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ON A RING PROPERTY GENERALIZING POWER-ARMENDARIZ AND CENTRAL ARMENDARIZ RINGS

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Abstract. We in this note consider a class of rings which is related to both power-Armendariz and central Armendariz rings, in the spirit of Armendariz and Kaplansky. We introduce *central power-Armendariz* as a generalization of them, and study the structure of central products of coefficients of zero-dividing polynomials. We also observe various sorts of examples to illuminate the relations between central power-Armendariz and related ring properties.

1. Central power-Armendariz rings

Throughout this note every ring is associative with identity unless otherwise stated. Let R be a ring. We use $R[x]$ to denote the polynomial ring with an indeterminate x over R. For $f(x) \in R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. $C(R)$ means the center of R, i.e., the set of all central elements in R. Denote the n by n full (resp., upper

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triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0.

Let R be a ring. $N_*(R)$, $N^*(R)$, $N_0(R)$, and $N(R)$ denote the prime radical, the upper nilradical (i.e., sum of all nil ideals), the Wedderburn radical (i.e., the sum of all nilpotent ideals), and the set of all nilpotent elements in R, respectively. It is well-known that $N_0(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$.

Kaplansky [18] proved that a division ring R is commutative if some power of a (depending on a) is central for every $a \in R$. Herstein [14] observed the structure of rings in which some power of each element is central. Based on these works, Jacobson [24] called a ring R a *K-ring* if for each element a of R there exists $n \geq 1$, depending on a, such that $a^n \in C(R)$. Every commutative ring is a K-ring clearly, but there exist many sorts of noncommutative K-rings (for examples, see [25]). Let R be a K-ring and $\{a_1, \ldots, a_n\}$ be a finite subset of R. Then there exist $m_i \geq 1$ such that $a_i^{m_i} \in C(R)$. Letting $m = m_1 \cdots m_n$, we have $a_i^m \in C(R)$ for all i. We will use this fact freely.

A ring (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. Let R be a reduced ring and suppose that $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$. In this situation, Armendariz [5, Lemma 1] proved that $ab = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$. Rege and Chhawchharia [27] called a ring (possibly without identity) *Armendariz* if it satisfies such property. So reduced rings are clearly Armendariz. This fact will be used freely in this note. A ring is usually called *Abelian* if every idempotent is central. Armendariz rings are Abelian by the proof of [3, Theorem 6] (or [21, Lemma 7]).

The classes of Armendariz rings and K-rings do not contain each other as we see in the following. In the literature we often see the following sort of subring of $Mat_n(R)$ which has a role in noncommutative ring theory:

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in U_n(R) \mid a, a_{ij} \in R \right\},$$

where R is a ring and $n \geq 2$.

Example 1.1. (1) There exists a commutative ring (hence a K-ring) which is not Armendariz in [27, Example 3.2].

There also exist noncommutative K-rings which are not Armendariz. We recall that $D_n(R)$ is a noncommutative K-ring for $n \geq 3$ if R is a K-ring of nonzero characteristic by [25, Theorem 2.3(2)]. But $D_n(R)$ cannot be Armendariz for any ring R and all $n \geq 4$ by [21, Example 3].

(2) (i) Let F be a field of characteristic zero and $R = D_3(F)$. Then R is Armendariz by [21, Proposition 2], but R is not a K-ring by [25, Theorem 2.3(1)].

(ii) By Kaplansky [18], a division ring, which is not a field, is not a K-ring. But division rings are clearly Armendariz.

(iii) There exists a reduced (hence Armendariz) ring which is neither a K-ring nor a division ring. Let F be a field and consider a monomorphism $\sigma : F[x] \to F[x]$ defined by $\sigma(f(x)) = f(x^2)$. Next set $R = F[x][t;\sigma]$ be the skew polynomial ring with an indeterminate t over $F[x]$, only subject to $tf(x) = \sigma(f(x))t$ for $f(x) \in F[x]$. Then R is a reduced ring. We claim that $t^n \notin C(R)$ for all $n \geq 1$. In fact, $t^n x = x^{2^n} t^n \neq x t^n$ for $n \geq 1$.

We next introduce a new ring property which generalize both Armendariz rings and K-rings. Following Bell [6], a ring R is said to satisfy the *Insertion-of-Factors-Property* (simply, an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Reduced rings are clearly IFP by simple computation. It is also easily checked that IFP rings are Abelian. It is easily checked that $N_*(R) = N^*(R) = N(R)$ for an IFP ring R. The classes of IFP rings and K-rings do not contain each other. Recall that $D_n(R)$ is a noncommutative K-ring for $n \geq 3$ if R is a K-ring of nonzero characteristic by [25, Theorem 2.3(2)]. But $D_n(R)$ cannot be IFP for any ring R and all $n \geq 4$ by [22, Example 1.3]. Note that any division ring, which is not a field, is an example of an IFP ring which is not K-ring. There also exists a non-reduced IFP ring, which is not a K-ring, by help of [22, Proposition 1.2] and [25, Theorem 2.3(1)], considering $D_3(R)$ when we take a reduced ring R of characteristic zero.

Due to Han et al. [13], a ring R (possibly without identity) is called *power-Armendariz* if whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$, there exist $m, n \geq 1$ (depending on a and b, repectively) such that $a^m b^n = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$. Here note that given each pair (a, b), $a^m b^n = 0$ for some $m, n \geq 1$ if and only if there exists $l \geq 1$ such that $a^l b^l = 0$

for all (a,b), because the number of (a,b)'s is finite. Armendariz rings are clearly power-Armendariz, but the converse need not be true by [13, Theorem 1.6] and [21, Example 3]. Power-Armendariz rings are Abelian by [13, Proposition 1.3(5)].

We see an interesting fact in the following when (power-)Armendariz rings and K-rings are combined.

Proposition 1.2. (1) *Let R be a power-Armendariz K-ring and suppose that $f(x)g(x)=0$ for $f(x),g(x)\in R[x]$. Then, for all $a\in C_{f(x)}$ and $b\in C_{g(x)}$, there exists $s\geq 1$ (depending on a and b) such that $a^sRb^s=0$.*

(2) *Let R be a power-Armendariz K-ring and suppose that $f(x)g(x)=0$ for $f(x),g(x)\in R[x]$. If R is a prime ring then $a\in N(R)$ or $b\in N(R)$ for every tuple $(a,b)\in C_{f(x)}\times C_{g(x)}$.*

Proof. (1) Suppose that $f(x)g(x)=0$ for $f(x),g(x)\in R[x]$. Since R is power-Armendariz, there exist $p\geq 1$ such that $a^pb^p=0$ for all $a\in C_{f(x)}$ and $b\in C_{g(x)}$. Moreover since R is a K-ring, there exist $q\geq 1$, depend on a and b, such that $a^q,b^q\in C(R)$. Set $s=pq$. Then we have $a^sRb^s=0$, combining $a^q,b^q\in C(R)$ and $a^pb^p=0$.

(2) Suppose that $f(x)g(x)=0$ for $f(x),g(x)\in R[x]$. Let R be a power-Armendariz K-ring. Then we obtain by (1) that there exists $s\geq 1$ such that $a^sRb^s=0$ for some $s\geq 1$ and for every tuple $(a,b)\in C_{f(x)}\times C_{g(x)}$. Here if R is a prime ring, then $a^s=0$ or $b^s=0$, entailing $a\in N(R)$ or $b\in N(R)$. □

Armendariz rings are power-Armendariz. So we can obtain the following by Proposition 1.2, using $ab=0$ in place of $a^pb^p=0$.

Corollary 1.3. (1) *Let R be an Armendariz K-ring and suppose that $f(x)g(x)=0$ for $f(x),g(x)\in R[x]$. Then, for all $a\in C_{f(x)}$ and $b\in C_{g(x)}$, there exist $m\geq 1$ such that $a^mRb=0$ and $aRb^m=0$.*

(2) *Let R be an Armendariz K-ring and suppose that $f(x)g(x)=0$ for $f(x),g(x)\in R[x]$. If R is a prime ring, then $a\in N(R)$ or $b\in N(R)$ for every tuple $(a,b)\in C_{f(x)}\times C_{g(x)}$.*

Following Agayev et al. [1], a ring R is called *central Armendariz* if $ab\in C(R)$ for all $a\in C_{f(x)}$ and $b\in C_{g(x)}$ whenever $f(x)g(x)=0$ for $f(x),g(x)\in R[x]$. Armendariz rings are clearly central Armendariz, but the converse need not hold by [1, Example 2.1]. Commutative rings are clearly central Armendariz but there exists a commutative ring which is not Armendariz by [27, Example 3.2]. Central Armendariz rings are Abelian by [1, Proposition 2.1].

We now introduce a new concept which combine power-Armendariz and central Armendariz, considering the results in Proposition 1.2 over centers.

Definition 1.4. A ring R (possibly without identity) will be called *central power-Armendariz* provided that if $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$ then there exist $m, n \geq 1$ such that

$$a^m b^n \in C(R) \text{ for all } a \in C_{f(x)} \text{ and } b \in C_{g(x)}.$$

The class of central power-Armendariz rings contains cenrtal Armendariz rings obvously. But the converse need not be true as we see in the following.

Moreover the class of central power-Armendariz rings contains both K-rings and Armendariz rings obvously. But the converses also need not be true as we see in the following.

Example 1.5. (1) There exists a central power-Armendariz ring but not central Armendariz. Let R be a non-reduced non-commutative ring which is both IFP and Armendariz (e.g., $D_3(\mathbb{Z})$ is such a ring by [21, Proposition 2] and [22, Proposition 1.2]). Then $D_3(R)$ is power-Armendariz (hence central power-Armendariz) by [13, Theorem 1.6(1)], but $D_3(R)$ is not (central) Armendariz by [1, Theorem 2.11]. Here $D_3(R)$ is also not a K-ring by [25, Theorem 2.3(1)].

(2) We use the ring in [16, Example 2]. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$, and B be the set of polynomials of zero constant term in A.

Let I be the ideal of A generated by $a_0rb_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$, and $r_1r_2r_3r_4$, where $r \in A$ and $r_1, r_2, r_3, r_4 \in B$. Then clearly $B^4 \in I$. Let $R = (\mathbb{Z}_2 + A)/I$. Then R is an IFP ring by [16, Example 2].

We identify $a_0, a_1, a_2, b_0, b_1, b_2, c$ with their images in R for simplicity. Then $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) = 0$. But $a_0b_1c \neq ca_0b_1$, so R is not (central) Armendariz.

Next let $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$. Then since R is IFP, $\frac{R}{N_*(R)}[x] \cong \frac{R[x]}{N_*(R)[x]}$ is a reduced (hence Armendariz) ring. Thus $f(x)g(x) = 0$ implies that $st \in N_*(R)$ for all $s \in C_{f(x)}$ and $t \in C_{g(x)}$. It is easily checked that $N_*(R) = B$ (i.e., $R/N_*(R) \cong \mathbb{Z}_2$) and $B^4 = 0$. Here, if

$s_0 \notin B$ and $t_0 \notin B$ for some $s_0 \in C_{f(x)}$ and $t_0 \in C_{g(x)}$, then $s_0t_0 \notin B$, a contradiction to $s_0t_0 \in N_*(R)$. Thus $s_0 \in B$ or $t_0 \in B$. This yields that $s^4t^4 = 0$ for all $s \in C_{f(x)}$ and $t \in C_{g(x)}$. Therefore R is central power-Armendariz.

(3) There exists a polynomial ring over an IFP ring which is a K-ring but not IFP. Let R the IFP ring in (2). Then $R[x]$ is not an IFP ring by the argument in [16, Example 2]. But we claim that $R[x]$ is a K-ring. Let $f \in R[x]$. Then f can be expressed by

$$f = f_0 + f_1 \text{ with } f_0 \in \mathbb{Z}_2[x] \text{ and } f_1 \in B[x].$$

We compute f^4. Since $f_0 \in C(R[x])$, $B^4 = 0$, and the characteristic of $R[x]$ is 2, we have

$$f^4 = (f_0 + f_1)^4 = f_0^4 + f_1^4 = f_0^4 \in C(R[x]).$$

This implies that $R[x]$ is a K-ring.

We observe next the basic properties of central power-Armendariz rings.

LEMMA 1.6. (1) *Central power-Armendariz rings are Abelian.*

(2) *The class of central power-Armendariz rings is closed under subrings (possibly without identity).*

(3) *Let R be a ring and $m, n, k \geq 1$. If $a^mb^n \in C(R)$ then $a^{mk}b^{nk} \in C(R)$.*

Proof. (1) We apply the proof of [13, Proposition 1.3(5)]. Let e be any idempotent in R. Consider $f(x) = e + (e + er(1-e))x$, $g(x) = (1-e)+((1-e)-er(1-e))x \in R[x]$ for any $r \in R$. Then $f(x)g(x) = 0$. Since R is central power-Armendariz,

$$(e + er(1 - e))^m(1 - e)^n = er(1 - e)$$

is central for some $m, n \geq 1$. So $er(1 - e) = 0$ for all $r \in R$. Similarly consider $h(x) = (1-e)+((1-e)+(1-e)re)x$ and $t(x) = e+(e-(1-e)re)x$ in $R[x]$ for any $r \in R$. Then $h(x)t(x) = 0$ and so $(1 - e)re = 0$ because R is central power-Armendariz. Thus R is Abelian.

(2) is obvious.

(3) Suppose that $a^mb^n \in C(R)$ for some $a, b \in R$ and $m, n \geq 1$. Then we have

$$\begin{aligned}(a^{mk}b^{nk})r &= a^{m(k-1)}(a^m b^n)b^{n(k-1)}r = a^{m(k-1)}b^{n(k-1)}r(a^m b^n)\\ &= a^{m(k-2)}(a^m b^n)b^{n(k-2)}ra^m b^n = a^{m(k-2)}b^{n(k-2)}ra^m(a^m b^n)b^n\\ &= \cdots = r(a^{mk}b^{nk}).\end{aligned}$$

for all $r \in R$. □

The following example shows that Abelian rings need not be central power-Armendariz. This illuminates Lemma 1.6(1).

EXAMPLE 1.7. We adapt [23, Example 18]. Let K be a field and $A = K\langle a_0, a_1, a_2\rangle$ be the free algebra generated by the noncommuting indeterminates a_0, a_1, a_2. Let I be the ideal of A generated by

$$a_0^2, a_0a_1 + a_1a_0, a_0a_2 + a_1^2 + a_2a_0, a_1a_2 + a_2a_1, a_2^2$$

and $R = A/I$. Then R is Abelian by [23, Example 18]. We identify a_i with their images in R for simplicity. For $f(x) = a_0 + a_0a_2x$, $g(x) = a_0a_2 - a_2a_0a_2x \in R[x]$, we get $f(x)g(x) = 0$ and a_1^2 is central by the definition of I. But $(a_0a_2)^k = (-1)^{k-1}a_0a_2a_1^{2(k-1)}$ is not central for all $k \geq 2$. Thus R is not central power-Armendariz.

It is natural to ask whether the class of central power-Armendariz rings is closed under direct products, as compared with the affirmative situation of subrings. But the answer is negative as follows.

EXAMPLE 1.8. The class of central power-Armendariz rings is not closed under direct products. We use the ring and the argument in [13, Example 1.7(3)]. Let $R_u = D_{8u}(S)$ for $u = 1, 2, \ldots$ where S is a reduced ring. Then R_u is (central) power-Armendariz by [13, Theorem 1.6(2)]. Set $E = \prod_{u=1}^{\infty} R_u$, the direct product of R_u's. Consider

$$f(x) = (e_{13} + e_{35} + \cdots + e_{(8u-3)(8u-1)})_{u=1}^{\infty} + (e_{12} + e_{34} + \cdots + e_{(8u-1)8u})_{u=1}^{\infty}x$$

and

$$g(x) = (e_{25} + e_{47} + \cdots + e_{(8u-4)(8u-1)})_{u=1}^{\infty} - (e_{35} + e_{57} + \cdots + e_{(8u-3)(8u-1)})_{u=1}^{\infty}x$$

in $E[x]$. Then $f(x)g(x) = 0$. Use $(-)$ in place of $(-)_{u=1}^{\infty}$ for simplicity here. There cannot exist $m, n \geq 1$ such that $(e_{13} + e_{35} + \cdots +$

$e_{(8u-3)(8u-1)})^m(e_{35}+e_{57}+\cdots+e_{(8u-3)(8u-1)})^n$ is cenrtral. In fact, we have

$$\begin{aligned}&(e_{13}+e_{35}+\cdots+e_{(8u-3)(8u-1)})^m(e_{35}+e_{57}+\cdots+e_{(8u-3)(8u-1)})^n\\ =&(e_{1(1+2m)}+e_{3(3+2m)}+\cdots+e_{(8u-(1+2m))(8u-1)})(e_{3(3+2n)}+e_{5(5+2n)}\\ &\qquad+\cdots+e_{(8u-(1+2n))(8u-1)})\\ =&(e_{1(1+2m+2n)}+e_{3(3+2m+2n)}+\cdots+e_{(8u-(1+2m+2n))(8u-1)})\notin C(E).\end{aligned}$$

2. Properties in relation to other ring properties

In this section we push the study of central power-Armendariz rings further. We concentrate our works on the relations between central power-Armendariz rings and related ring properties.

If R is an Armendariz ring then $N_0(R)=N_*(R)=N^*(R)$ by [20, Lemma 2.3(5)]. We see a similar result for central Armendariz rings as follows.

Following the literature, the *index of nilpotency* of $a\in N(R)$ is the least positive integer n such that $a^n=0$, write $i(a)$ for n. The *index of nilpotency* of a subset S of R is the supremum of the indices of nilpotency of all nilpotent elements in S, write $i(S)$; and if such a supremum is finite, then S is said to be *of bounded index of nilpotency.* $[a]$ means the largest integer $\leq a$ for a real number a.

Proposition 2.1. (1) *Let R be a central Armendariz ring with $i(R)=2$. Then*

$$N_0(R)=N_*(R)=N^*(R).$$

(2) *Let R be a central Armendariz ring which is not Armendariz. Then R contains a nonzero ideal which is contained in $C(R)$.*

(3) *Let R be a central power-Armendariz ring and suppose that $a,b\in N(R)$ with $a^m=0$ and $b^n=0$. Then*

$$a^{[\frac{m+1}{2}]}b^h \text{ and } a^kb^{[\frac{n+1}{2}]}\in N(R)$$

for $1\leq h\leq n-1$ and $1\leq k\leq m-1$. Especially $ab\in N(R)$ when $a^2=0$ or $b^2=0$.

Proof. (1) Let $cd=0$ for $c,d\in R$ and $a\in N(R)$. Then $a^2=0$ and so $rc(1-ax)(1+ax)ds=rc1ds=rcds=0$ for all $r,s\in R$. Since R is central Armendariz, $rcads\in C(R)$. So $RcN(R)dR\subseteq C(R)$ since

$$rcads+r_1cads_1=rscad+r_1s_1cad=(rs+r_1s_1)cad\in C(R)$$

for all $r_1, s_1 \in R$.

Next let $a \in N^*(R)$. Then $a^2 = 0$ and $RaR \subseteq N^*(R)$; hence we obtain

$$(RaR)^3 = Ra(RaR)aR \subseteq C(R)$$

from the preceding result. It follows from this that

$$0 = (RaR)^3 a^2 = a(RaR)^3 a$$

and

$$(RaR)(RaR)^3(RaR) = (RaR)^5 = 0,$$

entailing $a \in N_0(R)$.

(2) Let R be a central Armendariz ring which is not Armendariz. Then there exist $f(x), g(x) \in R[x]$ such that $f(x)g(x) = 0$ but $ab \neq 0$ for some $a \in C_{f(c)}, b \in C_{g(x)}$. Since R is central Armendariz, $ab \in C(R)$. From $f(x)g(x) = 0$, we have $rf(x)g(x)s = 0$ for all $r, s \in R$. So $rabs \in C(R)$. But $ab \in C(R)$, so the subset

$$I = \{rabs \mid r, s \in R\}$$

of $C(R)$ is a nonzero ideal of R since

$$rabs + r_1abs_1 = rsab + r_1s_1ab = (rs + r_1s_1)ab \in I$$

for all $r_1, s_1 \in R$.

(3) Let $a, b \in N(R)$ such that $a^m = b^n = 0$. We apply the proof of [13, Proposition 1.10]. Consider two polynomials $f(x) = 1-bx, g(x) = 1 + bx + \cdots + b^{n-1}x^{n-1}$. Then $f(x)g(x) = 1$ since $b^n = 0$. Multiplying $f(x)g(x)$ on the left and right by $a^{[\frac{m+1}{2}]}$ and $a^{[\frac{m+1}{2}]}b$ respectively, we obtain

$$\begin{aligned} 0 &= a^{[\frac{m+1}{2}]}a^{[\frac{m+1}{2}]}b = a^{[\frac{m+1}{2}]}(1-bx)(1+bx+\cdots+b^{n-1}x^{n-1})a^{[\frac{m+1}{2}]}b \\ &= (a^{[\frac{m+1}{2}]} - a^{[\frac{m+1}{2}]}bx)(a^{[\frac{m+1}{2}]}b + ba^{[\frac{m+1}{2}]}bx + \cdots + b^{n-1}a^{[\frac{m+1}{2}]}bx^{n-1}). \end{aligned}$$

Since R is central power-Armendariz,

$$(a^{[\frac{m+1}{2}]}b)^s(a^{[\frac{m+1}{2}]}b)^t \in C(R)$$

for some $s, t \geq 1$. Then we obtain

$$(a^{[\frac{m+1}{2}]}b)^{s+t+1} = (a^{[\frac{m+1}{2}]}b)^{s+t}a^{[\frac{m+1}{2}]}b = a^{[\frac{m+1}{2}]}(a^{[\frac{m+1}{2}]}b)^{s+t}b = 0.$$

This yields $a^{[\frac{m+1}{2}]}b \in N(R)$. Similarly we get

$$b^{[\frac{n+1}{2}]}a \in N(R)$$

and so $ab^{[\frac{n+1}{2}]} \in N(R)$.

Use now a^k and b^h in place of a and b respectively in the computation above, where $1 \leq h \leq n-1$ and $1 \leq k \leq m-1$. Then we can obtain $a^{[\frac{m+1}{2}]}b^h$ and $a^k b^{[\frac{n+1}{2}]} \in N(R)$. The remainder follows immediately. $\square$

We can see an example satisfying Proposition 2.1(1) in [4, Example 4.10]. The converse of Proposition 2.1(1) is not true in general as can be seen by $R = U_2(A)$ over a reduced ring A, noting that R is non-Abelian and $N_0(R) = N_*(R) = N^*(R) = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$.

We next observe a kind of algebraic structure in which the central power-Armendariz and (weak) Armendariz are equivalent. Let K be a field and R_1, R_2 be K-algebras. Use $R_1 *_K R_2$ to denote the ring coproduct of R_1 and R_2 (see Antoine [4] and Bergman [7,8] for details). The following is an extension of [13, Corollary 1.11].

Corollary 2.2. *Let K be a field and A be a K-algebra. Let $C = K[b]$ be the polynomial ring with an indeterminate b over K, and I be the ideal of C generated by b^2. Set $B = C/I$ and $R = A *_K B$. Then the following conditions are equivalent:*

(1) *R is Armendariz;*
(2) *R is power-Armendariz;*
(3) *R is weak Armendariz;*
(4) *R is central power-Armendariz;*
(5) *$N(R)$ is multiplicatively closed;*
(6) *A is a domain and $U(A) = K\backslash\{0\}$.*

Proof. The equivalence of the conditions (1) and (6) are shown by [4, Theorem 4.7]. The equivalences of the conditions (1), (3), and (5) are shown by [17, Theorem 1]. (1)⇒(2) and (2)⇒(4) are obvious. It suffices to prove (4)⇒(5). But $i(R) = 2$ by help of the computation in [17, page 5]. So $N(R)$ is multiplicatively closed by Proposition 2.1(3). $\square$

In Corollary 2.2, let $A = K[[a]]$ be the power series ring with an indeterminate a over a field K. Then $U(A) \neq K\backslash\{0\}$ since $1-a \in U(A)$. Let $f(x) = u^{-1}bu^{-1}b + u^{-1}bu^{-1}x, g(x) = bub - ubx \in R[x]$ as in the proof of [17, Theorem 1], where $u = 1-a$. Then $f(x)g(x) = 0$ but $u^{-1}bu^{-1}bub = (1-a)^{-1}b(1-a)^{-1}b(1-a)b$ is not central in R since $a(1-a)^{-1}b(1-a)^{-1}b(1-a)b \neq (1-a)^{-1}b(1-a)^{-1}b(1-a)ba$. This illuminates the details of Corollary 2.2.

It is natural to conjecture that R is a central power-Armendariz ring if for any nonzero proper ideal I of R, R/I and I are central power-Armendariz, where I is considered as a central power-Armendariz ring without identity. However we have a negative answer to this situation by the following example.

EXAMPLE 2.3. Let D be any division ring and consider $R = U_2(D)$. Then, by [21, Example 14], R/I and I are Armendariz for any nonzero proper ideal I of R, where I is considered as a subring of R without identity. So R/I and I are also central power-Armendariz. Now we show that R is not central power-Armendariz. Let $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$ be the polynomials in $R[x]$. Then $f(x)g(x) = 0$ but, for any $m, n \geq 1$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^m \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not central.

Moerover Example 2.3 illuminates that the ring R is not central power-Armendariz any more, if we take the stonger condition "I is Armendariz" instead of the condition "I is central power-Armendariz". However if we take a stronger condition "I is reduced (i.e., I contains no nonzero nilpotent elements)" then we may have an affirmative answer as in the following.

PROPOSITION 2.4. *Let R be a ring and assume that R/I is a central power-Armendariz ring for some proper ideal I of R. If I contains no nonzero nilpotent elements, then R is central power-Armendariz.*

Proof. Assume that $f(x)g(x) = 0$ for $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$. Then $a_iIb_j = b_jIa_i = 0$ for all i, j by [13, Theorem 1.12(1)].

Since R/I is central power Armendariz, there exist $s, t \geq 1$ such that $\overline{a_i}^s\overline{b_j}^t \in C(R/I)$ for all $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$. So $a_i{}^sb_j{}^tr - ra_i{}^sb_j{}^t \in I$ for all $r \in R$. Combining this with the result that $a_iIb_j = b_jIa_i = 0$, we can obtain

$$(a_i{}^sb_j{}^tr - ra_i{}^sb_j{}^t)I(a_i{}^sb_j{}^tr - ra_i{}^sb_j{}^t) = 0.$$

But since I contains no nonzero nilpotent elements, we have $a_i{}^sb_j{}^tr - ra_i{}^sb_j{}^t = 0$, entailing $a_i{}^sb_j{}^tr = ra_i{}^sb_j{}^t$. This implies that $a_i{}^sb_j{}^t$ is central, proving that R is central power-Armendariz. $\square$

As a converse of Proposition 2.4, one may ask whether R/I is central power-Armendariz for a given central power-Armendariz ring R and a proper ideal I of R which is reduced as a subring of R without identity. The answer is negative by the following. Let R be the Hamilton quaternions over the field of real numbers. Consider the ideal pR for a prime integer ≥ 3. Then R/I is isomorphic to $Mat_2(\mathbb{Z}_p)$ by [12, Exercise 2A], entailing that R/I is not central power-Armendariz by help of Lemma 1.6(1). We see an application of Proposition 2.4 in the following.

Example 2.5. Let K be a field and $A = K\langle a_i, b_j, c \mid i \in I, j \in J\rangle$ be the free algebra generated by the noncommuting indeterminates c, a_i's and b_j's over K, where I, J are index sets. Let B be the ideal of A generated by

$$a_iAc, cAa_i, b_jAc, cAb_j, d_1d_2d_3d_4$$

where $d_1, d_2, d_3, d_4 \in \{a_i, b_j\}$. Set $R = A/B$.

Consider the ideal $C = R\bar{c}R$ of R. Then C is a nonzero proper ideal of R that is reduced as a ring since C is isomorphic to $\bar{c}K[\bar{c}]$. Consider R/C. Then R/C is isomorphic to $K\langle a_i, b_j\rangle/D$ where D is the ideal of $K\langle a_i, b_j\rangle$ generated by

$$d_1d_2d_3d_4 \text{ with } d_1, d_2, d_3, d_4 \in \{a_i, b_j\}.$$

We identify $\bar{a}, \bar{b}$ with their images in $D = R/C$ for simplicity. Note that

$$N(D) = N_*(D) = N^*(D) = D\bar{a}D + D\bar{b}D \text{ and } D/N_*(D) \cong K.$$

Let $f(x)g(x) = 0$ for $f(x), g(x) \in D[x]$. Then $h \in N_*(D)$ or $k \in N_*(D)$ for all $h \in C_{f(x)}, k \in C_{g(x)}$ since $D/N_*(D)$ is a field. This implies that $h^4k^4 = 0$ in D for all $h \in C_{f(x)}, k \in C_{g(x)}$, so D is (central) power-Armendariz. Thus R is (central) power-Armendariz by Corollary 2.2(4).

In the proof of Lemma 1.6(1), we use linear polynomials whose product is zero. This naturally induces the following definition. We will call a ring R *linearly central power-Armendariz* if whenever linear polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, there exist $m, n \geq 1$ such that $a^mb^n \in C(R)$ for all $a \in C_{f(x)}, b \in C_{g(x)}$. Thus linearly central power-Armendariz rings are Abelian by the proof of Lemma 1.6(1), but not conversely by Example1.7.

Following Agayev et al. [2], a ring R is called *central semicommutative* if $ab = 0$ for $a, b \in R$ implies $arb \in C(R)$ for all $r \in R$. It is clear that every IFP ring is central semicommutative. By [2, Lemma 2.6] central semicommutative rings are Abelian.

In the following two propositions we consider the cases of polynomials of degree ≤ 2. We actually do no know whether these results are true for the general case of degree n polynomials. Thus we deal with in this note the special case of degree ≤ 2.

In the following we prove that central semicommutative rings are linearly central power-Armendariz. We obtain the following relation between central semicommutative rings and linearly central power-Armendariz rings.

PROPOSITION 2.6. *Every central semicommutative ring is linearly central power-Armendariz.*

Proof. Let R be a central semicommutative ring, and take two linear polynomials $f(x) = a_0 + a_1x, g(x) = b_0 + b_1x \in R[x]$ with $f(x)g(x) = 0$. Then $a_iRb_i \subseteq C(R)$ for $i = 0, 1$ since $a_0b_0 = 0 = a_1b_1$. From the equality $a_0b_1 + a_1b_0 = 0$, we have $(a_0b_1 + a_1b_0)b_0 = 0$ and $(a_0b_1 + a_1b_0)b_1 = 0$. So

$$a_1b_0^2 = -a_0b_1b_0 \in C(R) \text{ and } a_0b_1^2 = -a_1b_0b_1 \in C(R).$$

Thus R is linearly central power-Armendariz. □

The converse of Proposition 2.6 does not hold in general by the following.

EXAMPLE 2.7. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Set I be the ideal of A generated by b^2. Then $R = A/I$ is isomorphic to $K[a] *_K \frac{K[b]}{b^2K[b]}$. So R is (linearly central power-)Armendariz by Corollary 2.2, but not central semicommutative as can be seen by $b^2 = 0$ and $bab \notin C(R)$. In fact, $abab \neq baba$.

In the following we see a similar result for the case of polynomials of degree 2, considering a stronger condition than central semicommutative. We deal here with rings without identity.

PROPOSITION 2.8. *Let R be a ring without idenity and assume that R satisfies the condition $(*)$:*

$$arb \in C(R) \text{ for all } r \in R \text{ whenever } ab \in C(R) \text{ for } a, b \in R.$$

Then there exist $m, n \geq 1$ such that $a_i^m b_j^n \in C(R)$ for all $i, j = 0, 1, 2$ whenever $f(x)g(x) = 0$ for $f(x) = a_0+a_1x+a_2x^2, g(x) = b_0+b_1x+b_2x^2 \in R[x]$.

Proof. Suppose that $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{2} a_i x^i, g(x) = \sum_{j=0}^{2} b_j x^j$ in $R[x]$. Then we have

$$\text{(I)} \qquad a_0 b_0 = 0,$$

$$\text{(II)} \qquad a_0 b_1 + a_1 b_0 = 0,$$

$$\text{(III)} \qquad a_0 b_2 + a_1 b_1 + a_2 b_0 = 0,$$

$$\text{(IV)} \qquad a_2 b_1 + a_1 b_2 = 0,$$

$$\text{(V)} \qquad a_2 b_2 = 0.$$

We will use freely the condition $(*)$. Multiplying the equality (II) by b_0 on the right (resp., by a_0 on the left), we have

$$\text{(VI)} \qquad a_1 b_0^2 = -a_0 b_1 b_0 \in C(R) \text{ (resp., } a_0^2 b_1 = -a_0 a_1 b_0 \in C(R))$$

by the equality (I).

Multiplying the equality (III) by b_0^2 on the right (resp., by a_0^2 on the left), we have

(VII)

$$a_2 b_0^3 = -a_0 b_2 b_0^2 - a_1 b_1 b_0^2 \in C(R) \text{ (resp., } a_0^3 b_2 = -a_0^2 a_1 b_1 - a_0^2 a_2 b_0 \in C(R))$$

by the equalities (I) and (VI).

Multiplying the equality (IV) by b_2 on the right (resp., by a_2 on the left), we have

$$\text{(VIII)} \qquad a_1 b_2^2 = -a_2 b_1 b_2 \in C(R) \text{ (resp., } a_2^2 b_1 = -a_2 a_1 b_2 \in C(R))$$

by the equality (V).

Multiplying the equality (III) by b_2^2 on the right (resp., by a_2^2 on the left), we have

(IX)

$$a_0 b_2^3 = -a_1 b_1 b_2^2 - a_2 b_0 b_2^2 \in C(R) \text{ (resp., } a_2^3 b_0 = -a_2^2 a_0 b_2 - a_2^2 a_1 b_1 \in C(R))$$

by the equalities (V) and (VII).

Lastly we will find $s, t \geq 1$ such that $a_1^s b_1^t \in C(R)$. From the equalities (I) $\sim$ (V), we have $(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 0$. This equality yields that $(a_0 + a_1 + a_2) r (b_0 + b_1 + b_2) \in C(R)$ for all $r \in R$. Taking $r = a_1^2 b_1^2$ here, we get

$$\begin{aligned} a_1^3 b_1^3 = &-a_0(a_1^2 b_1^2) b_0 - a_0(a_1^2 b_1^2) b_1 - a_0(a_1^2 b_1^2) b_2 - a_1(a_1^2 b_1^2) b_0 - a_1(a_1^2 b_1^2) b_2 \\ &- a_2(a_1^2 b_1^2) b_0 - a_2(a_1^2 b_1^2) b_1 - a_2(a_1^2 b_1^2) b_2. \end{aligned}$$

Letting $\alpha\beta = 0$ for $\alpha, \beta \in R$ implies $r\alpha\beta t = 0$ for all $r, t \in R$, and so we have $r\alpha s\beta t \in C(R)$ for all $r, s, t \in R$. This result yields the following.

$$\begin{aligned}
&-a_0(a_1^2b_1^2)b_0 \in C(R) \ (\because a_0b_0 = 0),\\
&-a_0(a_1^2b_1^2)b_1 = a_0a_1(a_0b_2 + a_2b_0)b_1^2 = \underline{a_0}a_1\underline{a_0}b_2\underline{b_1^2} + \underline{a_0}a_1a_2\underline{b_0}b_1^2 \in C(R)\\
&\qquad\qquad\qquad (\because a_0^2b_1 \in C(R)),\\
&-a_0(a_1^2b_1^2)b_2 = \underline{a_0}a_1\underline{a_0}b_2\underline{b_1b_2} + \underline{a_0}a_1a_2\underline{b_0}b_1b_2 \in C(R)\\
&\qquad\qquad\qquad (\because a_0b_1 + a_1b_0 = 0 \Rightarrow a_0(a_0b_1 + a_1b_0)b_2 = 0\\
&\qquad\qquad\qquad\quad \Rightarrow a_0^2b_1b_2 = -a_0a_1b_0b_2 \in C(R)),\\
&-a_1(a_1^2b_1^2)b_0 = a_1^2\underline{a_0}b_2b_1\underline{b_0} + \underline{a_1}a_1a_2\underline{b_0}b_1\underline{b_0} \in C(R) \ (\because a_1b_0^2 \in C(R)),\\
&-a_1(a_1^2b_1^2)b_2 = \underline{a_1}a_1a_0\underline{b_2}b_1\underline{b_2} + a_1^2\underline{a_2}b_0b_1\underline{b_2} \in C(R)\\
&\qquad\qquad\qquad (\because a_1b_2^2 \in C(R) \text{ and } a_2b_2 = 0),\\
&-a_2(a_1^2b_1^2)b_0 = a_2a_1\underline{a_0}b_2b_1\underline{b_0} + \underline{a_2}a_1\underline{a_2}b_0\underline{b_1b_0} \in C(R)\\
&\qquad\qquad\qquad (\because a_1b_2 + a_2b_1 = 0 \Rightarrow a_2(a_1b_2 + a_2b_1)b_0 = 0\\
&\qquad\qquad\qquad\quad \Rightarrow a_2^2b_1b_0 \in C(R)),\\
&-a_2(a_1^2b_1^2)b_1 = \underline{a_2}a_1a_0\underline{b_2}b_1^2 + \underline{a_2}a_1\underline{a_2}b_0b_1\underline{b_1} \in C(R) \ (\because a_2^2b_1 \in C(R)),\\
&-a_2(a_1^2b_1^2)b_2 \in C(R) \ (\because a_2b_2 = 0).
\end{aligned}$$

Thus we obtain finally $a_1^3b_1^3 \in C(R)$. □

We observe next some conditions under which related concepts are equivalent. In [19], *Baer rings* are introduced as rings in which the right(left) annihilator of every nonempty subset generated by an idempotent. According to Clark [10], a ring is said to be *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. These definitions are left-right symmetric. A ring R is called *right principally quasi-Baer* (or simply, *right p.q.-Baer*) [9] if the right annihilator of a principal right ideal of R is generated by an idempotent. Finally, a ring R is called *right principal projective*(or simply, *right p.p.-ring*) if the right annihilator of an element of R is generated by an idempotent.

THEOREM 2.9. *For a right p.p.-ring R the following conditions are equivalent:*

(1) *R is Armendariz;*

(2) *R is power-Armendariz;*

(3) *R is central Armendariz;*
(4) *R is central power-Armendariz;*
(5) *R is linearly central power-Armendariz;*
(6) *R is Abelian;*
(7) *R is reduced.*

Proof. (1)⇒(2), (1)⇒(3), (2)⇒(4), (3)⇒(4) and (4)⇒(5) are trivial. (5) ⇒(6) holds by the proof of Lemma 1.6(1). Abelian right p.p. rings are reduced (hence Armendariz) by the argument prior to [21, Theorem 9], showing (6)⇒(7). (7)⇒(1) is obvious. □

A ring R is called *weak Armendariz* [26, Definition 2.1] if whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$ we have $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. IFP rings are weak Armendariz, but not conversely by [26, Corollary 3.4 and Example 3.5]. Armendariz rings are obviously weak Armendariz. There exists IFP (hence weak Armendariz) rings but not Armendariz by [16, Example 2], and the ring $R = U_2(F)$ for a field F is weak Armendariz by [26, Proposition 2.2], but R is neither central power-Armendariz nor IFP.

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$.

Theorem 2.10. *For a regular ring R the following conditions are equivalent:*
(1) *R is central power-Armendariz;*
(2) *R is linearly central power-Armendariz;*
(3) *R is weak Armendariz;*
(4) *R is power-Armendariz;*
(5) *R is central Armendariz;*
(6) *R is Armendariz;*
(7) *R is Abelian;*
(8) *R is IFP;*
(9) *R is reduced.*

Proof. (6) ⇒ (5) ⇒ (1) ⇒ (2) are clear. (6) ⇔ (7) ⇔ (8) ⇔ (9) and (3) ⇔ (4) ⇔ (6) by [11, Theorem 3.2] and [13, Theorem 1.9]. Also (2) ⇒ (7) holds by the proof of Lemma 1.6(1). □

Recall that a ring R is called *π-regular* if for each $a \in R$ there exists a positive integer n, depending on a, $b \in R$ such that $a^n = a^n ba^n$. Regular rings are clearly π-regular. However central power-Armendariz π-regular rings need not be reduced as can be seen by $R = D_n(D)$ where $n \geq 2$ and D is a division ring. In fact, R is power-Armendariz by [13, Theorem 1.6(2)] and so R is central power-Armendariz. Moreover R is π-regular by [15, Lemma 5], but R is not reduced.

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