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ON (m, n)-IDEALS OF AN ORDERED ABEL-GRASSMANN GROUPOID

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Abstract. In this paper, we introduce the concept of (m, n)-ideals in a non-associative ordered structure, which is called an ordered Abel-Grassmann's groupoid, by generalizing the concept of (m, n)-ideals in an ordered semigroup [14]. We also study the (m, n)-regular class of an ordered $\mathcal{AG}$-groupoid in terms of (m, n)-ideals.

1. Introduction

The concept of a left almost semigroup ($\mathcal{LA}$-semigroup) was first given by Kazim and Naseeruddin in 1972 [3]. In [2], the same structure is called a left invertive groupoid. Protić and Stevanović called it an Abel-Grassmann's groupoid ($\mathcal{AG}$-groupoid) [13].

An $\mathcal{AG}$-groupoid is a groupoid S satisfying the left invertive law $(ab)c = (cb)a$ for all $a, b, c \in S$. This left invertive law has been obtained by introducing braces on the left of ternary commutative law $abc = cba$. An $\mathcal{AG}$-groupoid satisfies the medial law $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$. Since $\mathcal{AG}$-groupoids satisfy medial law, they belong to the class of entropic groupoids which are also called abelian quasigroups [15]. If an $\mathcal{AG}$-groupoid S contains a left identity (unitary $\mathcal{AG}$-groupoid), then it satisfies the paramedial law $(ab)(cd) = (dc)(ba)$ and the identity $a(bc) = b(ac)$ for all $a, b, c, d \in S$ [6].

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An $\mathcal{AG}$-groupoid is a useful algebraic structure, midway between a groupoid and a commutative semigroup. An $\mathcal{AG}$-groupoid is non-associative and non-commutative in general, however, there is a close relationship with semigroup as well as with commutative structures. It has been investigated in [6] that if an $\mathcal{AG}$-groupoid contains a right identity, then it becomes a commutative semigroup. The connection of a commutative inverse semigroup with an $\mathcal{AG}$-groupoid has been given by Yousafzai et. al. in [17] as, a commutative inverse semigroup $(S, .)$ becomes an $\mathcal{AG}$-groupoid $(S, *)$ under $a * b = ba^{-1}r^{-1}$ $\forall$ $a, b, r \in S$. An $\mathcal{AG}$-groupoid S with left identity becomes a semigroup under the binary operation "$\circ_e$" defined as, $x \circ_e y = (xe)y$ for all x,$y \in S$ [18]. An $\mathcal{AG}$-groupoid is the generalization of a semigroup theory [6] and has vast applications in collaboration with semigroups like other branches of mathematics. Many interesting results on $\mathcal{AG}$-groupoids have been investigated in [6, 9, 10]. The structure of $\mathcal{AG}$-groupoids and other generalizations have been recently considered and studied by Mushtaq and Khan in [11, 12], respectively. Minimal ideals of an $\mathcal{AG}$-groupoid were also considered and studied in [4].

If S is an $\mathcal{AG}$-groupoid with product $\cdot : S \times S \to S$, then $ab \cdot c$ and $(ab)c$ both denote the product $(a \cdot b) \cdot c$.

Definition 1.1. [19] An $\mathcal{AG}$-groupoid $(S, \cdot)$ together with a partial order $\leq$ on S that is compatible with an $\mathcal{AG}$-groupoid operation, meaning that for $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy \text{ and } xz \leq yz,$$

is called an ordered $\mathcal{AG}$-groupoid.

Let $(S, \cdot, \leq)$ be an ordered $\mathcal{AG}$-groupoid. If A and B are nonempty subsets of S, we let

$$AB = \{xy \in S \mid x \in A, y \in B\},$$

and

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Definition 1.2. [19] Let $(S, \cdot, \leq)$ be an ordered $\mathcal{AG}$-groupoid. A nonempty subset A of S is called a left (resp. right) ideal of S if the followings hold:

(i) $SA \subseteq A$ (*resp.* $AS \subseteq A$);
(ii) $x \in A$ *and* $y \in S$, $y \leq x$ *implies* $y \in A$.

Equivalently, $(SA] \subseteq A$ (*resp.* $(AS] \subseteq A$). If A is both a left and a right ideal of S, then A is called a two-sided ideal or an ideal of S.

A nonempty subset A of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ is called an $\mathcal{AG}$-subgroupoid of S if $xy \in A$ for all $x, y \in A$.

It is clear to see that every left and and right ideals of an ordered $\mathcal{AG}$-groupoid is an $\mathcal{AG}$-subgroupoid.

Let $(S, \cdot, \leq)$ be an ordered $\mathcal{AG}$-groupoid and let A and B be nonempty subsets of S, then the following was proved in [16]:

(i) $A \subseteq (A]$;
(ii) If $A \subseteq B$, then $(A] \subseteq (B]$;
(iii) $(A](B] \subseteq (AB]$;
(iv) $(A] = ((A]]$;
(v) $((A](B]] = (AB]$.

Also for every left (resp. right) ideal T of S, $(T] = T$.

The concept of (m, n)-ideals in ordered semigroups were given by J. Sanborisoot and T. Changphas in [14] which was obtained by generalizing the idea of (m, n)-ideals in semigroup *[5]*. It's natural to ask whether the concept of (m, n)-ideals in ordered $\mathcal{AG}$-groupoids is valid or not? The aim of this paper is to deal with (m, n)-ideals in ordered $\mathcal{AG}$-groupoids. We introduce the concept of (m, n)-ideals in ordered $\mathcal{AG}$-groupoids as follows:

DEFINITION 1.3. Let $(S, \cdot, \leq)$ be an ordered $\mathcal{AG}$-groupoid and let m, n be non-negative integers. An $\mathcal{AG}$-subgroupoid A of S is called an (m, n)-ideal of S if the followings hold:

(i) $A^m S \cdot A^n \subseteq A$;
(ii) for $x \in A$ and $y \in S, y \leq x$ implies $y \in A$.

Here, A^0 is defined as $A^0 S \cdot A^n = SA^n = S$ if $n = 0$ and $A^m S \cdot A^0 = A^m S = S$ if $m = 0$. Equivalently, an $\mathcal{AG}$-subgroupoid A of S is called an (m, n)-ideal of S if

$$(A^m S \cdot A^n] \subseteq A.$$

If A is an (m, n)-ideal of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$, then $(A] = A$.

Note that the powers of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ are non-commutative and non associative, that is $aa \cdot a \neq a \cdot aa$ and $(aa \cdot a)a \neq a(aa \cdot a)$ for all $a \in S$. But a unitary ordered $\mathcal{AG}$-groupoid has associative powers, that is $(aa \cdot a)a = a(aa \cdot a)$ for all $a \in S$.

Assume that $(S, \cdot, \leq)$ is a unitary ordered $\mathcal{AG}$-groupoid. Let us define $a^{m+1} = a^m a$ and $a^m = ((((aa)a)a)...a)a = a^{m-1}a = aa^{m-1}$ for all $a \in S$ where $m \geq 4$. Also, we can show by induction, $(ab)^m = a^m b^m$ and $a^m a^n = a^{m+n}$ hold for all $a, b \in S$ and $m, n \geq 4$. Throughout this paper, we will use $m, n \geq 4$.

2. (m, n)-ideals in ordered $\mathcal{AG}$-groupoids

Definition 2.1. If there is an element 0 of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ such that $x \cdot 0 = 0 \cdot x = x\ \forall\ x \in S$, we call 0 a *zero element* of S.

Example 2.2. Let $S = \{a, b, c, d, e\}$ with a left identity d. Then the following multiplication table and order shows that $(S, \cdot, \leq)$ is a unitary ordered $\mathcal{AG}$-groupoid with a zero element a.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

$$\leq := \{(a, a), (a, b), (c, c), (a, c), (d, d), (a, e), (e, e), (b, b)\}.$$

Lemma 2.3. *If R and L are the right and the left ideals of a unitary ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ respectively, then $(RL]$ is an (m, n)-ideal of S.*

Proof. Let R and L be the right and the left ideals of S respectively, then

$$\begin{aligned}
(((RL)^m]S \cdot ((RL)^n]] &\subseteq (((RL)^m](S] \cdot ((RL)^n]] \subseteq ((RL)^m S \cdot ((RL)^n]] \\
&= ((R^m L^m \cdot S)(R^n L^n)] = ((R^m L^m \cdot R^n)(SL^n)] \\
&= ((L^m R^m \cdot R^n)(SL^n)] = ((R^n R^m \cdot L^m)(SL^n)] \\
&= ((R^m R^n \cdot L^m)(SL^n)] = ((R^{m+n} L^m)(SL^n)] \\
&= (S(R^{m+n} L^m \cdot L^n)] = (S(L^n L^m \cdot R^{m+n})] \\
&= ((SS] \cdot L^{m+n} R^{m+n}] \subseteq (SS \cdot L^{m+n} R^{m+n}]
\end{aligned}$$

$$
\begin{aligned}
&= (SL^{m+n} \cdot SR^{m+n}] = (R^{m+n}S \cdot L^{m+n}S] \\
&= ((R^m R^n \cdot (SS])(L^m L^n \cdot (SS])] \\
&\subseteq (((R^m R^n] \cdot (SS])((L^m L^n] \cdot (SS])] \\
&\subseteq ((R^m R^n \cdot SS)(L^m L^n \cdot SS)] \\
&= ((SS \cdot R^n R^m)(SS \cdot L^n L^m)] \\
&\subseteq (((SS] \cdot R^n R^m)((SS] \cdot L^n L^m)] = (SR^{m+n} \cdot SL^{m+n}].
\end{aligned}
$$

Now

$$
\begin{aligned}
(SR^{m+n} \cdot SL^{m+n}] &= ((S \cdot R^{m+n-1}R)(S \cdot L^{m+n-1}L)] \\
&= ((S(R^{m+n-2}R \cdot R))(S(L^{m+n-2}L \cdot L))] \\
&= (S(RR \cdot R^{m+n-2}))(S(LL \cdot L^{m+n-2})) \\
&\subseteq ((SS \cdot RR^{m+n-2})(SS \cdot LL^{m+n-2})] \\
&\subseteq ((SR \cdot SR^{m+n-2})(SL \cdot SL^{m+n-2})] \\
&\subseteq ((R^{m+n-2}S \cdot RS)(L \cdot SL^{m+n-2})] \\
&\subseteq ((R^{m+n-2}S \cdot (RS])(L \cdot SL^{m+n-2})] \\
&\subseteq ((R^{m+n-2}S \cdot R)(S \cdot LL^{m+n-2})] \\
&\subseteq (((RS] \cdot R^{m+n-2})(SL^{m+n-1})] \\
&\subseteq (RR^{m+n-2} \cdot SL^{m+n-1}] \\
&\subseteq (SR^{m+n-1} \cdot SL^{m+n-1}].
\end{aligned}
$$

Thus

$$
\begin{aligned}
(((RL)^m]S \cdot ((RL)^n]] &\subseteq (SR^{m+n} \cdot SL^{m+n}] \subseteq (SR^{m+n-1} \cdot SL^{m+n-1}] \\
&\subseteq \ldots \subseteq (SR \cdot SL] \subseteq (SR \cdot (SL]] \\
&\subseteq (SR \cdot L] \subseteq ((SS \cdot R)L] \\
&= ((RS \cdot S)L] \subseteq (((RS] \cdot S)L] \subseteq (RL].
\end{aligned}
$$

Also

$$
\begin{aligned}
(RL] \cdot (RL] &\subseteq (RL \cdot RL] = (LR \cdot LR] = ((LR \cdot R)L] \\
&= ((RR \cdot L)L] \subseteq (((RS] \cdot S)L] \subseteq (RL].
\end{aligned}
$$

This shows that $(RL]$ is an (m, n)-ideal of S. □

Definition 2.4. An (m, n)-ideal M of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ with zero is said to be *nilpotent* if $M^l = \{0\}$ for some positive integer l.

DEFINITION 2.5. An (m, n)-ideal A of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ with zero is said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only (m, n)-ideal of S properly contained in A.

THEOREM 2.6. *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid with zero. If S has the property that it contains no non-zero nilpotent (m, n)-ideals and if R (L) is a 0-minimal right (left) ideal of S, then either $(RL] = \{0\}$ or $(RL]$ is a 0-minimal (m, n)-ideal of S.*

Proof. Assume that R (L) is a 0-minimal right (left) ideal of S such that $(RL] \neq \{0\}$, then by Lemma 2.3, $(RL]$ is an (m, n)-ideal of S. Now we show that $(RL]$ is a 0-minimal (m, n)-ideal of S. Let $\{0\} \neq M \subseteq (RL]$ be an (m, n)-ideal of S. Note that since $(RL] \subseteq R \cap L$, we have $M \subseteq R \cap L$. Hence $M \subseteq R$ and $M \subseteq L$. By hypothesis, $M^m \neq \{0\}$ and $M^n \neq \{0\}$. Since $\{0\} \neq (SM^m] = (M^m S]$, therefore

$$\begin{aligned} \{0\} &\neq (M^m S] \subseteq (R^m S] = (R^{m-1}R \cdot S] = (SR \cdot R^{m-1}] \\ &= (SR \cdot R^{m-2}R] \subseteq (RR^{m-2} \cdot (RS]] \\ &\subseteq (RR^{m-2} \cdot R] = (R^m], \end{aligned}$$

and

$$\begin{aligned} (R^m] &\subseteq S(R^m] \subseteq (SR^m] \subseteq (SS \cdot RR^{m-1}] \\ &\subseteq (R^{m-1}R \cdot S] = ((R^{m-2}R \cdot R)S] \\ &= ((RR \cdot R^{m-2})S] \subseteq (SR^{m-2} \cdot (RS]] \\ &\subseteq (SR^{m-2} \cdot R] \subseteq ((SS \cdot R^{m-3}R)R] \\ &= ((RR^{m-3} \cdot SS)R] \subseteq (((RS] \cdot R^{m-3}S)R] \\ &\subseteq ((R \cdot R^{m-3}S)R] \subseteq ((R^{m-3} \cdot (RS])R] \\ &\subseteq (R^{m-3}R \cdot R] = (R^{m-1}], \end{aligned}$$

therefore $\{0\} \neq (M^m S] \subseteq (R^m] \subseteq (R^{m-1}] \subseteq ... \subseteq (R] = R$. It is easy to see that $(M^m S]$ is a right ideal of S. Thus $(M^m S] = R$ since R is 0-minimal. Also

$$\begin{aligned} \{0\} &\neq (SM^n] \subseteq (SL^n] = (S \cdot L^{n-1}L] \\ &\subseteq (L^{n-1} \cdot (SL]] \subseteq (L^{n-1}L] = (L^n], \end{aligned}$$

and

$$\begin{aligned}(L^n] &\subseteq (SL^n] \subseteq (SS \cdot LL^{n-1}] \subseteq (L^{n-1}L \cdot S] \\ &= ((L^{n-2}L \cdot L)S] \subseteq ((SL] \cdot L^{n-2}L] \\ &\subseteq (L \cdot L^{n-2}L] \subseteq (L^{n-2} \cdot SL] \\ &\subseteq (L^{n-2}L] = (L^{n-1}] \subseteq ... \subseteq (L],\end{aligned}$$

therefore $\{0\} \neq (SM^n] \subseteq (L^n] \subseteq (L^{n-1}] \subseteq ... \subseteq (L] = L$. It is easy to see that $(SM^n]$ is a left ideal of S. Thus $(SM^n] = L$ since L is 0-minimal. Therefore

$$\begin{aligned}M &\subseteq (RL] = ((M^mS] \cdot (SM^n]] = (M^nS \cdot SM^m] \\ &= ((SM^m \cdot S)M^n] \subseteq ((SM^m \cdot SS)M^n] \\ &\subseteq ((S \cdot M^mS)M^n] = ((M^m \cdot SS)M^n] \\ &\subseteq (M^mS \cdot M^n] \subseteq M.\end{aligned}$$

Thus $M = (RL]$, which means that $(RL]$ is a 0-minimal (m,n)-ideal of S. □

Note that if $(S, \cdot, \leq)$ is a unitary ordered $\mathcal{AG}$-groupoid and $M \subseteq S$, then it is easy to see that $(SM^2]$ and $(SM]$ are the left and the right ideals of S respectively.

Theorem 2.7. *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid with zero. If R (L) is a 0-minimal right (left) ideal of S, then either $(R^mL^n] = \{0\}$ or $(R^mL^n]$ is a 0-minimal (m,n)-ideal of S.*

Proof. Assume that R (L) is a 0-minimal right (left) ideal of S such that $(R^mL^n] \neq \{0\}$, then $R^m \neq \{0\}$ and $L^n \neq \{0\}$. Hence $\{0\} \neq R^m \subseteq R$ and $\{0\} \neq L^n \subseteq L$, which shows that $R^m = R$ and $L^n = L$ since R (L) is a 0-minimal right (left) ideal of S. Thus by lemma 2.3, $(R^mL^n] = (RL]$ is an (m,n)-ideal of S. Now we show that $(R^mL^n]$ is a 0-minimal (m,n)-ideal of S. Let $\{0\} \neq M \subseteq (R^mL^n] = (RL] \subseteq R \cap L$ be an (m,n)-ideal of S. Hence

$$\{0\} \neq (SM^2] \subseteq (MM \cdot SS] = (MS \cdot MS] \subseteq ((RS] \cdot (RS]] \subseteq R,$$

and $\{0\} \neq (SM] \subseteq (SL] \subseteq L$. Thus $R = (SM^2]$ and $(SM] = L$ since R (L) is a 0-minimal right (left) ideal of S. Since

$$(SM^2] \subseteq (MM \cdot SS] = (SM \cdot M] \subseteq (SM],$$

therefore

$$\begin{aligned} M &\subseteq (R^m L^n] \subseteq (((SM)^m]((SM)^n]] = ((SM)^m (SM)^n] \\ &= (S^m M^m \cdot S^n M^n] = (SS \cdot M^m M^n] \subseteq (M^n M^m \cdot S] \\ &\subseteq ((SS)(M^{m-1} M) \cdot M^n] = ((MM^{m-1})(SS) \cdot M^n] \\ &\subseteq (M^m S \cdot M^n] \subseteq M, \end{aligned}$$

Thus $M = (R^m L^n]$, which shows that $(R^m L^n]$ is a 0-minimal (m,n)-ideal of S. □

THEOREM 2.8. *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Assume that A is an (m,n)-ideal of S and B is an (m,n)-ideal of A such that B is idempotent. Then B is an (m,n)-ideal of S.*

Proof. It is trivial that B is an $\mathcal{AG}$-subgroupoid of S. Secondly, since $(A^m S \cdot A^n] \subseteq A$ and $(B^m A \cdot B^n] \subseteq B$, then

$$\begin{aligned} (B^m S \cdot B^n] &\subseteq ((B^m B^m \cdot S)(B^n B^n)] = ((B^n B^n)(S \cdot B^m B^m)] \\ &= (((S \cdot B^m B^m) B^n) B^n] \subseteq (((B^n \cdot B^m B^m)(SS)) B^n] \\ &= (((B^m \cdot B^n B^m)(SS)) B^n] = ((S(B^n B^m \cdot B^m)) B^n] \\ &= ((S(B^n B^m \cdot B^{m-1} B)) B^n] = ((S(BB^{m-1} \cdot B^m B^n)) B^n] \\ &= ((S(B^m \cdot B^m B^n)) B^n] \subseteq ((B^m (SS \cdot B^m B^n)) B^n] \\ &= ((B^m (B^n B^m \cdot SS)) B^n] \subseteq ((B^m (S B^m \cdot B^n)) B^n] \\ &\subseteq ((B^m ((SS \cdot B^{m-1} B) B^n)) B^n] \subseteq ((B^m (B^m S \cdot B^n)) B^n] \\ &\subseteq ((B^m (A^m S \cdot A^n]) B^n] \subseteq (B^m A \cdot B^n] \subseteq B, \end{aligned}$$

which shows that B is an (m,n)-ideal of S. □

LEMMA 2.9. *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Then $\langle a \rangle_{(m,n)} = (a^m S \cdot a^n]$ is an (m,n)-ideal of S.*

Proof. Assume that S is a unitary $\mathcal{AG}$-groupoid. It is easy to see that $(\langle a\rangle_{(m,n)})^n \subseteq \langle a\rangle_{(m,n)}$. Now

$$\begin{aligned}
(((\langle a\rangle_{(m,n)})^m S)(\langle a\rangle_{(m,n)})^n] &= (((((a^m S)a^n))^m]S \cdot (((a^m S)a^n)^n]] \\
&\subseteq (((a^m S)a^n)^m S \cdot ((a^m S)a^n)^n] \\
&= (((a^{mm} S^m)a^{mn})S \cdot (a^{mn} S^n)a^{nn}] \\
&= (a^{nn}(a^{mn} S^n) \cdot S((a^{mm} S^m)a^{mn})] \\
&= ((S((a^{mm} S^m)a^{mn}) \cdot a^{mn} S^n)a^{nn}] \\
&= ((a^{mn} \cdot (S((a^{mm} S^m)a^{mn}))S^n)a^{nn}] \\
&\subseteq (a^{mn} S \cdot a^{nn}] \subseteq (a^{mn} S^n \cdot a^{nn}] \\
&= ((a^m S \cdot a^n)^n] \subseteq (((a^m S \cdot a^n)]^n] \\
&= ((\langle a\rangle_{(m,n)})^n] \subseteq (\langle a\rangle_{(m,n)}],
\end{aligned}$$

which shows that $\langle a\rangle_{(m,n)}$ is an (m,n)-ideal of S. □

THEOREM 2.10. *Let $(S,\cdot,\leq)$ be a unitary ordered $\mathcal{AG}$-groupoid and $\langle a\rangle_{(m,n)}$ be an (m,n)-ideal of S. Then the following statements hold*:

(i) $((\langle a\rangle_{(1,0)})^m S] = (a^m S]$;
(ii) $(S(\langle a\rangle_{(0,1)})^n] = (Sa^n]$;
(iii) $((\langle a\rangle_{(1,0)})^m S \cdot (\langle a\rangle_{(0,1)})^n] = (a^m S \cdot a^n]$.

Proof. (i) As $\langle a\rangle_{(1,0)} = (aS]$, we have

$$\begin{aligned}
((\langle a\rangle_{(1,0)})^m S] &= (((aS])^m S] \subseteq (((aS)^m]S] \subseteq ((aS)^m S] \\
&= ((aS)^{m-1}(aS) \cdot S] = (S(aS) \cdot (aS)^{m-1}] \\
&\subseteq ((aS)(aS)^{m-1}] = ((aS) \cdot (aS)^{m-2}(aS)] \\
&= ((aS)^{m-2}(aS \cdot aS)] = ((aS)^{m-2}(a^2 S)] \\
&= ... = ((aS)^{m-(m-1)}(a^{m-1}S)] \text{ [if } m \text{ is odd]} \\
&= ... = ((a^{m-1}S)(aS)^{m-(m-1)}] \text{ [if } m \text{ is even]} \\
&= (a^m S].
\end{aligned}$$

Analogously, we can prove (ii) and (iii) as well. □

COROLLARY 2.11. *Let $(S,\cdot,\leq)$ be a unitary ordered $\mathcal{AG}$-groupoid and let $\langle a\rangle_{(m,n)}$ be an (m,n)-ideal of S. Then the following statements hold*:

(i) $((\langle a\rangle_{(1,0)})^m S] = (Sa^m]$;

(ii) $(S(\langle a\rangle_{(0,1)})^n] = (a^nS]$;
(iii) $((\langle a\rangle_{(1,0)})^mS \cdot (\langle a\rangle_{(0,1)})^n] = (Sa^m \cdot a^nS]$.

3. (m,n)-ideals in (m,n)-regular ordered $\mathcal{AG}$-groupoids

Definition 3.1. Let m,n be non-negative integers and $(S,\cdot,\leq)$ be an ordered $\mathcal{AG}$-groupoid. We say that S is *(m,n)-regular* if for every element $a \in S$ there exists some $x \in S$ such that $a \leq a^mx \cdot a^n$. Note that a^0 is defined as an operator element such that $a^mx \cdot a^0 = a^mx = x$ if $m = 0$, and $a^0x \cdot a^n = xa^n = x$ if $n = 0$.

Let $\mathfrak{L}_{(0,n)}$, $\mathfrak{R}_{(m,0)}$ and $\mathfrak{A}_{(m,n)}$ denote the sets of $(0,n)$-ideals, $(m,0)$-ideals and (m,n)-ideals of an ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ respectively.

Theorem 3.2. *Let $(S,\cdot,\leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Then the following statements hold*:
(i) S *is* $(0,1)$*-regular if and only if* $\forall\, L \in \mathfrak{L}_{(0,1)}, L = (SL]$;
(ii) S *is* $(2,0)$*-regular if and only if* $\forall R \in \mathfrak{R}_{(2,0)}$, $R = (R^2S]$ *such that every* R *is semiprime*;
(iii) S *is* $(0,2)$*-regular if and only if* $\forall U \in \mathfrak{A}_{(0,2)}$, $U = (U^2S]$ *such that every* U *is semiprime.*

Proof. (i) Let S be $(0,1)$-regular, then for $a \in S$ there exists $x \in S$ such that $a \leq xa$. Since L is $(0,1)$-ideal, therefore $(SL] \subseteq L$. Let $a \in L$, then $a \leq xa \in (SL] \subseteq L$. Hence $L = (SL]$. Converse is simple.

(ii) Let S be $(2,0)$-regular and R be $(2,0)$-ideal of S, then it is easy to see that $R = (R^2S]$. Now for $a \in S$ there exists $x \in S$ such that $a \leq a^2x$. Let $a^2 \in R$, then

$$a \leq a^2x \in RS = (R^2S] \cdot S \subseteq (R^2S \cdot S] = (SS \cdot R^2] \subseteq (R^2S] = R,$$

which shows that every $(2,0)$-ideal is semiprime.

Conversely, let $R = (R^2S]$ for every $R \in \mathfrak{R}_{(2,0)}$. Since $(Sa^2]$ is a $(2,0)$-ideal of S such that $a^2 \in (Sa^2]$, therefore $a \in (Sa^2]$. Thus

$$\begin{aligned} a &\in (Sa^2] = (((Sa^2)^2]S] \subseteq ((Sa^2 \cdot Sa^2)S] = ((a^2S \cdot a^2S)S] \\ &= ((a^2(a^2S \cdot S))S] \subseteq ((a^2 \cdot Sa^2)S] = ((S \cdot Sa^2)a^2] \\ &\subseteq (Sa^2] \subseteq (a^2S], \end{aligned}$$

which implies that S is $(2,0)$-regular.

Analogously, we can prove (iii). □

LEMMA 3.3. *Let $(S,\cdot,\leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Then the following statements hold*:

(i) *If S is $(0,n)$-regular, then $\forall L \in \mathfrak{L}_{(0,n)}$, $L = (SL^n]$*;
(ii) *If S is $(m,0)$-regular, then $\forall\, R \in \mathfrak{R}_{(m,0)}$, $R = (R^m S]$*;
(iii) *If S is (m,n)-regular, then $\forall\, U \in \mathfrak{A}_{(m,n)}$, $U = (U^m S \cdot U^n]$.*

Proof. It is simple. □

COROLLARY 3.4. *Let $(S,\cdot,\leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Then the following statements hold*:

(i) *If S is $(0,n)$-regular, then $\forall\, L \in \mathfrak{L}_{(0,n)}$, $L = (L^n S]$*;
(ii) *If S is $(m,0)$-regular, then $\forall\, R \in \mathfrak{R}_{(m,0)}$, $R = (SR^m]$*;
(iii) *If S is (m,n)-regular, then $\forall\, U \in \mathfrak{A}_{(m,n)}$, $U = (U^{m+n}S] = (SU^{m+n}]$.*

THEOREM 3.5. *Let $(S,\cdot,\leq)$ be a unitary (m,n)-regular ordered $\mathcal{AG}$-groupoid. Then for every $R \in \mathfrak{R}_{(m,0)}$ and $L \in \mathfrak{L}_{(0,n)}$, $R \cap L = (R^m L] \cap (RL^n]$.*

Proof. It is simple. □

THEOREM 3.6. *Let $(S,\cdot,\leq)$ be a unitary (m,n)-regular ordered $\mathcal{AG}$-groupoid. If M (N) is a 0-minimal $(m,0)$-ideal ($(0,n)$-ideal) of S such that $(MN] \subseteq M \cap N$, then either $(MN] = \{0\}$ or $(MN]$ is a 0-minimal (m,n)-ideal of S.*

Proof. Let M (N) be a 0-minimal $(m,0)$-ideal ($(0,n)$-ideal) of S. Let $O = (MN]$, then clearly $O^2 \subseteq O$. Moreover

$$\begin{aligned}
(O^m S \cdot O^n] &= ((MN]^m S \cdot (MN]^n] \subseteq (((MN)^m]S \cdot ((MN)^n]] \\
&\subseteq ((MN)^m S \cdot (MN)^n] = ((M^m N^m)S \cdot M^n N^n] \\
&\subseteq ((M^m S)S \cdot SN^n] \subseteq (SM^m \cdot SN^n] \\
&\subseteq (M^m S \cdot SN^n] \subseteq ((M^m S] \cdot (SN^n]] \\
&\subseteq (MN] = O,
\end{aligned}$$

which shows that O is an (m,n)-ideal of S. Let $\{0\} \neq P \subseteq O$ be a non-zero (m,n)-ideal of S. Since S is (m,n)-regular, therefore by using

Lemma 3.3, we have

$$\begin{aligned}
\{0\} &\neq P = (P^m S \cdot P^n] \subseteq ((P^m \cdot SS)P^n] = ((S \cdot P^m S)P^n] \\
&\subseteq ((P^n \cdot P^m S)(SS)] = ((P^n S)(P^m S \cdot S)] \\
&\subseteq (P^n S \cdot SP^m] = (P^m S \cdot SP^n] \\
&= ((P^m S] \cdot (SP^n]].
\end{aligned}$$

Hence $(P^m S] \neq \{0\}$ and $(SP^n] \neq \{0\}$. Further $P \subseteq O = (MN] \subseteq M \cap N$ implies that $P \subseteq M$ and $P \subseteq N$. Therefore $\{0\} \neq (P^m S] \subseteq (M^m S] \subseteq M$ which shows that $(P^m S] = M$ since M is 0-minimal. Likewise, we can show that $(SP^n] = N$. Thus we have

$$\begin{aligned}
P &\subseteq O = (MN] = ((P^m S] \cdot (SP^n]] = (P^m S \cdot SP^n] \\
&= (P^n S \cdot SP^m] \subseteq ((SP^m \cdot SS)P^n] \\
&\subseteq ((S \cdot P^m S)P^n] \subseteq (P^m S \cdot P^n] \subseteq P.
\end{aligned}$$

This means that $P = (MN]$ and hence $(MN]$ is 0-minimal. □

THEOREM 3.7. *Let $(S, \cdot, \leq)$ be a unitary (m, n)-regular ordered $\mathcal{AG}$-groupoid. If M (N) is a 0-minimal $(m, 0)$-ideal $((0, n)$-ideal$)$ of S, then either $M \cap N = \{0\}$ or $M \cap N$ is a 0-minimal (m, n)-ideal of S.*

Proof. Once we prove that $M \cap N$ is an (m, n)-ideal of S, the rest of the proof is the same as in Theorem 3.5. Let $O = M \cap N$, then it is easy to see that $O^2 \subseteq O$. Moreover

$$(O^m S \cdot O^n] \subseteq ((M^m S] \cdot N^n] \subseteq (MN^n] \subseteq (SN^n] \subseteq N.$$

But, we also have

$$\begin{aligned}
(O^m S \cdot O^n] &\subseteq (M^m S \cdot N^n] \subseteq ((M^m \cdot SS)N^n] = ((S \cdot M^m S)N^n] \\
&= ((N^n \cdot M^m S)S] \subseteq ((M^m \cdot N^n S)(SS)] \\
&= ((M^m S)(N^n S \cdot S)] \subseteq (M^m S \cdot SN^n] \\
&\subseteq (M^m S \cdot N^n S] = (N^n (M^m S \cdot S)] \\
&\subseteq (N^n \cdot SM^m] \subseteq (N^n \cdot M^m S] = (M^m \cdot N^n S] \\
&\subseteq (M^m \cdot (SN^n]] \subseteq (M^m N] \subseteq (M^m S] \subseteq M.
\end{aligned}$$

Thus $(O^m S \cdot O^n] \subseteq M \cap N = O$ and therefore O is an (m, n)-ideal of S. □

4. Conclusions

All the results of this paper can be obtain for an $\mathcal{AG}$-groupoid without order which will give us the extension of the carried out in [1] on (m, n)-ideals in an $\mathcal{AG}$-groupoid. Also the results of this paper can be trivially followed for a locally associative ordered $\mathcal{AG}$-groupoid which will generalize and extend the concept of a locally associative $\mathcal{AG}$-groupoid [7].

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