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ON THE ANTICYCLOTOMIC $\mathbb{Z}_p$-EXTENSION OF AN IMAGINARY QUADRATIC FIELD

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Abstract. We prove that if a subfield of the Hilbert class field of an imaginary quadratic field k meets the anticyclotomic $\mathbb{Z}_p$-extension k^a_∞ of k, then it is contained in k^a_∞ . And we give an example of an imaginay quadratic field k with $\lambda_3(k^a_\infty) \geq 8$.

1. Introduction

An abelian extension L of k is called an anti-cyclotomic extension of k if it is Galois over $\mathbb{Q}$, and $Gal(k/\mathbb{Q})$ acts on $Gal(L/k)$ by -1. For each prime number p, the compositum K of all $\mathbb{Z}_p$-extensions over k becomes a ${\mathbb{Z}_p}^2$-extension, and K is the compositum of the cyclotomic $\mathbb{Z}_p$-extension k^c_∞ and the anti-cyclotomic $\mathbb{Z}_p$-extension k^a_∞ of k.

The layers k^c_n of the cyclotomic $\mathbb{Z}_p$-extension are well understood. Since the Hilbert class field of k is an anti-cyclotomic extension of k, determination of the first layer of the anti-cyclotomic $\mathbb{Z}_p$-extension becomes complicated as the p-rank of the p-Hilbert class field of k becomes larger. In the papers [3,5,6], using Kummer theory and class field theory, we constructed the first layer k^a_1 of the anti-cyclotomic $\mathbb{Z}_3$-extension of k under the assumption that the 3-part of Hilbert class field H_k of k is 3-elementary. A criterion on linearly disjointness of k^a_1 and H_k over k is

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proved in [4] under the assumption. In this paper,we prove the criterion without the assumption. See Corollary 1 of this paper.

Contrary to the case of the cyclotomic $\mathbb{Z}_p$-extension, the lambda invariant $\lambda_p(k^a_\infty)$ of the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field is not well known. Few examples of computation of $\lambda_p(k^a_\infty)$ are given. Following the idea of Fujii [1], we give an example of k with $\lambda_3(k^a_\infty) \geq 8$.

2. Proof of Theorems

Let p be an odd prime number. Throughout this section, we denote by H_k, h_k, A_k, and M_k the p-part of Hilbert class field, the p-class number, p-part of ideal class group, and the maximal abelian p-extension of an imaginary quadratic field k unramified outside above p, respectively. The first layer of the anti-cyclotomic $\mathbb{Z}_p$-extension of k may be or may not be contained in the p-Hilbert class field of k. The following theorem and the criterion in [4] gives an answer for this question. We define $\mathrm{rank}_{\mathbb{Z}/p}A$ to be the dimension of A/A^p over $\mathbb{Z}/p\mathbb{Z}$ for any abelian group A. Note that $K \cap H_k = k^a_\infty \cap H_k$.

Theorem 1. *Let $d \not\equiv 3$ mod 9 be a square free positive integer, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field. Let L be a subfield of H_k which satisfies the following properties:*

$$H_k \cap k^a_\infty = k^a_n \leq L (n \geq 1), \quad Gal(L/k) \text{ is cyclic.}$$

Then

$$L = k^a_n.$$

Proof. Assume that $k^a_n \neq L$. Then there exists a ramified extension of k of degree p which becomes unramifed over k^a_∞. By class field theory, we see that

$$Gal(M_k/H_k) \simeq (\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}}),$$

where $U_{1,\mathfrak{p}}$ is the local units of k which is congruent to 1 $mod\mathfrak{p}$. However, by the condition of Theorem 1, there is no p-torsion point in $\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}}$, which contradicts to the fact that the ramified extension of k of degree p exists. This completes the proof. □

By Theorem 1 one can easily prove the following corollary, which was proved in [4] with the assumption that $A_{\mathbb{Q}(\sqrt{-d})}$ is 3-elementary, without the assumption. In fact, the following equivalence

$$H_k \cap k_\infty^a = k \Longleftrightarrow rank_{\mathbb{Z}/p} X_{k,\chi} = 1 + rank_{\mathbb{Z}/p} A_k$$

in [4] holds without the assumption by Theorem 1. Here

$$X_k := Gal(M_k/k)/pGal(M_k/k)$$

and $X_{k,\chi}$ be the χ-component of X_k for the nontrivial character χ of $Gal(k/\mathbb{Q})$.

Corollary 1. *Let $d \not\equiv 3 \bmod 9$ be a square free positive integer, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field and k_∞^a the anti-cyclotomic $\mathbb{Z}_3$-extension over k. Then*

$$H_k \cap k_\infty^a = k \Longleftrightarrow$$

$$\mathrm{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} = \mathrm{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})}.$$

By following the idea of Fujii [1], we give an example of an imaginary quadratic field with large invariant $\lambda_3(k_\infty^a)$.

Theorem 2.

$$\lambda_3(k_\infty^a) \geq 8,$$

where $k = \mathbb{Q}(\sqrt{-1423})$,

Proof. Denote by K_2^a the compositum of all $\mathbb{Z}_3$-extensions of k_2^a. First note that the class number of $\mathbb{Q}(\sqrt{3 * 1423})$ is one. Hence, by Theorem 3 below, $H_k \subset k_\infty^a$. Since the class number of k is 9, $H_k = k_2^a$. By simple computation, we see that 3 stays prime in k. The definition of anti-cyclotomic extension and class field theory shows that $\mathfrak{p}_3$, the prime of k above 3, splits completely in k_2^a. Note that the $\mathbb{Z}_3$-rank of $Gal(K_2^a/k_2^a)$ is 10. Since the inertia group of primes of k_2^a above 3 is isomorphic to $\mathbb{Z}_3^2$ and K/k is abelian, the extension K_2^a/K is unramified everywhere. Hence the maximal abelian 3-extension of k_∞^a contains K_2^a, and the galois group of K_2^a over K is isomorphic to $\mathbb{Z}_3^8$. This completes the proof. □

The following theorem is given in [2].

Theorem 3. *If $p = 3$ and $d \not\equiv 3 \bmod 9$, then $H_k \subset k_\infty^a$ if and only if the class number of $\mathbb{Q}(\sqrt{3d})$ is not divisible by 3.*

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