{n+1} - x_0\|^2 \\
&= \|\alpha_n(u - x_0) + \beta_n(x_n - x_0) + \gamma_n(Dx_n - x_0)\|^2 \\
&\le \|\beta_n(x_n - x_0) + \gamma_n(Dx_n - x_0)\|^2 + 2\alpha_n\langle u - x_0, j(x_{n+1} - x_0)\rangle \\
&\le (\beta_n\|x_n - x_0\| + \gamma_n\|x_n - x_0\|)^2 + 2\alpha_n\langle u - x_0, j(x_{n+1} - x_0)\rangle \\
&\le (1 - \alpha_n)\|x_n - x_0\|^2 + 2\alpha_n\langle u - x_0, j(x_{n+1} - x_0)\rangle. \qquad (3.15)
\end{aligned}$$

From the condition (ii), (3.14) and Lemma 2.6 to (3.15), we obtain that

$$\lim_{n\to\infty} \|x_n - x_0\| = 0.$$

This completes the proof. □

REMARK 3.1. (1) If we take $a = 0$, then the iterative scheme (3.1) reduces to the following scheme:

(3.16)
$$\begin{cases} x_1, u \in C, \\ y_n = Q_C(I - \lambda_2 B)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta S x_n + (1-\delta)Q_C(y_n - \lambda_1 A y_n)], \quad \forall n \geq 1, \end{cases}$$

From Theorem 3.1, we obtain that the sequence $\{x_n\}$ generated by (3.16) converges strongly to $x_0 = Q_{\cap_{i=1}^N F(T_i) \cap F(G)} u$, where the mapping $G : C \to C$ defined by $G(x) = Q_C(I-\lambda_1 A)Q_C(I-\lambda_2 B)x$ for all $x \in C$ and (x_0, y_0) is a solution of (1.2), where $y_0 = Q_C(I - \lambda_2 B)x_0$.

(2) If we take $x_1 = u$, $A = B$, $N = 1$, $\eta_1 = 1$ and $T_1 = S : C \to C$ is a nonexpansive mapping, then the iterative scheme (3.16) reduces to the following scheme:

(3.17)
$$\begin{cases} x_1 = u \\ y_n = Q_C(I - \lambda_2 A)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta S x_n + (1-\delta)Q_C(y_n - \lambda_1 A y_n)], \quad \forall n \geq 1, \end{cases}$$

which is (1.6). From Theorem 3.1, we obtain that the sequence $\{x_n\}$ generated by (3.17) converges strongly to $x_0 = Q_{F(S)\cap F(G)} u$, where the mapping $G : C \to C$ defined by $G(x) = Q_C(I - \lambda_1 A)Q_C(I - \lambda_2 A)x$ for all $x \in C$ and (x_0, y_0) is a solution of (1.3), where $y_0 = Q_C(I - \lambda_2 A)x_0$.

REMARK 3.2. (i) We note that all Hilbert spaces and $L^p (p \geq 2)$ spaces are 2-uniformly smooth.

(ii) If $E = H$ is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection P_C from H onto C.

(iii) It is well known that the 2-uniformly smooth constant $K = \frac{\sqrt{2}}{2}$ in Hilbert spaces.

From Theorem 3.1 and Remark 3.3, we can obtain the following result immediately.

COROLLARY 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and P_C be the metric projection from H onto C. Let $A, B : C \to H$ be ζ_1, ζ_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by*

$$G(x) = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$$

for all $x \in C$, $\lambda_1, \lambda_2 > 0$ *and* $a \in [0,1)$. *Let* $S : C \to C$ *be the* K*-mapping generated by* $T_1, T_2, \cdots, T_N$ *and* $\eta_1, \eta_2, \cdots, \eta_N$, *where* $\eta_i \in (0,1)$ *for* $i = 1, 2, \cdots, N-1$ *and* $\eta_N \in (0,1]$ *with* $\mathcal{F} = \cap_{i=1}^N F(T_i) \cap F(G) \neq \phi$. *Suppose that* $\{x_n\}$ *is the sequence generated by*

$$\begin{cases} x_1, u \in C, \\ y_n = P_C(I - \lambda_2 B)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta S x_n \\ \qquad +(1-\delta)P_C(ax_n + (1-a)y_n - \lambda_1 A(ax_n + (1-a)y_n))], \quad \forall n \geq 1, \end{cases}$$

where $\lambda_1 \in (0, 2\zeta_1)$, $\lambda_2 \in (0, 2\zeta_2)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$. *Assume that the following conditions hold:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $x_0 = P_{\mathcal{F}} u$ *and* (x_0, y_0) *is a solution of (1.5), where* $y_0 = P_C(I - \lambda_2 B)x_0$.

Remark 3.3. We can see easily that Aoyama et al. [1], Iiduka and Takahashi [4], Yao and Yao [14], Qin et al. [6], Wang and Yang [12]'s results are special cases of Theorem 3.1.

Completing interests

The author declares that he has no competing interests.

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