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ANALYSIS ON GENERALIZED IMPACT ANGLE CONTROL GUIDANCE LAW

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ABSTRACT. In this paper, a generalized guidance law with an arbitrary pair of guidance coefficients for impact angle control is proposed. Under the assumptions of a stationary target and a lag-free missile with constant speed, necessary conditions for the guidance coefficients to satisfy the required terminal constraints are obtained by deriving an explicit closed-form solution. Moreover, optimality of the generalized impact-angle control guidance law is discussed. By solving an inverse optimal control problem for the guidance law, it is found that the generalized guidance law can minimize a certain quadratic performance index. Finally, analytic solutions of the generalized guidance law for a first-order lag system are investigated. By solving a third-order linear time-varying ordinary differential equation, the blowing-up phenomenon of the guidance loop as the missile approaches the target is mathematically proved. Moreover, it is found that terminal misses due to the system lag are expressed in terms of the guidance coefficients, homing geometry, and the ratio of time-to-go to system time constant.

1. INTRODUCTION

Some guided weapons have additional requirements to achieve a proper flight path angle at impact for enhanced warhead effect, better target information, and so on. For this reason, impact angle control guidance laws have been widely studied. Most of these studies, optimal guidance laws for various performance indices are suggested by solving the linear quadratic optimal control problem based on linearized models of pursuit kinematics. The energy-optimal guidance law with the impact angle constraint for lag-free system [1], which is identical to the optimal solution of the simple rendezvous problem solved by Bryson and Ho [3], is representative. Ryoo et al. proposed a time-to-go weighted optimal guidance law that was obtained by the solution of a linear quadratic optimal control problem with the energy cost weighted by a power of the time-to-go [6]. Most of these methods need time-to-go estimations for implementation of the guidance laws. On the other hand, there are some different approaches concerned with additional bias terms of the conventional proportional navigation guidance (PNG) to control the impact angle. Kim et al. [8] have considered a time-varying bias which is intuitively chosen as a combination of the state variables such as the line-of-sight angle (LOS), the relative range, and the flight path angle.

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According to the previous works, lots of impact-angle control guidance laws have two time-varying feedback terms in common: one is a position error term and the other is a velocity error term. Especially, the energy optimal impact-angle-control guidance law for lag-free system has two time-varying feedback terms with fixed guidance coefficients. Here, a question arises: what would happen if a guidance law for an arbitrary pair of guidance coefficients rather than the fixed values is adopted. In order to clarify it, the generalized guidance law is introduced by replacing the fixed guidance coefficients of the optimal guidance laws with an arbitrary pair of guidance coefficients.

Every pair of the guidance coefficients, however, cannot guarantee that the missile achieves guidance goals. For example, if the two guidance coefficients are the same, the guidance law may become conventional PNG which cannot satisfy the impact angle constraint. In order to find all feasible guidance coefficients for impact angle control, necessary conditions for the guidance coefficients to satisfy the required impact angle as well as zero miss distance are studied by obtaining explicit closed-form solutions for lag-free system. In the guidance law, furthermore, time-to-go appears explicitly but it cannot be directly measured from any device. Hence, a suitable time-to-go estimation method is required to implement the guidance law. In this paper, a practical and precise time-to-go estimation method for the generalized guidance law is discussed by considering the curved trajectory depending on the guidance coefficients.

Optimality of the generalized guidance law is another main concern of this paper. It is found that the guidance law with an arbitrary pair of guidance coefficients can minimize a certain quadratic performance index subject to the terminal constraints by solving an inverse optimal control problem whose purpose is to find the performance index for which the given guidance law is optimal. The solution of the inverse optimal problem furnishes us with the relationship between the guidance coefficients and the corresponding quadratic performance index which provides physical insights into the maneuvering characteristics of the missile. Above all, it serves as a theoretical foundation to choose the adequate values of the guidance coefficients to improve robustness to external disturbances and uncertainties of the missile system. In addition, it will provide a chance for the missile to fulfill additional requirements such as small angle-of-attack at impact for warhead effect, flight time constraint for salvo-attack, look angle limitation for strap-down seeker, and so on.

Finally, performance degradation of the generalized guidance law due to the system lag is investigated by obtaining analytic solutions for a first-order lag system. While the generalized guidance law with feasible guidance coefficients for a lag-free system satisfies the terminal constraints, it can hardly be expected to provide the perfect performance in practice because there are a lot of error sources in realistic environment. In order to investigate the performance degradation due to system lag which is one of the critical error sources, explicit closed-form solutions for the first-order lag system are obtained. If system lag is considered, however, it is hard to derive analytic solutions because the governing equation becomes a high-order non-equidimensional linear time-varying differential equation. Under the assumption that the missile is given by a first-order lag system, analytic solutions are derived for the generalized guidance law with an arbitrary pair of guidance coefficients. The analytic solutions provide an insight into the behavior of the missile near to target: the guidance command, the acceleration of the missile, and the velocity component perpendicular to the collision course tend to diverge as the missile approaches the target. Terminal misses due to the system lag are discussed by using the analytic solutions, and effects of guidance coefficients on the terminal misses are examined. In addition, homing geometries which are advantageous in terms of the terminal misses are proposed. It is important that the analytic solutions explain blowing-up phenomenon of the homing loop mathematically and provide useful clues in designing an impact-angle-control guidance law.

2. Generalized Impact-Angle-Control Guidance Law

In this chapter, a generalized guidance law for impact angle control is introduced. The guidance law is initially assumed as a similar form with arbitrary guidance coefficients to other impact-angle-control guidance laws. By obtaining closed-form solutions of the guidance law for a lag-free system, feasible sets of the guidance coefficients to satisfy the impact angle constraint as well as zero miss distance are demonstrated. In order to implement the guidance law, a time-to-go calculation method which considers the curved-path depending on the guidance coefficients is proposed.

2.1. **Previous Works on Impact Angle Control.** Consider the homing guidance geometry for a stationary or a slowly moving target as shown in Fig. 1.

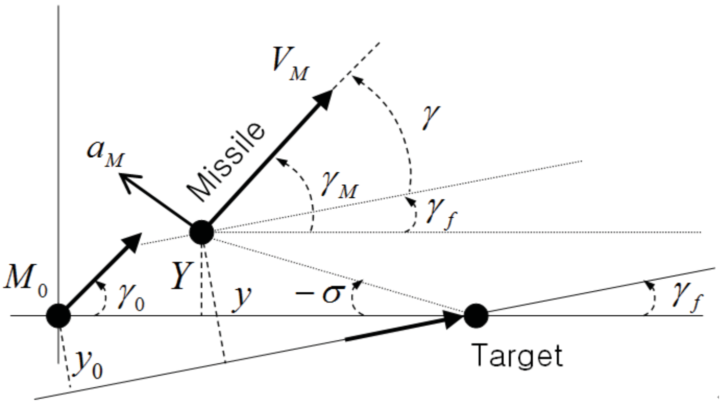


FIGURE 1. Homing guidance geometry.

Here, V_M and a_M denote the missile velocity and the acceleration applied normal to the velocity vector, respectively. Moreover, Y, γ_M, and γ_f denote the cross range, flight path angle, and impact angle with respect to the initial LOS, respectively. For the purpose of simplification, a new cross range and flight path angle γ with respect to the predetermined collision course are introduced. The remaining variables in Fig. 1 are self-explanatory.

The equations of motion for this homing problem are given by

$$\dot{y} = V_M \sin\gamma,\ y(t_0) = y_0,$$

$$V_M \dot{\gamma} = a_M, \ \gamma(t_0) = \gamma_0 \tag{2.1}$$

If the autopilot dynamics of the system is neglected, then the control input equals to the acceleration; i.e., $u(t) = a_M(t)$. Under the assumption that V_M is constant and γ is small, the following linear differential equation can be obtained:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2.2}$$

where

$$\mathbf{x} = \begin{bmatrix} y \\ v \end{bmatrix}, \ \mathbf{x}_0 = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{2.3}$$

and the velocity component perpendicular to the predetermined collision course is defined by

$$v(t) = V_M \gamma(t). \tag{2.4}$$

Note that the velocity component at impact is directly proportional to the impact angle error.

2.1.1. *Energy Optimal Guidance Law (EOGL)[1].* Consider the following optimal problem: Find $u(t)$ that minimizes J defined by

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2 d\tau \tag{2.5}$$

subject to (2.2), (2.3), and terminal constraints given by $\mathbf{x}(t_f) = \mathbf{0}$. The optimal guidance command can be obtained using Pontryagin's minimum principle:

$$u(t) = -\left[\frac{6}{(t_f - t)^2}x_1(t) + \frac{4}{(t_f - t)}x_2(t)\right]. \tag{2.6}$$

This is one of the most widely used impact-angle-control guidance law for stationary or slowly moving target because it is not only optimal but easy to analyze the performance of the guidance loop.

2.1.2. *Optimal Guidance with Time-to-go Weighted Cost (TOGL) [6].* Consider following optimal problem with the time-to-go weightings in the cost defined by

$$J = \frac{1}{2}\int_{t_0}^{t_f} (t_f - t)^p u^2 d\tau, \ \rho \leq 0, \tag{2.7}$$

subject to (2.2), (2.3), and terminal constraints given by $\mathbf{x}(t_f) = \mathbf{0}$. The optimal guidance command can be obtained as

$$u(t) = 0\left[\frac{(2-\rho)(3-\rho)}{(t_f - t)^2}x_t(t) + \frac{2(2-\rho)}{t_f - t}x_t(t)\right]. \tag{2.8}$$

When $\rho = 0$, it is easily seen that TOGL is identical to EOGL. When $\rho < 0$, on the other hand, the time-to-go weighted cost becomes increasingly expensive as $t \to t_f$ so that the guidance command eventually becomes zero at impact. This property is important for ensuring some operational margin for guidance command to handle external disturbances, model uncertainties, and command saturation. Moreover, it may enhance the warhead effect of the missile because the zero control command at impact provides a very small angle-of-attack.

2.1.3. *Time-Varying Biased PNG (TBPNG) [8].* The biased PNG described in [8] is mainly concerned with an additional time-varying bias term of conventional PNG required to control the impact angle, which can be expressed as

$$u(t) = NV_M\{\dot{\sigma}(t) - \dot{\sigma}_b(t)\}, \tag{2.9}$$

where N is a navigation constant, $\dot{\sigma}(t)$ is the LOS rate, and $\dot{\sigma}_b(t)$ denotes the time-varying bias term to control the impact angle defined by

$$\dot{\sigma}_b(t) = \frac{\eta V_M\{\gamma_f - \sigma(t)}{NR(t)\cos(\gamma_M(t) - \sigma(t))}. \tag{2.10}$$

Here, η is an arbitrary positive constant. Under the assumptions of stationary target and small flight path angle, (2.9) is rewritten by simple manipulations as

$$u(t) = -\left[\frac{N+\eta}{(t_f-t)^2}x_1(t) + \frac{N}{(t_f-t)}x_t(t)\right]. \tag{2.11}$$

2.2. **Generalized Impact-Angle-Control Guidance Law [20].** According to the overview of previous works, it is seen that lots of impact-angle control laws have two time-varying feedback loops in common. Based on this observation, a generalized form of impact-angle-control guidance law is introduced as

$$u(t) = -\left[\frac{k_1}{(t_f-t)^2}x_1(t) + \frac{k_2}{(t_f-t)}x_2(t)\right] \tag{2.12}$$

where both of the guidance coefficients, k_1 and k_2, are positive constants. Here, a question now arises: Is it possible that the control law with arbitrary k_1 and k_2 fulfill the guidance goals?

Consider obtaining explicit closed-form solutions to find feasible pairs of the guidance coefficients that produce zero miss distances and zero impact angle errors. When the guidance law (2.12) is applied to (2.2), the system equation can be expressed by

$$\ddot{y} + \frac{k_2}{(t_f-t)}\dot{y} + \frac{k_1}{(t_f-t)^2}y = 0. \tag{2.13}$$

This is the second-order Cauchy equation. The characteristic equation is obtained by letting $y = (t_f - t)^\lambda$ as

$$\lambda^2 - (k_2+1)\lambda + k_1 = 0, \tag{2.14}$$

and the roots of (2.14) are

$$\lambda_1, \lambda_2 = \frac{k_2+1}{2} \pm \sqrt{\left(\frac{k_2+1}{2}\right)^2 - k_1}. \tag{2.15}$$

From (2.15), another relational expression between the roots and guidance coefficients are derived as

$$k_1 = \lambda_1\lambda_2 \quad \text{and} \quad k_2 = \lambda_1 + \lambda_2 - 1. \tag{2.16}$$

Since k_1 and k_2 are real, the characteristic equation may have three different kinds of roots: two distinct real roots, a real double root, or two complex conjugate roots. These cases are classified using the discriminant of (2.14) as

$$d = \left(\frac{k_2+1}{2}\right)^2 - k_1. \tag{2.17}$$

For convenience, the guidance coefficient vector denoted by $\mathbf{z} = \left[\begin{array}{cc} k_1 & k_2 \end{array}\right]$ in $\mathbb{Z} = \{\mathbf{z} \in \mathbb{R}^2 | k_1 > 0 \text{ and } k_2 > 0\}$ is defined. The set $\mathbb{Z}$ is divided into three clear subsets: $\mathbb{Z} = \{\mathbf{z} \in \mathbb{R}^2 | d > 0\}$, $\mathbb{Z} = \{\mathbf{z} \in \mathbb{R}^2 | d = 0\}$, and $\mathbb{Z} = \{\mathbf{z} \in \mathbb{R}^2 | d < 0\}$ according to the discriminant. Now, the closed-form solutions are found case-by-case according to which subset contains the guidance coefficient vector.

2.2.1. *In Case of* $\mathbf{z} \in \mathbb{Z}_1$. In this case, λ_1 and λ_2 given by (2.15) are distinct real. By considering the initial conditions $y(0) = y_0$ and $v(0) = v_0$, the closed-form solutions are

$$y(t) = c_{11}(t_f - t)^{\lambda_1} + c_{12}(t_f - t)^{\lambda_2}, \tag{2.18}$$

$$v(t) = -c_{11}\lambda_1(t_f - t)^{\lambda_1 - 1} - c_{12}\lambda_2(t_f - t)^{\lambda_2 - 1}, \tag{2.19}$$

$$u(t) = c_{11}\lambda_1(\lambda_1 - 1)(t_f - t)^{\lambda_1 - 2} + c_{12}\lambda_2(\lambda_2 - 1)(t_f - t)^{\lambda_2 - 2}, \tag{2.20}$$

where $c_{11} = -(\lambda_2 y_0 + t_f v_0)/[(\lambda_1 - \lambda_2)t_f^{\lambda_1}]$ and $c_{12} = (\lambda_1 y_0 + t_f v_0)/[(\lambda_1 - \lambda_2)t_f^{\lambda_2}]$. Note that the closed-loop solutions consist of two distinct time-to-go polynomial terms which are highly dependent on the roots of the characteristic equation determined by the guidance coefficients.

Next, the solutions are arranged in terms of the initial states to find the necessary conditions that satisfy the terminal constraints regardless of the initial conditions. It is expressed in terms of y_0 and v_0 by (2.18) and (2.19) as

$$y(\tau) = \frac{\tau^{\lambda_2}}{\lambda_1 - \lambda_2}\left[(\lambda_1 - \lambda_2\tau^{\lambda_1 - \lambda_2})y_0 + t_f(1 - \tau^{\lambda_1 - \lambda_2})v_0\right], \tag{2.21}$$

$$v(\tau) = \frac{\tau^{\lambda_2 - 1}}{(\lambda_1 - \lambda_2)t_f}\left[\lambda_1\lambda_2(\tau^{\lambda_1 - \lambda_2} - 1)y_0 + t_f(\lambda_1\tau^{\lambda_1 - \lambda_2} - \lambda_2)v_0\right], \tag{2.22}$$

$$\begin{aligned} u(\tau) = \frac{\tau^{\lambda_2 - 2}}{(\lambda_1 - \lambda_2)t_f^2}\Big[&\lambda_1\lambda_2\{(\lambda_2 - 1) - (\lambda_1 - 1)\tau^{\lambda_1 - \lambda_2}\}y_0 \\ &+ t_f\{\lambda_2(\lambda_2 - 1) - \lambda_1(\lambda_1 - 1)\tau^{\lambda_1 - \lambda_2}\}v_0\Big], \end{aligned} \tag{2.23}$$

where $\tau = 1 - t/t_f$.

Since $\lambda_1 > \lambda_2$, from (2.21) and (2.22), $\lambda_2 \geq 1$ is required for a zero miss distance with a finite velocity regardless of the initial conditions; i.e.,

$$\frac{k_2 - 1}{2} \geq \sqrt{\left(\frac{k_2+1}{2}\right)^2 - k_1}. \tag{2.24}$$

Furthermore, the positive discriminant yields

$$k_1 < \left(\frac{k_2+1}{2}\right)^2. \tag{2.25}$$

From (2.24), it can be intuitively seen that $k_2 > 1$. According to (2.23), the guidance coefficient vector that satisfies the above conditions cannot guarantee the finite control input as $\tau \to 0$ when $1 < \lambda_2 < 2$. In this case, the acceptable performance cannot be expected in real situations where many nonlinearities and uncertainties are included. Thus, the condition of finite control inputs should be also considered. For $\lambda_2 \geq 2$, it is easy to see that $u(\tau)$ remains finite for all τ. Additionally, it is seen that $u(\tau)$ can remain finite when $\lambda_2 = 1$. By substituting $\lambda_2 = 1$, (2.23) is represented as

$$u(\tau)|_{\lambda_2=1} = -\frac{\lambda_1 \tau^{\lambda_1-2}}{t_f^2}(y_0 + t_f v_0). \tag{2.26}$$

Since $\lambda_2 = 1$, it is easy to see that $\lambda_1 = k_1 = k_2$ from(2.15). This is the well known PNG whose navigation constant is λ_1; for $u(\tau)$ to remain finite at $\tau = 0$, $k_2 \geq 2$ is required, as is evident in (2.26).

Now, return to the impact angle control. In order to control the impact angle, the velocity component v that is perpendicular to the predetermined collision course should approach zero as $\tau \to 0$. From the solutions of (2.21) and (2.22), it is seen that the impact angle error is zero, as well as the zero miss distance being zero, for $\lambda_2 > 1$; for $u(\tau)$ to remain finite, $\lambda_2 \geq 2$ is required; i.e.,

$$\frac{k_2-3}{2} \geq \sqrt{\left(\frac{k_2+1}{2}\right)^2 - k_1}. \tag{2.27}$$

Then, a feasible set of guidance coefficient vectors can be obtained from (2.25) and (2.27) as

$$\mathbb{F}_1 = \{\mathbf{z} \in \mathbb{Z} \big| 2(k_2-1) \leq k_1 < \left(\frac{k_2+1}{2}\right)^2 \text{ and } k_2 > 3\}. \tag{2.28}$$

The guidance coefficient vector $\mathbf{z} = \begin{bmatrix} 6 & 4 \end{bmatrix}$, which is that of the energy optimal guidance law for impact angle control, is an element of the set $\mathbb{F}_1$. In this case, the roots of the characteristic equation are $\lambda_1 = 3$ and $\lambda_2 = 2$. From (2.18), it is easy to see that the profile of $y(t)$ is the third-order polynomial of t.

It is interesting to note here that zero control input at impact can be achieved by choosing guidance coefficients which satisfy the condition of $\lambda_2 > 2$; i.e., $k_1 > 2(k_2 - 1)$. In general, the energy optimal guidance law for impact angle control converges to a nonzero value as the missile approaches the target so that sometimes the maneuver acceleration can be easily saturated in real situations. The saturation of the maneuvering acceleration in the terminal phase may cause large terminal errors. From this point of view, it is useful if magnitude of the terminal acceleration command is reduced by tuning the guidance coefficients k_1 and k_2 so that $\lambda_2 > 2$.

2.2.2. *In Case of* $\mathbf{z} \in \mathbb{Z}_2$. In this case, the double root of the characteristic equation becomes

$$\lambda_d = \frac{k_2 + 1}{2} \tag{2.29}$$

The closed-form solutions are then

$$y(t) = (t_f - t)^{\lambda_d}[c_{21} + c_{22}\ln(t_f - t)], \tag{2.30}$$

$$v(t) = -(t_f - t)^{\lambda_d - 1}[c_{21}\lambda_d + c_{22}\{1 + \lambda_d \ln(t_f - t)\}], \tag{2.31}$$

$$u(t) = (t_f - t)^{\lambda_d - 2}[c_{21}\lambda_d(\lambda_d - 1) + c_{22}\{(2\lambda_d - 1) + \lambda_d(\lambda_d - 1)\ln(t_f - t)\}], \tag{2.32}$$

where, $c_{21} = [\{1 + \lambda_d \ln(t_f)\}y_0 + t_f \ln(t_f)v_0]/t_f^{\lambda_d}$ and $c_{22} = -(\lambda_d y_0 + t_f v_0)/t_f^{\lambda_d}$. It is seen that the profiles of the closed-form solutions are the combinations of the polynomial and logarithmic functions of t.

The closed-form solutions are then arranged in terms of the initial states to find the necessary conditions that satisfy the terminal constraints regardless of the initial conditions. This is expressed in terms of y_0 and v_0 by (2.30) to (2.32) as

$$y(\tau) = \tau^{\lambda_d}[(1 - \lambda_d \ln\tau)y_0 - t_f \ln\tau v_0], \tag{2.33}$$

$$v(\tau) = \frac{\tau^{\lambda_d - 1}}{t_f}[\lambda_d{}^2 \ln\tau y_0 + t_f(1 + \lambda_d \ln\tau)v_0], \tag{2.34}$$

$$u(\tau) = -\frac{\tau^{\lambda_d - 2}}{t_f^2}[\lambda_d{}^2\{1 + (\lambda_d - 1)\ln\tau\}y_0 + t_f\{(2\lambda_d - 1) + \lambda_d(\lambda_d - 1)\ln\tau\}v_0]. \tag{2.35}$$

From the solutions for $y(\tau)$, $v(\tau)$, and $u(\tau)$ of (2.33), (2.34), and (2.35), respectively, it is seen that $\lambda_d > 2$ is required for $u(\tau \to 0)$ to remain finite, as well as for the two terminal constraints, $y(\tau \to 0) = 0$ and $v(\tau \to 0) = 0$, to be satisfied regardless of the initial conditions of y_0 and v_0. From the condition of $\lambda_d > 2$ as well as $d = 0$, a new feasible set of guidance coefficients can be obtained:

$$\mathbb{F}_2 = \{\mathbf{z} \in \mathbb{Z} \big| k_1 = \left(\frac{k_2 + 1}{2}\right)^2 \text{ and } k_2 > 3\}. \tag{2.36}$$

From (2.35), it is seen that $\lambda_d > 2$ causes the control input at impact to be zero since $\lim_{\tau \to 0} \tau^p \ln\tau$ for positive p. Thus, all pairs of guidance coefficients in $\mathbb{F}_2$ have the acceleration commands converge to zero.

2.2.3. *In Case of* $\mathbf{z} \in \mathbb{Z}_3$. In this case, two complex conjugate roots of the characteristic equation become

$$\lambda_1,\ \lambda_2 = \alpha \pm i\beta, \tag{2.37}$$

where $\alpha = (k_2 + 1)/2$, $\beta = \sqrt{k_1 - \alpha^2}$ and $i = \sqrt{-1}$. Hence, the closed-form solutions are

$$y(t) = (t_f - t)^{\alpha}[c_{31}\cos(\beta \ln(t_f - t)) + c_{32}\sin(\beta \ln(t_f - t))], \tag{2.38}$$

$$v(t) = -(t_f - t)^{\alpha - 1}[(c_{31}\alpha + c_{32}\beta)\cos(\beta \ln(t_f - t)) + (c_{32}\alpha + c_{31}\beta)\sin(\beta \ln(t_f - t))], \tag{2.39}$$

$$u(t) = (t_f - t)^{\alpha-2} \begin{bmatrix} \{c_{31}(\alpha^2 - \alpha - \beta^2) + c_{32}\beta(2\alpha - 1)\} \cos(\beta \ln(t_f - t)) \\ +\{c_{32}(\alpha^2 - \alpha - \beta^2) - c_{31}\beta(2\alpha - 1)\} \sin(\beta \ln(t_f - t)) \end{bmatrix}, \tag{2.40}$$

where

$$c_{31} = \frac{[\alpha \sin(\beta \ln t_f) + \beta \cos(\beta \ln t_f)]y_0 + t_f \sin(\beta \ln t_f) v_0}{\beta t_f^\alpha},$$
$$c_{32} = \frac{[\alpha \cos(\beta \ln t_f) - \beta \sin(\beta \ln t_f)]y_0 + t_f \cos(\beta \ln t_f) v_0}{\beta t_f^\alpha}.$$

From (2.38) to (2.40), it is seen that the profiles of the closed-form solutions are combinations of the polynomial, logarithmic, and trigonometric functions. This means that the trajectory of the missile provides an oscillatory motion.

The closed-form solutions are represented in terms of the initial states to find the necessary conditions that satisfy the terminal constraints regardless of the initial conditions. From (2.38) to(2.40), the followings are obtained:

$$y(\tau) = -\frac{\tau^\alpha}{\beta}[\{\alpha \sin(\beta \ln \tau) - \beta \cos(\beta \ln \tau)\}y_0 + t_f \sin(\beta \ln \tau) v_0], \tag{2.41}$$

$$v(\tau) = \frac{\tau^{\alpha-1}}{\beta t_f}[\{(\alpha^2 + \beta^2) \sin(\beta \ln \tau) y_0 + t_f\{\alpha \sin(\beta \ln \tau) + \beta \cos(\beta \ln \tau)\}v_0], \tag{2.42}$$

$$u(\tau) = -\frac{\tau^{\alpha-2}}{\beta t_f^2} \begin{bmatrix} (\alpha^2 + \beta^2)\{(\alpha - 1) \sin(\beta \ln \tau) + \beta \cos(\beta \ln \tau)\}y_0 \\ +t_f\{(\alpha^2 - \alpha - \beta^2) \sin(\beta \ln \tau) + \beta(2\alpha - 1) \cos(\beta \ln \tau)\}v_0 \end{bmatrix}. \tag{2.43}$$

From (2.41) to (2.43), it is seen that $\alpha \geq 2$ is required for $u(\tau \to 0)$ to remain finite, as well as for the two terminal constraints to be satisfied regardless of the initial conditions. From the condition of $\alpha \geq 2$ as well as $d < 0$, another feasible set can be defined:

$$\mathbb{F}_3 = \{\mathbf{z} \in \mathbb{Z} \big| k_1 > \left(\frac{k_2 + 1}{2}\right)^2 \text{ and } k_2 \geq 3\}. \tag{2.44}$$

From (2.43), we see that zero control input at impact can be achieved by choosing guidance coefficients which satisfy the condition of $\alpha > 2$, i.e., $k_2 > 3$. In this case, while the magnitude of the control input tends to decrease, its oscillating frequency gradually increase as the missile approaches the target.

2.3. **Feasible Sets of Guidance Coefficients.** As previously discussed, there are three kinds of feasible guidance coefficient sets according to the classes of the solutions: One is $\mathbb{F}_1$ where the guidance coefficients lead to the time-to-go polynomial form trajectories as

$$\mathbb{F}_1 = \{\mathbf{z} \in \mathbb{Z} \big| 2(k_1 - 1) \leq k_1 < \left(\frac{k_2 + 1}{2}\right)^2 \text{ and } k_2 > 3\}, \tag{2.45}$$

another is $\mathbb{F}_2$ where the guidance coefficients provide the trajectory profiles combined with polynomial and logarithmic functions as

$$\mathbb{F}_2 = \{\mathbf{z} \in \mathbb{Z} \big| k_1 = \left(\frac{k_2+1}{2}\right)^2 \text{ and } k_2 > 3\}, \tag{2.46}$$

and the other is where the guidance coefficients provide the oscillatory motion as

$$\mathbb{F}_3 = \{\mathbf{z} \in \mathbb{Z} \big| k_1 > \left(\frac{k_2+1}{2}\right)^2 \text{ and } k_2 > 3\}, \tag{2.47}$$

Consider the coefficients of the impact-angle-control guidance laws summarized in section 2.1. Let $\mathbb{S}_{EOGL}$ and $\mathbb{S}_{TOGL}$ be the guidance coefficient sets of EOGL and TOGL, respectively. Then, we see that $\mathbb{S}_{EOGL} \subset \mathbb{S}_{TOGL} \subset \mathbb{F}_1$ since

$$\mathbb{S}_{EOGL} = \{[6, 4]\}, \tag{2.48}$$

$$\mathbb{S}_{TOGL} = \{\mathbf{z} \in \mathbb{Z} \big| k_1 = \left(\frac{k_2+1}{2}\right)^2 - \frac{1}{4} \quad \text{and} \quad k_2 > 4\}, \tag{2.49}$$

Figure 2 illustrates the three feasible sets of guidance coefficient vectors for impact angle control with finite control inputs. It is shown that an entire feasible set of the guidance coefficient vectors is the union of $\mathbb{F}_1$, $\mathbb{F}_2$, and $\mathbb{F}_3$. In Fig. 2, it is seen that $\mathbb{F}_3$ may provide a wider choice of the guidance coefficients than others. In addition, zero control input at impact can be achieved by choosing an interior point of $\mathbb{F}$ since the boundary of the feasible area is determined by the condition of finite control input. It is also seen that $\mathbb{S}_{TOGL}$ is a subset of $\mathbb{F}_1$ but very close to $\mathbb{F}_2$. This implies that the missile's behavior when TOGL is employed is similar to that for $\mathbf{z} \in \mathbb{F}_2$.

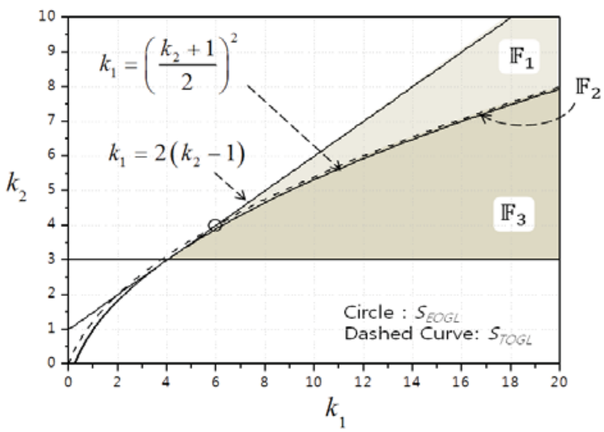


FIGURE 2. Feasible sets of guidance coefficients.

2.4. **Time-to-go Calculation.** The generalized form of impact-angle-control guidance law proposed in this section becomes

$$u(t) = -\left[\frac{k_1}{(t_f - t)^2} x_1(t) + \frac{k_2}{(t_f - t)} x_2(t)\right] \tag{2.50}$$

where the guidance coefficients, k_1 and k_2, are arbitrary but contained in the feasible set F.

For implementing the guidance law, the missile only requires a built-in navigation system. If an additional seeker is provided to measure the missile-to-target LOS angle $\sigma(t)$, we can use $\sigma(t)$ instead of $x_1(t)$. In this case, $x_1(t)$ can be approximated by $V_M(t_f - t)(\gamma_f - \sigma)$ and $x_2(t)$ becomes $V_M(\gamma_M - \gamma_f)$. Hence, (2.50) is rewritten as

$$u(t) = -\frac{V_M}{(t_f - t)}[-k_1\sigma(t) + k_2\gamma_M(t) + (k_1 - k_2)\gamma_f]. \tag{2.51}$$

If the seeker is provided to measure the LOS rate $\dot{\sigma}(t)$ as well, since $\dot{\sigma}(t)$ can be approximated by $-\frac{1}{V_M}\left[\frac{x_1(t)}{(t_f - t)^2} + \frac{x_2(t)}{(t_f - t)}\right]$, (2.50) is rewritten as

$$u(t) = k_2 V_M \dot{\sigma}(t) + \frac{(k_1 - k_2)V_M}{(t_f - t)}[\sigma(t) - \gamma_f]. \tag{2.52}$$

Here, $t_f - t$, named time-to-go, appears explicitly in all implementation methods, but it may not be directly measured by the sensors. Hence, to implement the guidance laws, a suitable time-to-go estimation is required.

The most widely used time-to-go estimation method is the range over closing velocity, i.e., R/V. This method provides good estimates of time-to-go when the trajectory is near the collision course. For the impact angle control laws, however, this method is not adequate because the trajectory may be curved and distant from the collision course.

Here, a time-to-go estimation method that considers the curved trajectory generated by the generalized guidance law with an arbitrary $\mathbf{z}$ in F is proposed. From the previous section, the closed-form solutions can be represented in terms of time t. Under the small angle assumption, solutions can be approximated to functions of range x by substituting $t \approx x/V$. The flight path angle γ_M with respect to the initial LOS can be represented as functions of range x. Since $v = V_M(\gamma_M - \gamma_f)$, $y_0 = R\gamma_f$, and $t_f \approx R/V_M$, the flight path angle is represented by

$$\gamma_M(\xi) - \gamma_f = \begin{cases} \frac{1}{2}\xi^{(\alpha-1)}\left[\Phi_0\left(\xi^{\sqrt{d}} + \xi^{-\sqrt{d}}\right) + \frac{\Psi_0}{\sqrt{d}}\left(\xi^{\sqrt{d}} - \xi^{-\sqrt{d}}\right)\right] & \text{, if } d > 0 \\ \xi^{(\alpha-1)}(\Phi_0 + \Psi_0 \ln \xi) & \text{, if } d = 0 \\ \xi^{(\alpha-1)}\left[\Phi_0 \cos\left(\sqrt{-d}\ln\xi\right) + \frac{\Psi_0}{\sqrt{-d}}\sin\left(\sqrt{-d}\ln\xi\right)\right] & \text{, if } d < 0 \end{cases} \tag{2.53}$$

where $\xi = 1 - \frac{x}{R}$, $\alpha = \frac{k_2 + 1}{2}$, $\Phi_0 = \gamma_0 - \gamma_f$, and $\Psi_0 = \alpha\gamma_0 - (\alpha - k_1)\gamma_f$.

Since the velocity $V_{_M}$ is constant, the final time t_f can be calculated by the total length of the curved path S over the velocity $V_{_M}$ as

$$t_f = \frac{S}{V_{_M}} = \frac{R}{V_{_M}} \int_0^t \sqrt{1+y'^2} d\xi. \tag{2.54}$$

Since

$$\sqrt{1+y'^2} = \sqrt{1+\tan^2\gamma_{_M}} = \sec\gamma_{_M} = 1 + \frac{{\gamma_{_M}}^2}{2!} + \frac{5{\gamma_{_M}}^4}{4!} + \frac{61{\gamma_{_M}}^6}{6!} + \cdots\ , \ |\gamma_{_M}| < \frac{\pi}{2},$$

t_f can be represented as

$$t_f = \frac{R}{V_{_M}}(1+h), \tag{2.55}$$

where the length increment factor due to the path curvature, h, is

$$h = \frac{1}{2}\int_0^1 {\gamma_{_M}}^2 d\xi + \frac{5}{24}\int_0^1 {\gamma_{_M}}^4 d\xi + \frac{61}{720}\int_0^1 {\gamma_{_M}}^6 d\xi + \cdots\ .$$

For simplification, assuming that $\gamma_{_M}$ is sufficiently small for the high order terms of $\gamma_{_M}$ to be near zero, h can be calculated approximately; that is,

$$h \approx \frac{1}{2}\int_0^1 {\gamma_{_M}}^2 d\xi \tag{2.56}$$

Even if considering the higher order terms in the Taylor series produces more accurate results, they are too complicated to implement in real situations.

Now, $\int_0^1 {\gamma_{_M}}^2 d\xi$ can be calculated for three cases; $d > 0$, $d = 0$ and $d < 0$. By manipulating three integration results for (2.53), it can be expressed as one equation regardless of the discriminant d:

$$\int_0^1 {\gamma_{_M}}^2 d\xi = \frac{(2k_1-k_2)\gamma_0^2 - 2(k_1-k_2)\gamma_0\gamma_{_f} + 2(k_1-k_2)^2{\gamma_{_f}}^2}{k_2(4k_1-2k_2-1)}. \tag{2.57}$$

For any $\mathbf{z}$ in F, numerical singularity is not a relevant issue because $k_2 \geq 2(k_2-1)$ and $k_2 \geq 3$.

The final time given by (2.55) is replaced by the time-to-go with regard to the present LOS with an angle of σ as the initial LOS; let $\bar{\gamma}_M = \gamma_{_M} - \sigma$ and $\bar{\gamma}_f = \gamma_{_f} - \sigma$, then the time-to-go is represented by

$$t_{go} = \frac{R}{V_{_M}}\left(1 + \frac{\epsilon}{2}\right), \tag{2.58}$$

where

$$\epsilon = \frac{(2k_1-k_2)\bar{\gamma}_M^2 - 2(k_1-k_2)\bar{\gamma}_M\bar{\gamma}_f + 2(k_1-k_2)^2\bar{\gamma}_f^2}{k_2(4k_1-2k_2-1)}. \tag{2.59}$$

2.5. **Numerical Simulations.** In this section, the performance of the impact angle control laws with various guidance coefficients is studied through nonlinear simulations. In the simulations, it is assumed that the target is stationary and the speed of the missile is constant. It is also assumed that the missile is a lag-free system without control input limits and has perfect measurements.

TABLE 1. Initial conditions for nonlinear simulations.

Parameters	Values
Missile position (x_0, y_0)	(0m, 0m)
Target position $(x_t y_t)$	(5000m, 0m)
Missile velocity V_M	300m/s
Launch angle γ_0	10°
Impact angle γ_f	−20°

The simulations are conducted for five guidance coefficient vectors denoted by $\mathbf{z}_1 \cdots \mathbf{z}_5$ which are shown in Figure 3.

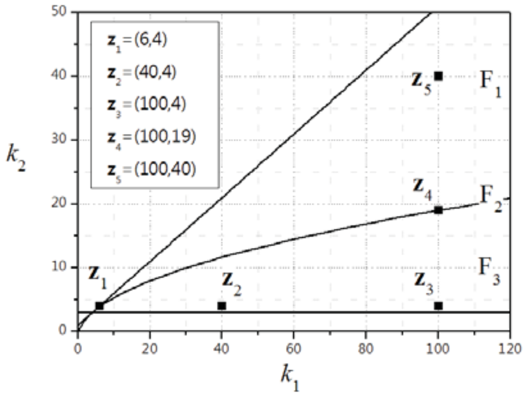


FIGURE 3. Guidance coefficient vectors for simulations.

The missile trajectories, flight path angles, control inputs and time-to-go estimation errors are shown in Fig. 4 through Fig. 7. As shown in Fig. 4 and Fig. 5, all cases do not produce terminal errors. It is also observed that the trajectory for $\mathbf{z}_3$, which has the highest k_1 with $k_2 = 4$, shows the most oscillatory behavior and the trajectory for $\mathbf{z}_4$ contained in shows a rapid approach to the collision course without oscillation.

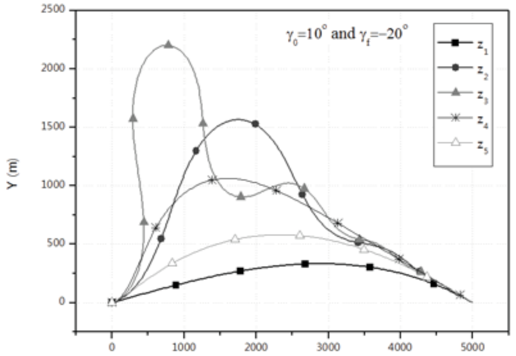


FIGURE 4. Trajectories ($\gamma_0 = 10°,\ \gamma_f = -20°$).

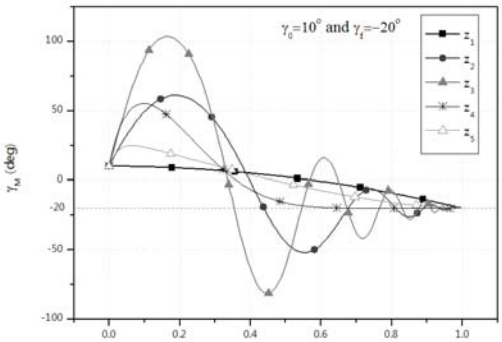


FIGURE 5. Flight path angles ($\gamma_0 = 10°, \gamma_f = -20°$).

As shown in Fig. 6, the case of high k_1 and low k_2 leads to a very large control input with fast oscillating behavior, but high k_2 reduces the required control input as well as the oscillatory behavior as a damping ratio increases.

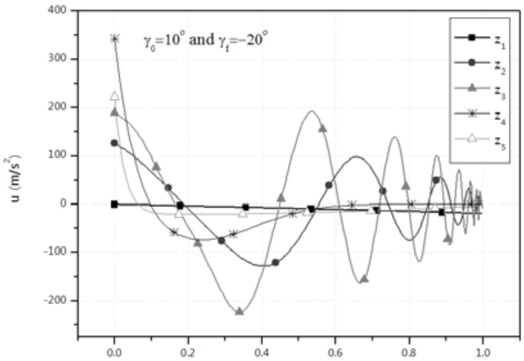


FIGURE 6. Control inputs ($\gamma_0 = 10°, \gamma_f = -20°$).

As shown in Fig. 7, the guidance law with $\mathbf{z}_3$ produces the largest time-to-go estimation error due to the nonlinearity effect enhanced by the oscillatory motion.

From these simulation results, it is found that various guidance coefficient vectors ensure zero impact angle errors as well as zero miss distances. However, the guidance law with high guidance coefficients may produce a significantly worse performance when error sources such as system lag or command saturation are considered. Therefore, a special concern is required in choosing adequate guidance coefficients when implementing the guidance law in practice.

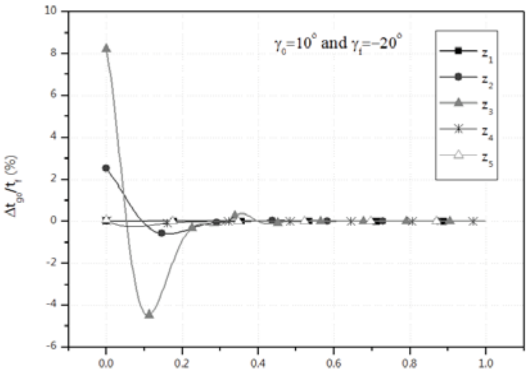


FIGURE 7. Time-to-go calculation errors ($\gamma_0 = 10°, \gamma_f = -20°$).

2.6. **Closing Remarks.** In this chapter, a generalized impact-angle control guidance law with arbitrary guidance coefficients is proposed. All feasible sets of the guidance coefficients that satisfy the terminal constraints are investigated. In order to find the feasible sets, explicit closed-form solutions for lag-free system are derived. The closed-form solutions have three classes of trajectories depending on the guidance coefficients: one is the time-to-go polynomial trajectory, another is the trajectory combined with time-to-go polynomial and logarithmic functions, and the third is the oscillatory trajectory combined with time-to-go polynomial, logarithmic, and harmonic functions. Moreover, based on the closed-form solutions, practical and precise time-to-go calculation methods that can be widely used to implement the generalized guidance law are established.

3. OPTIMALITY OF GENERALIZED GUIDANCE LAW [20]

In this chapter, optimality of the generalized impact-angle-control guidance law is discussed. Under the assumptions of a stationary target and a lag-free missile with constant speed and small flight path angle, optimality of the guidance law with arbitrary guidance coefficients is demonstrated by using the inverse optimal control theory. This result may be a theoretical foundation to choose the adequate values of the guidance coefficients to improve robustness to external disturbances or uncertainties of the missile system.

3.1. **Inverse Problem of Linear Optimal Control.** The inverse problem of linear optimal control is to find necessary and sufficient conditions for a given class of linear feedback system to determine all members of this class of indices. The time-invariant case of the inverse problem was solved for scalar control by Kalman [9], and the results were generalized to the multi-input and time-varying case by Jameson and Kreindler [12]. In this section, the main results of [12] are summarized to be applied to the guidance problem.

Consider a linear system given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{x}(t_0) = \mathbf{x}_0, \tag{3.1}$$

$$\mathbf{u} = \mathbf{K}\mathbf{x}, \tag{3.2}$$

and a performance index given by

$$\mathbf{J} = \frac{1}{2}\mathbf{x}^T(t_f)\mathbf{F}\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt, \tag{3.3}$$

where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{u} \in \mathbf{R}^m$, and t_f is a fixed terminal time. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{K}$, $\mathbf{Q}$, and $\mathbf{R}$ may be time-varying, and are assumed to be uniformly bounded and continuous on $[t_0, t_f]$.

The inverse problem of the linear optimal control is to find necessary and sufficient conditions on the system matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{K}$ so that some performance index of the type (3.1) is minimized, and to determine all such $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{E}$.

The direct problem and its solution are well known. The optimal feedback gain $\mathbf{K}$ is given by

$$\mathbf{K} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}, \tag{3.4}$$

where symmetric matrix $\mathbf{P}$ is the solution of the Riccati equation:

$$-\dot{\mathbf{P}} = \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q},\ \mathbf{P}(t_f) = \mathbf{F}. \tag{3.5}$$

For the existence of a unique solution, it is assumed that $\mathbf{R}$ is positive definite(denoted by $\mathbf{R} > 0$), and it is usually assumed that $\mathbf{Q}$ and $\mathbf{F}$ are nonnegative definite($\mathbf{Q} \geq 0$, $\mathbf{F} \geq 0$)as a *sufficient* condition for the existence of a solution $\mathbf{P}(t)$ of (3.5). The minimal value J^* of J is then nonnegative for all $\mathbf{x}_0$ and t_0, and since

$$J^* = \frac{1}{2}\mathbf{x}_0^T\mathbf{P}(t_0)\mathbf{x}_0, \tag{3.6}$$

and $\mathbf{P}$ is nonnegative definite.

Jameson and Kreindler [12] found out, however, that a nonnegative definite $\mathbf{Q}$ condition might not be necessary under some conditions as follows.

Theorem 3.1. *Consider a closed-loop linear system (3.1) and (3.2) satisfying that* $\mathbf{B}$ *and* $\mathbf{K}$ *are differentiable on* $[t_0, t_f]$ *and are of constant rank. It is possible to construct a performance index (3.3) with*

$$\mathbf{F} = \mathbf{F}^T,\ \mathbf{Q} = \mathbf{Q}^T,\ \mathbf{R} = \mathbf{R}^T > 0,$$

that attains its absolute minimum J^* *over all square-integrable controls, for all* $\mathbf{x}_0$ *and* $t_0 < t_f \leq \infty$*, if and only if for all* T*,* $t_0 \leq t \leq t_f$*, the following conditions hold:*

$$\mathbf{KB} \text{ has } m \text{ linearly independent real eigenvectors}, \tag{3.7}$$

and

$$\text{rank } \mathbf{BK} = \text{rank } \mathbf{K}. \tag{3.8}$$

The minimal value J^* can be negative. An index (3.3) such that $J^* \geq 0$ for all $\mathbf{x}_0$ and $t_0 < t_f \leq \infty$ can be constructed if and only if, in addition to (3.7), for all t, $t_0 < t_f \leq \infty$:

$$\text{all eigenvalues of } \mathbf{KB} \text{ are nonpositive,} \tag{3.9}$$

and (3.8) is strengthened to

$$\textit{rank}\, \mathbf{KB} = \textit{rank}\, \mathbf{K}. \tag{3.10}$$

An index (3.3) such that $J^* > 0$ for all $\mathbf{x}_0$ and $t_0 < t_f \leq \infty$ can be constructed if and only if in addition to (3.7) and (3.9), the rank condition (3.10) is strengthened to

$$\text{rank}\, \mathbf{KB} = \text{rank}\, \mathbf{K} = \text{rank}\, \mathbf{B}. \tag{3.11}$$

Let's construct a performance index (3.3) when the above conditions hold. We first construct $\mathbf{R} = \mathbf{R}^T > 0$ so that $\mathbf{RKB}$ is symmetric. From (3.4), if $\mathbf{P}$ is symmetric, symmetric condition of $\mathbf{RKB}$ holds. Since the eigenvectors of $\mathbf{KB}$ are real and linearly independent, the matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{B}^T\mathbf{K}^T$ is real and nonsingular. Thus,

$$\mathbf{B}^T\mathbf{K}^T\mathbf{V} = \mathbf{V\Lambda} \tag{3.12}$$

where is the diagonal matrix of eigenvalues of . Then we have following theorem.

Theorem 3.2. *Let the eigenvector condition (3.7) hold. Then every given real $\mathbf{R} = \mathbf{R}^T > 0$ such that $\mathbf{RKB}$ is symmetric is necessarily given by*

$$\mathbf{R} = \mathbf{VV}^T, \tag{3.13}$$

where the columns of $\mathbf{V}$ are suitably chosen eigenvectors of $\mathbf{B}^T\mathbf{K}^T$.

We next solve (3.4) for a real symmetric $\mathbf{P}$. From (3.4),

$$\mathbf{B}^T\mathbf{P} = -\mathbf{RK}. \tag{3.14}$$

Let $\mathbf{W}$ be any real $m \times n$ matrix such that

$$\mathbf{B}^T\mathbf{W}^T\mathbf{RK} = \mathbf{RK}. \tag{3.15}$$

By inspection of (3.15), $-\mathbf{W}^T\mathbf{RK}$ is a solution of (3.14) for $\mathbf{P}$, which, however, is not necessarily symmetric. To obtain a symmetric solution, set

$$\mathbf{P}_0 = -\mathbf{W}^T\mathbf{RK} - \mathbf{K}^T\mathbf{RW} + \mathbf{W}^T\mathbf{RKBW}. \tag{3.16}$$

By (3.15) and the symmetry of $\mathbf{RKB}$,

$$\mathbf{B}^T\mathbf{P}_0 = -\mathbf{RK} - \mathbf{B}^T\mathbf{K}^T\mathbf{RW} + \mathbf{RKBW} = -\mathbf{RK}.$$

Further, if $\mathbf{P}$ is any real symmetric solution of(3.14), then

$$\mathbf{B}^T(\mathbf{P} - \mathbf{P}_0) = 0,$$

whence the general solution of (3.4) for a real symmetric $\mathbf{P}$ is

$$\mathbf{P} = -\mathbf{W}^T\mathbf{RK} - \mathbf{K}^T\mathbf{RW} + \mathbf{W}^T\mathbf{RKBW} + \mathbf{Y}, \tag{3.17}$$

where $\mathbf{Y}$ is any real matrix such that

$$\mathbf{B}^T\mathbf{Y} = \mathbf{0},\ \mathbf{Y} = \mathbf{Y}^T. \tag{3.18}$$

Under the rank condition (3.10) on $\mathbf{KB}$, an additional representation of $\mathbf{P}$ is available as follows.

Theorem 3.3. *Let* $\mathbf{R}$ *be a real, symmetric and positive definite matrix such that* $\mathbf{RKB}$ *is symmetric. If*

$$\text{rank}\ \mathbf{KB} = \text{rank}\ \mathbf{K},$$

then all real symmetric $\mathbf{P}$ *satisfying (3.14) are represented in terms of the given* $\mathbf{R}$ *by*

$$\mathbf{P} = -\mathbf{K}^T\mathbf{R}(\mathbf{RKB})^{\dagger}\mathbf{RK} + \mathbf{Y}, \tag{3.19}$$

where † *denotes the Penrose generalized inverse and* $\mathbf{Y}$ *are all real matrices that satisfy (3.18).*

The solutions of (3.4) for $\mathbf{R}$ and $\mathbf{P}$ are pointwise in time, but $\mathbf{P}$ can be constructed to be differentiable, so that we have $\dot{\mathbf{P}}$. Then $\mathbf{F} = \mathbf{P}(t_1)$, and $\mathbf{Q}$ is given by

$$\mathbf{Q} = \dot{\mathbf{P}} - \mathbf{PA} - \mathbf{A}^T\mathbf{P} + \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P}. \tag{3.20}$$

It is remarked that $\mathbf{Q}$ so determined may not be nonnegative definite even if $\mathbf{P}$ is positive definite.

3.2. **Inverse Optimal Problem of Generalized Guidance Law.** In this section, an inverse problem for the linear time-varying guidance law with an impact angle constraint is proposed. The purpose of the inverse problem is to find the performance indices in the general class to be optimized

$$J = \frac{1}{2}\mathbf{x}^T(t_f)\mathbf{F}\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(\mathbf{x}^T\mathbf{Q}\mathbf{x} + ru_d^2t \tag{3.21}$$

subject to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u,\ \mathbf{x}(t_0) = \mathbf{x}_0, \tag{3.22}$$

where

$$\mathbf{x} = \begin{bmatrix} y \\ v \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.23}$$

that result in

$$u(t) = -\left[\frac{k_1}{(t_f - t)^2}x_1(t) + \frac{k_2}{(t_f - t)}x_2(t)\right] \tag{3.24}$$

for arbitrary coefficients k_1 and k_2 in the feasible set. The guidance law (3.24) is rewritten as

$$u = \mathbf{K}(t)\mathbf{x}(t) \tag{3.25}$$

where the linear time-varying guidance gain matrix, $\mathbf{K}(t)$, is

$$\mathbf{K}(t) = \left[-\frac{k_1}{(t_f - t)^2} \quad -\frac{k_2}{(t_f - t)}\right]. \tag{3.26}$$

3.2.1. *General Solution.* Next, construction of the performance index (3.21) so that the control law (3.25) is optimal is considered. For the closed-loop linear system (3.22) with a single input of (3.25), rank $\mathbf{KB}$ = rank $\mathbf{K}$ = rank $\mathbf{B} = 1$ and the eigenvalue of $\mathbf{KB}$ is non-positive. As described in 3.1, therefore, the eigenvector condition (3.7), the eigenvalue condition (3.9) and the rank condition (3.11) hold. Thus, by Theorem 1, the performance index can be constructed with symmetric $\mathbf{F}$, $\mathbf{Q}$, and a positive scalar r that attains its positive absolute minimum. From Theorem 2, every positive r satisfies the necessary condition of the symmetry of $r\mathbf{KB}$ since $\mathbf{KB}$ is scalar. Given r, Theorem 3 provides the real symmetric solution $\mathbf{P}$. Since the symmetric matrix $\mathbf{Y}$, such that $\mathbf{B}^T\mathbf{Y} = 0$, is given by

$$\mathbf{Y} = \begin{bmatrix} y_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where y_{11} is arbitrarily real, it is found that (3.19) becomes

$$\mathbf{P}(t) = -\mathbf{K}^T r(r\mathbf{KB})^{-1} r\mathbf{K} + \mathbf{Y} = \begin{bmatrix} \frac{k_1^2/k_2}{(t_f-t)^3} r + y_{11} & \frac{k_1}{(t_f-t)^2} r \\ \frac{k_1}{(t_f-t)^2} r & \frac{k_2}{(t_f-t)} r \end{bmatrix}. \tag{3.27}$$

For convenience, let

$$y_{11} \triangleq \frac{\mu - k_1^2/k_2}{(t_f - t)^3} r, \tag{3.28}$$

where μ is an arbitrary constant. Then, $\mathbf{P}$ is represented as

$$\mathbf{P} = \begin{bmatrix} \dfrac{\mu}{(t_f - t)^3} r & \dfrac{k_1}{(t_f - t)^2} r \\ \dfrac{k_1}{(t_f - t)^2} r & \dfrac{k_2}{(t_f - t)} r \end{bmatrix} \tag{3.29}$$

Here, if the performance index is positive for all $\mathbf{x}_0$ and t_0, additional condition that $\mathbf{P}$ should be a positive definite is required in view of (3.6); i.e., $\mu > k_1^2/k_2$. Substituting $\mathbf{P}$ and $\dot{\mathbf{P}}$ into (3.20) yields

$$\begin{aligned} \mathbf{Q} &= \dot{\mathbf{P}} - \mathbf{PA} - \mathbf{A}^T\mathbf{P} + \mathbf{PB}r^{-1}\mathbf{B}^T\mathbf{P} \\ &= \begin{bmatrix} \dfrac{k_1^2 - 3\mu}{(t_f - t)^4} r - \dfrac{\mu}{(t_f - t)^3}\dot{r} & \dfrac{k_1(k_2 - 2) - \mu}{(t_f - t)^3} r - \dfrac{k_1}{(t_f - t)^2}\dot{r} \\ \dfrac{k_1(k_2 - 2) - \mu}{(t_f - t)^3} r - \dfrac{k_1}{(t_f - t)^2}\dot{r} & \dfrac{k_2^2 - k_2 - 2k_1}{(t_f - t)^2} r - \dfrac{k_2}{(t_f - t)}\dot{r} \end{bmatrix}. \end{aligned} \tag{3.30}$$

This completes the solution of the inverse problem. Many different $[\mathbf{F},\ \mathbf{Q},\ r]$ can be obtained from (3.29) and (3.30) by selecting different μ and r. For our understanding better, special cases will be considered next.

3.2.2. *Solution for Time-to-go Polynomial Weighting.* From Theorem 2, every positive r satisfies the necessary condition of the symmetry of $r\mathbf{KB}$ since $\mathbf{KB}$ is scalar. Thus, consider the simple form of the weighting factor r:

$$r = (t_f - t)^\rho \tag{3.31}$$

where ρ is an arbitrarily real constant. Then the following is obtained

$$\mathbf{P} = \begin{bmatrix} \dfrac{\mu}{(t_f - t)^{3-\rho}} & \dfrac{k_1}{(t_f - t)^{2-\rho}} \\ \dfrac{k_1}{(t_f - t)^{2-\rho}} & \dfrac{k_2}{(t_f - t)^{1-\rho}} \end{bmatrix}, \tag{3.32}$$

and, since $\dot{r} = -\rho(t_f - t)^{\rho-1}$,

$$\mathbf{Q} = \begin{bmatrix} \dfrac{k_1^2 + (\rho - 3)\mu}{(t_f - t)^{4-\rho}} & \dfrac{k_1(k_2 + \rho - 2) - \mu}{(t_f - t)^{3-\rho}} \\ \dfrac{k_1(k_2 + \rho - 2) - \mu}{(t_f - t)^{3-\rho}} & \dfrac{k_2(k_2 + \rho - 1) - 2k_1}{(t_f - t)^{2-\rho}} \end{bmatrix} \tag{3.33}$$

This is the special solution of the inverse problem for time-to-go polynomial r. From (3.33), however, we see that the simplest solution may be obtained by choosing adequate μ.

3.2.3. *Simplest Solution.* In the case of the simple form of the weighting factor r of (3.31), choose

$$\mu = k_1(k_2 + \rho - 2), \tag{3.34}$$

then the following is obtained by (3.32) and (3.33) as

$$\mathbf{P} = \begin{bmatrix} \dfrac{k_1(k_2 + \rho - 2)}{(t_f - t)^{3-\rho}} & \dfrac{k_1}{(t_f - t)^{2-\rho}} \\ \dfrac{k_1}{(t_f - t)^{2-\rho}} & \dfrac{k_2}{(t_f - t)^{1-\rho}} \end{bmatrix}, \tag{3.35}$$

and

$$\mathbf{Q} = \begin{bmatrix} \dfrac{k_1\{k_1 + k_2(\rho - 3) + (\rho - 2)(\rho - 3)\}}{(t_f - t)^{4-\rho}} & 0 \\ 0 & \dfrac{k_2(k_2 + \rho - 1) - 2k_1}{(t_f - t)^{2-\rho}} \end{bmatrix}. \tag{3.36}$$

Consider a special case given by choosing

$$k_1 = (2 - \rho)(2 - \rho) \text{ and } k_2 = 2(2 - \rho). \tag{3.37}$$

Then, $\mathbf{Q} = 0$ with

$$\mathbf{P} = (2 - \rho) \begin{bmatrix} \frac{(2-\rho)(3-\rho)}{(t_f-t)^{3-\rho}} & \frac{3-\rho}{(t_f-t)^{2-\rho}} \\ \frac{3-\rho}{(t_f-t)^{2-\rho}} & \frac{2}{(t_f-t)^{1-\rho}} \end{bmatrix}$$

where $\rho < 1$ for positive $\mathbf{P}$; i.e., the guidance command (3.25) whose coefficients are (3.37) minimizes the performance index:

$$J = \frac{1}{2}\int_{t_0}^{t_f} (t_f - t)^{\rho} u^2 dt \tag{3.38}$$

which is subject to (3.22) with terminal constraints $\mathbf{x}(t_f) = \mathbf{0}$ since $\mathbf{F} = \mathbf{P}(t_f) = \infty$ for $\rho < 1$. This is consistent with the time-to-go weighted optimal guidance proposed by Ryoo *et al.* [6], who studied the direct optimal guidance problem with the index (3.38) for $\rho \leq 0$. According to the approach proposed in this chapter, it is seen that a solution exists even if $0 < \rho < 1$. In that case, however, the control law may not be practical because the boundedness of control input is not guaranteed.

Note that a number of performance indices can be constructed by(3.31), (3.35) and (3.36) for the given guidance coefficients k_1 and k_2. For example, consider one simple case given by

$$\mathbf{F} = \mathbf{0},\ r = (t_f - t)^4. \tag{3.39}$$

and

$$\mathbf{Q} = \begin{bmatrix} k_1(k_1 + k_2 + 2) & 0 \\ 0 & (k_2^2 + 3k_2 - 2k_1)(t_f - t)^2 \end{bmatrix}, \tag{3.40}$$

for $\rho = 4$. Recall the solution of EOGL with $k_1 = 6$ and $k_2 = 4$, which minimizes the performance index with

$$\mathbf{F} = \infty,\ \mathbf{Q} = \mathbf{0},\ r = 1,$$

for $\rho = 0$. From (3.35) and (3.36), however, the solution also minimizes the index with

$$\mathbf{F} = \mathbf{0},\ \mathbf{Q} = \begin{bmatrix} 72 & 0 \\ 0 & 16(t_f - t)^2 \end{bmatrix},\ r = (t_f - t)^4.$$

Thus, it is significant that a performance index which is minimized by the control law for any given k_1 and k_2 can be constructed even if it is not unique. Moreover, another important point of these results is that the state weighting matrix $\mathbf{Q}$ does not require the nonnegative condition. This means that the missile may move away from the collision course or have an oscillating motion in the middle of the homing leg while maintaining optimal characteristics.

3.3. **Closing Remarks.** It is important that the generalized guidance law with any $\mathbf{z} = [k_1\ k_2]$ contained in F minimizes the quadratic performance index with the corresponding weightings $\mathbf{F}$, $\mathbf{Q}$ and r and also satisfies the terminal constraints with the finite control input. From a practical viewpoint, these are very useful in designing new guidance laws that meet the additional requirements or improve the performance of the guidance loop in realistic environment.

4. Analytic Solutions of Generalized Guidance Law for Single Lag System [21]

In this chapter, closed-form solutions of the generalized guidance law for a first-order lag system are derived. Under some, they are obtained by solving a third-order linear time-varying ordinary differential equation with arbitrary guidance coefficients. Moreover, terminal misses due to the system lag and homing geometries for zero miss-distance are studied. Nonlinear simulations are performed to verify the proposed results.

Under the assumptions of a stationary target, constant missile velocity, and small flight path angles on planar homing geometry, we have the linear differential equations with a first-order system dynamics

$$\begin{aligned} \dot{y}(t) &= v(t), \qquad y(t_0) = y_0 \\ \dot{v}(t) &= a_M(t), \quad v(t_0) = v_0, \\ \dot{a_M} &= \frac{1}{T}(u - a_M), \ a_M(t_0) = a_0 \end{aligned} \tag{4.1}$$

where y, v, a_M, and u are the cross-range, the velocity component perpendicular to the collision course, the normal acceleration, and the control command, respectively. T denotes the time constant of the missile control system. The control command is generated by the generalized impact-angle-control guidance law

$$u(t) = -\frac{k_1}{t_{go}^2} y(T) - \frac{k_2}{t_{go}} v(t), \tag{4.2}$$

where $t_{go} = t_f - t$ and t_f is the final time.

By manipulating (4.1) and (4.2), a linear time-varying third-order differential equation without a forcing term is obtained as

$$T\dddot{y} + \ddot{y} + \frac{k_2}{t_f - t}\dot{y} + \frac{k_1}{(t_f - t)^2} y = 0. \tag{4.3}$$

Thus, seeking the analytic solution of the guidance loop with the first-order lag system is down to finding the solution of (4.3).

If T is zero, then (4.3) becomes the second-order Cauchy equation as shown in chapter 2. By using relational expression between the characteristic roots and guidance coefficients for lag-free systems

$$k_1 = \lambda_1 \lambda_2 \quad \text{and} \quad k_2 = \lambda_1 + \lambda_2 - 1, \tag{4.4}$$

(4.3) can be rewritten as

$$T\dddot{y} + \ddot{y} + \frac{\lambda_1 + \lambda_2 - 1}{t_f - t}\dot{y} + \frac{\lambda_1 \lambda_2}{(t_f - t)^2} y = 0. \tag{4.5}$$

From the trajectory solutions for lag free system shown in chapter 2, it is seen that the terminal misses become always zero. In practice, however, it is hard to expect zero misses when the system lag is considered since the guidance command tends to blow up as the missile approaches the target. To investigate this phenomenon analytically, we attempt to solve (4.5) for nonzero T.

4.1. **Basis Solutions.** For simplification, we change the independent variable from flight time x to normalized time x to have the new equation

$$Ly \equiv D^3y - cD^2y + \frac{(\lambda_1 + \lambda_2 - 1)\zeta}{x}Dy - \frac{\lambda_1\lambda_2\zeta}{x^2}y = 0 \tag{4.6}$$

where $x = 1 - 1/t_f$, $\zeta = t_f/T$, D is a differentiation operator. In addition, L denotes a linear operator defined by (4.6). It is seen that (4.6) is a third-order linear differential equation with variable coefficients and has a regular singular point at $x = 0$.

In order to find nontrivial solutions which are in the form of a power series in x. Let

$$y = x^s \sum_{k=0}^{\infty} A_k x^k. \tag{4.7}$$

where s is to be determined. By substituting (4.7) into(4.6),

$$Ly = f(s)A_0x^{s-3} + \sum_{k=1}^{\infty}[f(s+k)A_k + g_1(s+k)A_{k-1}]x^{s-3+k}, \tag{4.8}$$

where

$$f(x) = s(s-1)(s-2), \tag{4.9}$$

$$g_1(s) = -\zeta(s - \lambda_1 - 1)(s - \lambda_2 - 1). \tag{4.10}$$

In order that Ly equals to zero in an interval including $x = 0$, the coefficients of all powers of x in (4.8) must vanish independently. It gives an indicial equation

$$f(s) = s(s-1)(s-2) = 0, \tag{4.11}$$

which determines three values of s; $s_1 = 2$, $s_2 = 1$, and $s_3 = 0$, and what is known as the recurrence formula

$$\begin{aligned} f(s+k)A_k &= -g_1(s+k)A_{k-1} \\ &= \zeta(s + k - \lambda_1 - 1)(s + k - \lambda_2 - 1)A_{k-1},\ (k \geq 1) \end{aligned} \tag{4.12}$$

which determines each A_k in terms of A_0.

4.1.1. *First Basis Solution.* If $s = s_1 = 2$, the recurrence formula can determine each A_k in terms of A_0; i.e.,

$$\begin{aligned} A_k &= \frac{(-\lambda_1 + 1 + k)(-\lambda_2 + 1 + k)}{k(k+1)(k+2)}\zeta A_{k-1} \\ &= \frac{(-\lambda_1 + 1 + k)(-\lambda_2 + 1 + k)}{k(k+1)(k+2)} \frac{(-\lambda_1 + k)(-\lambda_2 + k)}{(k-1)(k)(k+1)}\zeta A_{k-2} \\ &\vdots \\ &= \frac{(-\lambda_1 + 2)_k(-\lambda_2 + 2)_k\zeta^k}{(2)_k(3)_k k!}A_0 \end{aligned} \tag{4.13}$$

By substituting the determined $A_k's$ into (4.7) and neglecting A_0, the solution corresponding to s_1 becomes

$$y_1(x) = x_2^2F_2(2-\lambda_1, 2-\lambda_2; 2, 3; \zeta x), \tag{4.14}$$

where ${}_2F_2(2-\lambda_1, 2-\lambda_2; 2, 3; \zeta x)$ is known as the hypergeometric function,

$${}_mF_n(\alpha_1, \cdots, \alpha_M; \beta_1, \cdots, \beta_n; z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{m}(\alpha_i)_k}{\prod_{j=1}^{n}(\beta_j)_k} \frac{z^k}{k!}, \tag{4.15}$$

and (α) stands for the Pochhammer symbol as

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \; (\alpha)_0 = 1.$$

4.1.2. *Second Basis Solution.* If $s = s_2 = 1$, the recurrence formula for $k = 1$ cannot be satisfied for any nontrivial A_1 because $f(s_2+1) = 0$. In order to overcome the difficulty, a new setting of $y(x, s)$ with $A_k(s)$, as a function of s, and constant A_0 is introduced

$$y(x, s) = x^s \sum_{k=0}^{\infty} A_k(s) x^k. \tag{4.16}$$

Assuming that $A_k(s)$ satisfies the recurrence formula, we have

$$L\{(s-s_2)y(x,s)\} = A_0(s-s_3)(s-s_2)^2(s-s_1)x^{s-3},$$

and it is seen that the partial derivative of the right member with respect to s vanishes as $s \to s_2$. Since the operator $\partial/\partial s$ and the linear operator L are commutative, we conclude that

$$L\{\frac{\partial}{\partial s}[(s-s_2)y(x,s)]\}_{s=s_2} = 0$$

so that the function

$$y_2(x) = \lim_{s\to s_2}\left[\frac{\partial}{\partial s}[(s-s_2)y(x,s)]\right] \tag{4.17}$$

is another solution of (4.6). Equations (4.16) and (4.17) show that $y_2(x)$ is expressible as

$$\begin{aligned} y_2 &= \frac{\partial}{\partial s}\left[\sum_{k=0}^{\infty}(s-s_2)A_k(s)x^{s+k}\right]\Bigg|_{s=s_2} \\ &= \sum_{k=0}^{\infty}\left[(s-s_2)A_k(s)x^{s+k}\ln x + \frac{\partial}{\partial s}\{(s-s_2)A_k(s)\}x^{s+k}\right]\Bigg|_{s=s_2} \end{aligned}$$

Hence,

$$y_2(x) = \ln x \sum_{k=0}^{\infty} B_k(s_2)x^{k+s_2} + \sum_{k=0}^{\infty} C_k(s_2)x^{k+s_2}, \tag{4.18}$$

where

$$B_k(s_2) = \lim_{s\to s_2}(s-s_2)A_k(s), \tag{4.19}$$

$$C_k(s_2) = \lim_{s \to s_2} \frac{d}{ds}\{(s - s_2)A_k(s)\}. \tag{4.20}$$

Recall the recurrence formula

$$(s + k)(s + k - 1)(s + k - 2)A_k(s) = \zeta(s + k - \lambda_1 - 1)(s_k - \lambda_2 - 1)A_{k-1}(s), \tag{4.21}$$

for $k \geq 1$. By manipulating (4.19), (4.20), and (4.21), we can determine each $B_k(s_2)$ and $C_k(s_2)$ in terms of A_0 as follows;

$$B_0(s_2) = 0, \tag{4.22}$$

$$B_k(s_2) = \frac{(1 - \lambda_1)_k(1 - \lambda_2)_k}{(2)_k(1)_k(k - 1)!}\zeta A_0 \qquad (k \geq 1), \tag{4.23}$$

$$C_0(s_2) = A_0, \tag{4.24}$$

$$C_k(s_2) = B_k(s_2)\left[\sum_{j=1}^{k}\left(\frac{1}{j - \lambda_1} + \frac{1}{j - \lambda_2}\right) - \sum_{j=1}^{k}\left(\frac{1}{j + 1} + \frac{2}{j} + \frac{1}{k}\right)\right], \tag{4.25}$$

for $k \geq 1$. In (4.25), $B_k(s_2)/(j - \lambda_1)$ (for $l = 1, 2$ and $1 \leq j \leq k$) is obviously nonsingular since there is the multiplicative factor of $(j - \lambda_1)$ in the numerator of $B_k(s_2)$. After reduction of a fraction, we have

$$C_k(s_2) = \frac{p_k(\lambda_1, \lambda_2)\zeta^k A_0}{(2)_k(1)_k(k - 1)!} + \frac{B_k(s_2)}{k}\ (k \geq 1), \tag{4.26}$$

where

$$\begin{aligned} p_k(\lambda_1, \lambda_2) = &\sum_{j=1}^{k}\{2j - (\lambda_1 + \lambda_2)\}(1 - \lambda_1; j)_k(1 - \lambda_2; j)_k \\ &- (1 - \lambda_1)_k(1 - \lambda_2)_k\sum_{j=1}^{k}\left(\frac{1}{j + 1} + \frac{2}{j}\right), \end{aligned} \tag{4.27}$$

and $(\alpha; j)_n$ is defined by

$$(\alpha; j)_n = \alpha(\alpha + 1)\cdots(\alpha + j - 2)(\alpha + j)\cdots(\alpha + n - 1) = \prod_{\substack{i=1\cdots n \\ i \neq j}}(\alpha + i - 1), \tag{4.28}$$

and $(\alpha; j)_0 = 1$. By substituting these coefficients into(4.18), and neglecting the constant A_0, we have

$$\begin{aligned} y_2(x) = &\frac{1}{2}(1 - \lambda_1)(1 - \lambda_2)\zeta x^2 \ln x_2F_2(2 - \lambda_1, 2 - \lambda_2; 2, 3; \zeta x) \\ &+ x_2F_2(1 - \lambda_1, 1 - \lambda_2; 1, 2; \zeta x) + \frac{1}{\zeta}P(0; \zeta x) \end{aligned} \tag{4.29}$$

where

$$P(n;z)=\sum_{k=0}^{\infty}\frac{p_k(\lambda_1,\lambda_2)}{(1)_k(1)_{k-1}}\frac{z^{k+1-n}}{(k+1-n)!}. \tag{4.30}$$

4.1.3. *Third Basis Solution.* If $s=s_3=0$, the recurrence formula for $k=1$ or $k=2$ cannot be satisfied for any nontrivial A_k because $f(s_3+1)=f(s_3+2)=0$. In a similar way to 0, a new setting of $y(x,s)$ with $A_k(s)$, as a function of s, and constant A_0 is introduced

$$y(x,s)=x^s\sum_{k=0}^{\infty}A_k(s)x^k. \tag{4.31}$$

Assuming that $A_k(s)$ satisfies the recurrence formula, we have

$$L\{(s-s_3)^2y(x,s)\}=A_0(s-s_3)^3(s-s_2)(s-s_1)x^{s-3}.$$

and it is seen that the partial derivative of the right member with respect to s vanishes as $s\to s_3$. Since the operator $\partial^2/\partial s^2$ and the linear operator L are commutative, we conclude that

$$L\left\{\frac{\partial^2}{\partial s^2}[(s-s_3)^2y(x,s)]\right\}_{s=s_3}=0$$

so that the function

$$y_3(x)=\lim_{s\to s_3}\left[\frac{\partial^2}{\partial s^2}[(s-s_3)^2y(x,s)]\right] \tag{4.32}$$

is the other basis solution of(4.6). Equations (4.31) and (4.32) show that $y_3(s)$ is expressible as

$$\begin{aligned} y_3(x)&=\frac{\partial^2}{\partial s^2}\left[\sum_{k=0}^{\infty}(s-s_3)^2A_k(s)x^{s+k}\right]\Bigg|_{s=s_3}\\ &=\sum_{k=0}^{\infty}\left[\begin{array}{l}(s-s_3)^2A_k(s)x^{s+k}(\ln x)^2\\ +2\dfrac{\partial}{\partial s}\{(s-s_3)^2A_k(s)\}x^{s+k}\ln x+\frac{\partial^2}{\partial s^2}\{(s-s_3)^2A_k(s)\}xs+k\end{array}\right]\Bigg|_{s=s_3} \end{aligned} \tag{4.33}$$

Hence,

$$y_3(x)=(\ln x)^2\sum_{k=0}^{\infty}B_k(s_3)x^{k+s_3}+2\ln x\sum_{k=0}^{\infty}C_k(s_3)x^{k+s_3}+\sum_{k=0}^{\infty}D_k(s_3)x^{k+s_3}, \tag{4.34}$$

where

$$B_k(s_3)=\lim_{s\to s_3}(s-s_3)^2A_k(s), \tag{4.35}$$

$$C_k(s_3)=\lim_{s\to s_3}\frac{d}{ds}\{(s-s_3)^2A_k(s)\}, \tag{4.36}$$

$$D_k(s_3)=\lim_{s\to s_3}\frac{d^2}{ds^2}\{(s-s_3)^2A_k(s)\}. \tag{4.37}$$

By manipulating(4.35), (4.36), (4.37), and the recurrence formula, we can determine each $B_k(s_3)$, $C_k(s_3)$, and $D_k(s_3)$ in terms of A_0 as follows;

$$B_0(s_3) = B_1(s_3) = 0, \tag{4.38}$$

$$B_k(s_3) = -\frac{(-\lambda_1)_k(-\lambda_2)_k}{(1)_k(1)_{k-1}(1)_{k-2}}\zeta^k A_0, \tag{4.39}$$

$$C_0(s_3) = 0,\ C_1(s_3) = -\lambda_1\lambda_2\zeta A_0, \tag{4.40}$$

$$C_k(s_3) = B_k(s_3)\Big\{\sum_{k=1}^{j}\left(\frac{1}{j-\lambda_1-1}+\frac{1}{j-\lambda_2-1}\right) - 3\sum_{j=1}^{k}\frac{1}{j}+\frac{2}{k}+\frac{1}{k-1}+1\Big\}, \tag{4.41}$$

$$D_0(s_3) = 2A_0,\ D_1(s_3) = 2(\lambda_1+\lambda_2)\zeta A_0, \tag{4.42}$$

$$D_k(s_3) = B_k(s_3)\begin{bmatrix} \{\sum_{k=1}^{j}\left(\frac{1}{j-\lambda_1-1}+\frac{1}{j-\lambda_2-1}\right) \\ -3\sum_{j=1}^{k}\frac{1}{j}+\frac{2}{k}+\frac{1}{k-1}+1\}^2 \\ -\sum_{k=1}^{j}\{\frac{1}{(j-\lambda_1-1)^2}+\frac{1}{(j-\lambda_2-1)^2}\} \\ +3\sum_{j=1}^{k}\frac{1}{j^2}-\frac{2}{k^2}-\frac{1}{(k-1)^2}+1 \end{bmatrix}, \tag{4.43}$$

for $k \geq 2$. In (4.41), $B_k(s_3)/(j-\lambda_1-1)$ $(l=1,2)$ for all j is nonsingular since there is always the multiplicative factor of in $(j-\lambda_1-1)(j-\lambda_2-1)$ the numerator of $B_k(s_3)$. In (4.43), on the other hand, $B_k(s_3)/(j-\lambda_1-1)^2$ may become singular when $j=\lambda_1+1$ even if $B_k(s_3)/(j-\lambda_1-1)$ is nonsingular. However, $D_k(s_3)$ is obviously nonsingular because $B_k(s_3)/(j-\lambda_1-1)^2$ for all j will cancel out by expanding the equation. After some manipulation of (4.41) and (4.43), polynomial expression on λ_1 without the rational terms of $1/(j-\lambda_1-1)$ and $1/(j-\lambda_1-1)^2$ can be obtained as

$$C_k(s_3) = \frac{q_k(\lambda_1,\lambda_2)\zeta^k A_0}{(1)_k(1)_{k-1}(k-2)!} + B_k(s_3)\left(\frac{2}{k}+\frac{1}{k-1}+1\right)\ (k\geq 2), \tag{4.44}$$

$$D_k(s_3) = \frac{r_k(\lambda_1,\lambda_2)\zeta^k A_0}{(1)_k(1)_{k-1}(k-2)!} + 2B_k(s_3)\left(\frac{1}{k}+\frac{3}{k-1}+1\right)\ (k\geq 2), \tag{4.45}$$

where

$$q_k(\lambda_1,\lambda_2) = -\sum_{j=1}^{k}\{2(j-1)-(\lambda_1+\lambda_2)\}(-\lambda_1;j)_k(-\lambda_2;j)_k + 3(-\lambda_1)_k(-\lambda_2)_k\sum_{j=1}^{k}\frac{1}{j}, \tag{4.46}$$

$$\begin{aligned} r_k(\lambda_1,\lambda_2) = -2 \sum_{\substack{\forall j_1,j_2=1,2,\cdots,k \\ j_1<j_2}} &\{2(j_1-1)-(\lambda_1+\lambda_2)\}\{2(j_2-1) \\ &-(\lambda_1+\lambda_2)\}(-\lambda_1;j_1,j_2)_k(-\lambda_2;j_1,j_2)_k \end{aligned}$$

$$- 2\left(-3\sum_{j=1}^{k}\frac{1}{j}+\frac{2}{k}+\frac{1}{k-1}+1\right)\sum_{j=1}^{k}\{2(j-1)-(\lambda_1+\lambda_2)\}$$

$$(-\lambda_1;j)_k(-\lambda_2;j)_k - 2\sum_{j=1}^{k}(-\lambda_1;j)_k(-\lambda_2;j)_k - 3(\lambda_1)_k(-\lambda_2)_k$$

$$\left\{3\left(\sum_{j=1}^{k}\frac{1}{j}\right)^2+\sum_{j=1}^{k}\frac{1}{j^2}-2\sum_{j=1}^{k}\frac{1}{j}\left(\frac{2}{k}+\frac{1}{k-1}+1\right)\right\} \tag{4.47}$$

and $(\alpha;j_1,j_2)_n$ is defined by

$$(\alpha;j_1,j_2)_n = \prod_{\substack{i=1\cdots n\\ i\neq j_1,j_2}}(\alpha+i-1),\ (\alpha;j_1,j_2)_0 = 1. \tag{4.48}$$

By substituting these coefficients into(4.34) and neglecting the constant A_0 and a dependent term on the first basis, we have

$$\begin{aligned} y_3(x) &= 2(3\lambda_1\lambda_2+\lambda_1+\lambda_2)\zeta x + 2{}_2F_2(-\lambda_1,-\lambda_2;1,1;\zeta x)\\ &- 2\lambda_1\lambda_2\zeta x[(\ln x+3){}_2F_2(1-\lambda_1,1-\lambda_2;1,2,\zeta x)+{}_2F_2(1-\lambda_1,1-\lambda_2;2,2;\zeta x)]\\ &- (-\lambda_1)_2(-\lambda_2)_2(\zeta x)^2\ln x\left[\begin{array}{l}\left(\frac{1}{2}\ln x+1\right){}_2F_2(2-\lambda_1,2-\lambda_2;2,3;\zeta x)\\ +{}_2F_2(2-\lambda_1,2-\lambda_2;3,3;\zeta x)\end{array}\right]\\ &+ 2\ln xQ(0;\zeta x)+R(0;\zeta x) \end{aligned} \tag{4.49}$$

where

$$Q(\bar{n},z) = \sum_{k=2}^{\infty}\frac{q_k(\lambda_1,\lambda_2)}{(1)_{k-1}(1)_{k-2}}\frac{z^{k-n}}{(k-n)!}, \tag{4.50}$$

$$R(\bar{n},z) = \sum_{k=2}^{\infty}\frac{r_k(\lambda_1,\lambda_2)}{(1)_{k-1}(1)_{k-2}}\frac{z^{k-n}}{(k-n)!}, \tag{4.51}$$

Even if the roots of the characteristic equation λ_1 and λ_2 are complex conjugates of each other, all of the basis solutions are absolutely real functions because all λ-relevant terms of the solutions are composed of multiplicative and additive terms of complex conjugates. Moreover, we can see that $P(n;z)$, $Q(n;z)$, and $R(n;z)$ are bounded because their partial sums are all Cauchy sequences. Since every Cauchy sequence in R converges to an element of R, the infinite series functions are bounded for all z.

4.2. **General Solutions.** A general solution of the third-order differential equation is obtained by a linear combination of the three basis solutions. For the purpose of simplification, the general solution is assumed as

$$y(\tilde{x}) = \zeta^2 c_1 y_1(\tilde{x}) + \zeta\lambda_1\lambda_2 c_2 y_2(\tilde{x}) + c_3 y_3(\tilde{x}), \tag{4.52}$$

with change of the independent variable such that

$$\tilde{x} = \zeta x = \frac{t_{go}}{T}, \tag{4.53}$$

where c_1, c_2, and c_3 are arbitrary constants. For more convenience, we introduce new notations of the hypergeometric functions

$$\tilde{\phi}_\lambda(m;\tilde{x}) = \frac{(2-\lambda_1)_{m-2}(2-\lambda_2)_{m-2}}{(1)_{m-1}(1)_m} {}_2F_(m-\lambda_1, m-\lambda_2; m, m+1; \tilde{x}),\ (m = 2, 3, \cdots), \tag{4.54}$$

$$\phi_\lambda(m;\tilde{x}) = \frac{(-\lambda_1)_m(-\lambda_2)_m}{(1)_{m-1}(1)_m} {}_2F_(m-\lambda_1, m-\lambda_2; m, m+1; \tilde{x}),\ (m = 1, 2, 3, \cdots), \tag{4.55}$$

$$\psi_\lambda(m;\tilde{x}) = \frac{(-\lambda_1)_m(-\lambda_2)_m}{(1)_m(1)_m} {}_2F_(m-\lambda_1, m-\lambda_2; m+1, m; \tilde{x}),\ (m = 0, 1, 2, \cdots), \tag{4.56}$$

then their derivatives become

$$\frac{\partial}{\partial \tilde{x}}\tilde{\phi}_\lambda(m;\tilde{x}) = \tilde{\phi}_\lambda(m+1;\tilde{x}),\ \frac{\partial}{\partial \tilde{x}}\phi_\lambda(m;\tilde{x}) = \phi_\lambda(m+1;\tilde{x}), \frac{\partial}{\partial \tilde{x}}\psi_\lambda(m;\tilde{x}) = \psi_\lambda(m+1;\tilde{x})$$

respectively.

By substituting (4.14), (4.29), and (4.49) into (4.52) and replacing some notations with (4.53)-(4.56), the closed-form trajectory solution of the generalized guidance law can be expressed as

$$\begin{aligned} y(\tilde{x}) &= 2\mu c_3\tilde{x} + 2c_1\tilde{x}^2\tilde{\phi}_\lambda(2;\tilde{x}) + \{c_2 - 2c_3(3+\ln x)\}\tilde{x}\phi_\lambda(1;\tilde{x}) \\ &\quad + \{c_2 - c_3(2+\ln x)\}(\ln x)\tilde{x}^2\phi_\lambda(2;\tilde{x}) + 2c_3\psi_\lambda(0;\tilde{x}) - 2c_3\tilde{x}\psi_\lambda(1;\tilde{x}), \\ &\quad - 4c_3\tilde{x}^2\ln x\psi_\lambda(2;\tilde{x}) + c_2\tilde{P}(0;\tilde{x}) + 2c_3\ln xQ(0;\tilde{x}) + c_3R(0;\tilde{x}) \end{aligned} \tag{4.57}$$

where

$$\mu = 3\lambda_1\lambda_2 + \lambda_1 + \lambda_2, \tag{4.58}$$

and the series functions of $\tilde{P}(n;\tilde{x})$, $Q(n;\tilde{x})$, and $R(n;\tilde{x})$ are

$$\tilde{P}(n;\tilde{x}) = \lambda_1\lambda_2\sum_{k=0}^{\infty}\frac{p_k(\lambda_1,\lambda_2)}{(1)_k(1)_{k-1}}\frac{\tilde{x}^{k+1-n}}{(k+1-n)!} \tag{4.59}$$

$$Q(n;\tilde{x}) = \lambda_1\lambda_2\sum_{k=0}^{\infty}\frac{q_k(\lambda_1,\lambda_2)}{(1)_{k-1}(1)_{k-2}}\frac{\tilde{x}^{k-n}}{(k-n)!} \tag{4.60}$$

$$R(n;\tilde{x}) = \lambda_1\lambda_2\sum_{k=0}^{\infty}\frac{r_k(\lambda_1,\lambda_2)}{(1)_{k-1}(1)_{k-2}}\frac{\tilde{x}^{k-n}}{(k-n)!} \tag{4.61}$$

respectively. Here, $p_k(\lambda_1,\lambda_2)$, $q_k(\lambda_1,\lambda_2)$, $r_k(\lambda_1,\lambda_2)$ are constants which depend only on λ_1 and λ_2 as shown in (4.27), (4.46), and (4.47). It is noted that derivatives of the series functions become

$$\frac{\partial}{\partial\tilde{x}}\tilde{P}(n;\tilde{x}) = \tilde{P}(n+1;\tilde{x}),\ \frac{\partial}{\partial\tilde{x}}Q(n;\tilde{x}) = Q(n+1;\tilde{x}),\ \text{and}\ \frac{\partial}{\partial\tilde{x}}R(n;\tilde{x}) = R(n+1;\tilde{x})$$

From (4.1), moreover, the velocity perpendicular to the predetermined collision course and the normal acceleration are explicitly expressible as

$$v(\tilde{x}) = -\frac{1}{T}\left[\begin{array}{l} 2\mu c_3 + 4c_1\tilde{x}\tilde{\phi}_\lambda(2;\tilde{x}) + 2c_1\tilde{x}^2\tilde{\phi}_\lambda(3;\tilde{x}) - \{c_2 - 2c_3(4+\ln x)\}\phi_\lambda(1;\tilde{x}) \\ +2\{c_2(1+\ln x) - c_3(2+\ln x)^2\}\tilde{x}\phi_\lambda(2;\tilde{x}) \\ +\{c_2 - c_3(2+\ln x)\}(\ln x)\tilde{x}^2\phi_\lambda(3;\tilde{x}) \\ -2c_3(3+4\ln x)\tilde{x}\psi_\lambda(2;\tilde{x}) - 4c_3\tilde{x}^2\ln x\psi_\lambda(3;\tilde{x}) \\ +c_2\tilde{P}(1;\tilde{x}) + 2c_3\tilde{x}^{-1}Q(0;\tilde{x}) + 2c_3\ln xQ(1;\tilde{x}) + c_3R(1;\tilde{x}) \end{array}\right] \tag{4.62}$$

$$a_M(\tilde{x}) = \frac{1}{T^2}\left[\begin{array}{l} 4c_1\tilde{\phi}_\lambda(2;\tilde{x}) + 8c_1\tilde{x}\tilde{\phi}_\lambda(3;\tilde{x}) + 2c_2\tilde{x}^2\tilde{\phi}_\lambda(4;\tilde{x}) - 2c_3\tilde{x}^{-1}\phi_\lambda(1;\tilde{x}) \\ +\{c_2(5+\ln x) - 2c_3(3+\ln x)(4+\ln x)\}\phi_\lambda(2;\tilde{x}) \\ +\{c_2(3+4\ln x) - 2c_3(1+\ln x)(5+2\ln x)\}\tilde{x}\phi_\lambda(3;\tilde{x}) \\ +\{c_2 - c_3(2+\ln x)\}(\ln x)\tilde{x}^2\phi_\lambda(4;\tilde{x}) - 2c_3(7+4\ln x)\psi_\lambda(2;\tilde{x}) \\ -2c_3(5+8\ln x)\tilde{x}\psi_\lambda(3;\tilde{x}) - 4c_3(\ln x)\tilde{x}^2\psi_\lambda(4;\tilde{x}) + c_2\tilde{P}(2;\tilde{x}) \\ -2c_3\tilde{x}^2Q(0;\tilde{x}) + 4c_3\tilde{x}^{-1}Q(1;\tilde{x}) + 2c_3\ln xQ(2;\tilde{x}) + c_3R(2;\tilde{x}) \end{array}\right]. \tag{4.63}$$

By using (4.2), (4.57), and (4.62), the control input $u(\tilde{x})$ can be easily derived. From (4.57), (4.62), and (4.63), it is seen that the analytic solutions are expressed by combinations of polynomial, logarithmic, and a lot of series functions. From (4.57), since $\lim_{x\to 0}(\tilde{x}\ln x) = 0$ and $\lim_{x\to 0}\{\ln xQ(0;\tilde{x})\} = 0$, the trajectory solution is bounded for all $\tilde{x} \in [0,\zeta]$. However, it is seen that there exist unbounded terms such as $(\ln x)\phi_\lambda(1;\tilde{x})$ and $\tilde{x}^{-1}\phi_\lambda(1;\tilde{x})$ in the solutions of $v(\tilde{x})$ and $a_M(\tilde{x})$. In addition, from (4.63), it is expected for the acceleration to blow up in proportion to $\tilde{x}^{-2}$ as $\tilde{x} \to 0$. Since the coefficients c_1, c_2, and c_3 remain constant regardless of the independent variable $\tilde{x}$, we can have the coefficients as functions of initial conditions by letting $y(\zeta) = y_0$, $v(\zeta) = v_0$, and $a(\zeta) = a_0$; i.e., $c_1(y_0, v_0, a_0)$, $c_2(y_0, v_0, a_0)$, and $c_3(y_0, v_0, a_0)$ for given ζ, T, and guidance coefficients.

4.3. **Special Cases.** Now consider that the characteristic roots of λ_1 and λ_2 are positive integer. In this case, most of the series functions can be rewritten as explicit polynomial functions with finite power terms. From (4.54) and (4.56), since $\lambda_1 \geq \lambda_2$ and $(m-\lambda_2)_{\lambda_2-m+1} = 0 (l = 1, 2, \cdots)$, we have

$$\tilde{\phi}_\lambda(m;\tilde{x}) = \begin{cases} \dfrac{(-\lambda_1)_{m-2}(-\lambda_2)_{m-2}}{(1)_{m-1}(1)_m}\sum_{k=0}^{\lambda_1-m}\dfrac{(m-\lambda_1)_k(m-\lambda_2)_k}{(m)_k(m+1)_k}\dfrac{\tilde{x}^k}{k!} & 2 \leq m \leq \lambda_1 \\ \dfrac{(-\lambda_{11})_{m-2}(-\lambda_2)_{m-2}}{(1)_{m-1}(1)_m}{}_2F_2(m-\lambda_1, m-\lambda_2; m, m+1;\tilde{x}), & \lambda_1 < m \leq \lambda_2 + 2 \\ 0, & m > \lambda_2 + 2 \end{cases} \tag{4.64}$$

$$\phi_\lambda(m;\tilde{x}) = \begin{cases} \dfrac{(-\lambda_1)_m(-\lambda_2)_m}{(1)_{m-1}(1)_m} \sum_{k=0}^{\lambda_2-m} \dfrac{(m-\lambda_1)_k(m-\lambda_2)_k}{(m)_k(m+1)_k}\dfrac{\tilde{x}^k}{k!} & 1 \le m \le \lambda_2 \\ 0, & m > \lambda_2 \end{cases} \tag{4.65}$$

$$\psi_\lambda(m;\tilde{x}) = \begin{cases} \dfrac{(-\lambda_1)_m(-\lambda_2)_m}{(1)_{m-1}(1)_m} \sum_{k=0}^{\lambda_2-m} \dfrac{(m-\lambda_1)_k(m-\lambda_2)_k}{(m+1)_k(m+1)_k}\dfrac{\tilde{x}^k}{k!} & 0 \le m \le \lambda_2 \\ 0, & m > \lambda_2 \end{cases} \tag{4.66}$$

In addition, from (4.27) and (4.46), we have $p_{\lambda_1+1}(\lambda_1,\lambda_2) = 0(l = 0,1,2,\cdots)$ and $q_{\lambda_1+1}(\lambda_1,\lambda_2) = 0(l = 0,1,2,\cdots)$, then

$$\tilde{P}(n,\tilde{x}) = \lambda_1\lambda_2 \sum_{k=1}^{\lambda_1-1} \frac{p_k(\lambda_1,\lambda_2)}{(1)_k(1)_{k-1}} \frac{\tilde{x}^{k+1-n}}{(k+1-n)!}, \tag{4.67}$$

$$Q(n,\tilde{x}) = \sum_{k=2}^{\lambda_1} \frac{q_k(\lambda_1,\lambda_2)}{(1)_{k-1}(1)_{k-2}} \frac{\tilde{x}^{k-n}}{(k-n)!} \tag{4.68}$$

On the other hand, $R(n;\tilde{x})$ may not be expressed as an explicit polynomial function. From (4.47), since

$$r_k(\lambda_1,\lambda_2) = -2(-\lambda_1)_{\lambda_1}(-\lambda_2)_{\lambda_2}(1)_{k-\lambda_1-1}(1)_{k-\lambda_2-1} \quad \text{for} \quad k > \lambda_1,$$

$R(n;\tilde{x})$ can be rewritten as a combination of finite power terms and hypergeometric function as

$$\begin{aligned} R(n,\tilde{x}) = &\sum_{k=2}^{\lambda_1} \frac{r_k(\lambda_1,\lambda_2)}{(1)_{k-1}(1)_{k-2}} \frac{\tilde{x}^{k-n}}{(k-n)!} \\ &- \frac{2(-1)^{\lambda_1+\lambda_2}\lambda_1(1)_{\lambda_2}}{(\lambda_1-\lambda_2+1)_{\lambda_2}(1)_{\lambda_1+1-n}} \tilde{x}^{\lambda_1+1-n} {}_3F_3(1,1,1+\lambda_1-\lambda_2;\lambda_1,\lambda_1+1,\lambda_1+2-n;\tilde{x}). \end{aligned} \tag{4.69}$$

Thus, by substituting (4.65)-(4.69) into (4.57) and (4.62)-(4.63), closed-form solutions of the guidance law with positive integer characteristic roots for a 1st-order lag system are calculated more easily.

Consider a special case given by choosing $\lambda_1 = 3$ and $\lambda_2 = 2$, then a pair of the guidance coefficient $[k_1,\ k_2]$ becomes $[6,\ 4]$ which is that for EOGL. In this case, solutions of the guidance law can be drawn by substituting $\lambda_1 = 3$ and $\lambda_2 = 2$ into the general solutions. From (4.57) and (4.64) to (4.69), the trajectory solution of EOGL for a single lag system becomes

$$\begin{aligned} y(\tilde{x}) = &2c_3 + (6c_2 + 10c_3)\tilde{x} + (c_1 - 18c - 2 - 86c_3)\tilde{x}^2 - \left(c_2\frac{40}{3}c_3\right)\tilde{x}^3 - 12c_3\tilde{x}\ln x \\ &+ (6c_2 + 46c_3)\tilde{x}^2\ln x + 2c_3\tilde{x}^3\ln x - 6c_3\tilde{x}^2(\ln x)^2 + \frac{1}{12}c_3\tilde{x}^4 {}_2F_2(1,1,2;3,4,5;\tilde{x}). \end{aligned} \tag{4.70}$$

As another example, consider the case of $\lambda_2 = 2$. Since $k_1 = \lambda_1\lambda_2$ and $k_2 = \lambda_1 + \lambda_2 - 1$, then we easily see that $k_1 = k_2 = \lambda_1$. Hence, the guidance law for $\lambda_2 = 1$ is equivalent to conventional PNG with navigation constant $N = \lambda_1$. Assuming that N is an integer larger than 3, solutions of the guidance law can be drawn by substituting $\lambda_1 = N$ and $\lambda_2 = 1$ into the general solutions. From (4.64) to (4.66), we have $\tilde{\phi}_\lambda(2;\tilde{x}) = \sum_{k=0}^{N-2} \frac{(2-N)_k}{(2)_k(2)_{k+1}}\tilde{x}^k$, $\tilde{\phi}_\lambda(3;\tilde{x}) = N\sum_{k=0}^{N-3} \frac{(3-N)_k(k+1)}{(1)_{k+2}(1)_{k+3}}\tilde{x}^k$, $\tilde{\phi}_\lambda(4;\tilde{x}) = \phi_\lambda(2;\tilde{x}) = \phi_\lambda(3;\tilde{x}) = \phi_\lambda(4;\tilde{x}) = \psi_\lambda(2;\tilde{x}) = \psi_\lambda(3;\tilde{x}) = \psi_\lambda(4;\tilde{x}) = 0$, $\phi_\lambda(1;\tilde{x}) = \psi_\lambda(1;\tilde{x}) = N$, and $\psi_\lambda(0;\tilde{x}) = 1$. Assuming that N is an integer larger than 2, from (4.67) to (4.69) with (4.27), (4.46), and (4.47), we have

$$\tilde{P}(n;\tilde{x}) = -\sum_{k=1}^{N-1} \frac{(-N)_{k-1}}{(1)_k}\frac{\tilde{x}^{k+1-n}}{(k+1-n)!},\ Q(n;\tilde{x}) = \sum_{k=2}^{N} \frac{(-N)_k}{(1)_{k-1}}\frac{\tilde{x}^{k-n}}{(k-n)!},$$

$$R(n;\tilde{x}) = 2\sum_{k=2}^{N} \frac{\tilde{r}_k}{(1)_{k-1}}\frac{\tilde{x}^{k-n}}{(k-n)!} - 2\frac{(-1)^{N+1}\tilde{x}^{N+1-n}}{(N+1-n)!}{}_2F_2(1,1;N+1,N+2-n;\tilde{x}),$$

where

$$\tilde{r}_k = \sum_{j_2=3}^{k} \frac{(2j_2-N-3)(-N)_{j_2-1}(j_2-N)_{k-j_2}}{j_2-2} + \left(-3\sum_{j=1}^{k}\frac{1}{j} + \frac{2}{k} + \frac{1}{k-1} + 1\right)(-N)_k - N(2-N)_{k-2}.$$

Here, we see that $\tilde{P}(n;\tilde{x}) = -Q(n;\tilde{x})$ by substituting $k = k'-1$ in $\tilde{P}(n;\tilde{x})$. Thus, by (4.57), the trajectory solution of PNG for a single lag system becomes

$$\begin{aligned} y(\tilde{x}) &= 2c_3 + \{Nc_2 + 2c_3(1 - N\ln x)\}\tilde{x} - (c_2 - 2c - 3\ln x)\sum_{k=2}^{N}\frac{(-N)_k}{(1)_{k-1}}\frac{\tilde{x}^k}{k!} \\ &+ 2c_1\tilde{x}^2\sum_{k=0}^{N-2}\frac{(2-N)_k}{(2)_k(2)_{k+1}}\tilde{x}^k + 2c_3\sum_{k=2}^{N}\frac{\tilde{r}_k}{(1)_{k-1}}\frac{\tilde{x}^k}{k!} \\ &- 2c_3\frac{(-1)^{N+1}\tilde{x}^{N+1}}{(N+1)!}{}_2F_2(1,1;N+1,N+2;\tilde{x}). \end{aligned} \tag{4.71}$$

4.4. **Terminal Miss.** Now consider terminal misses due to the system lag. From (4.57) and (4.62), we have the miss-distance and the impact angle error denoted by Δy_f and $\Delta\gamma_f$ as

$$\Delta y_f = y(\tilde{x} = 0) = 2c_3, \tag{4.72}$$

$$\Delta\gamma_f = \frac{v(\tilde{x}=0)}{V_M} = \frac{\lambda_1\lambda_2}{V_M T}[-c_2 + 2c_3(\Lambda + E)], \tag{4.73}$$

where

$$\Lambda = 1 - ({\lambda_1}^{-1} + {\lambda_2}^{-1}) - (\lambda_1\lambda_2)^{-1} \text{ and } E = \lim_{\omega\to 0}\ln\omega. \tag{4.74}$$

Here, it is seen that $\Delta\gamma_f$ has a slowly divergent element E as $\omega \to 0$ while Δy_f is bounded. If c_3 is zero, interestingly, we see that not only does Δy_f become zero, but $\Delta\gamma_f$ is bounded. Note that the coefficients c_2, c_3 which are closely related to the misses are functions of guidance coefficients as well as initial conditions. It implies that it is possible to design the guidance law to minimize the terminal misses due to the system lag.

In order to investigate sensitivities of the terminal misses to the initial homing geometry under the assumption of $\alpha_0 = 0$, the terminal misses are rewritten in terms of γ_0 and γ_f by substituting $y_0 = V_M t_f \gamma_f$ and $v_0 = V_M(\gamma_0 - \gamma_f)$ into the coefficients c_2, c_3 as

$$\frac{\Delta y_f}{V_M T} = \left(\left.\frac{\Delta y_f}{V_M T \gamma_0}\right|_{\gamma_f=0} \right) \gamma_0 + \left(\left.\frac{\Delta y_f}{V_M T \gamma_f}\right|_{\gamma_0=0} \right) \gamma_f, \tag{4.75}$$

$$\gamma_f = \left(\left.\frac{\Delta\gamma_f}{\gamma_0}\right|_{\gamma_f=0} \right) \gamma_0 + \left(\left.\frac{\Delta\gamma_f}{\gamma_f}\right|_{\gamma_0=0} \right) \gamma_f, \tag{4.76}$$

where

$$\left.\frac{\Delta y_f}{V_M T \gamma_0}\right|_{\gamma_f=0} = \frac{2}{V_M T} \left.\frac{\partial c_3}{\partial \gamma_0}\right|_{\gamma_f=0}, \quad \left.\frac{\Delta y_f}{V_M T \gamma_f}\right|_{\gamma_0=0} = \frac{2}{V_M T} \left.\frac{\partial c_3}{\partial \gamma_f}\right|_{\gamma_0=0}$$

$$\left.\frac{\Delta\gamma_f}{\gamma_0}\right|_{\gamma_f=0} = \frac{\lambda_1\lambda_2}{V_M T}\left[- \left.\frac{\partial c_2}{\partial \gamma_0}\right|_{\gamma_f=0} + 2(\Lambda + E \left.\frac{\partial c_3}{\partial \gamma_0}\right|_{\gamma_f=0}) \right],$$

$$\left.\frac{\Delta\gamma_f}{\gamma_f}\right|_{\gamma_0=0} = \frac{\lambda_1\lambda_2}{V_M T}\left[- \left.\frac{\partial c_2}{\partial \gamma_f}\right|_{\gamma_0=0} + 2(\Lambda + E \left.\frac{\partial c_3}{\partial \gamma_f}\right|_{\gamma_0=0}) \right],$$

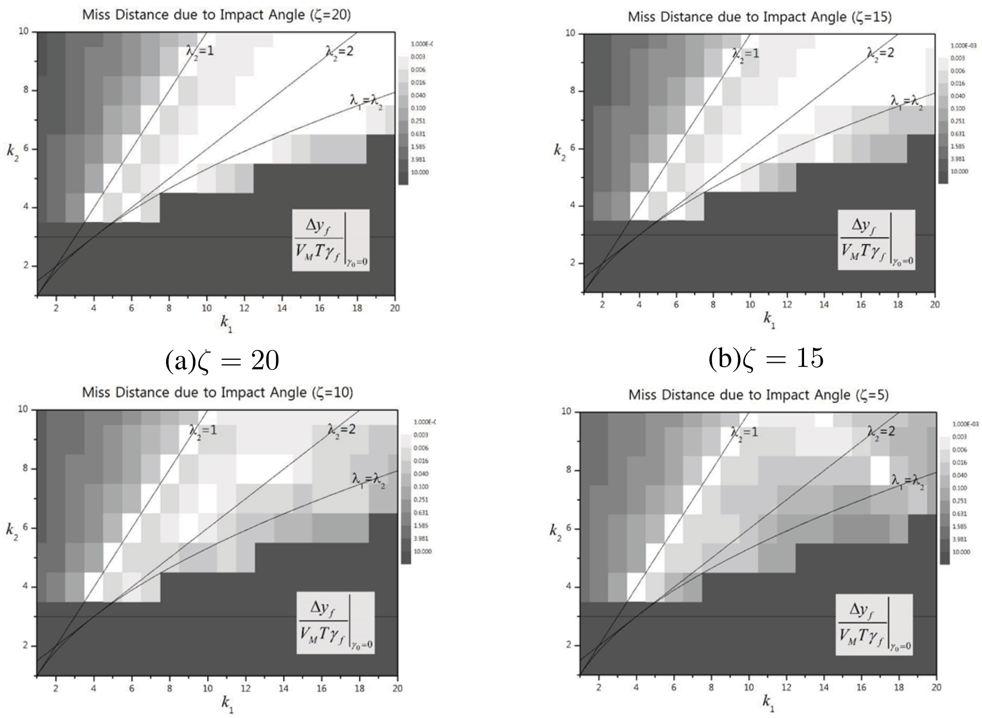
From (4.75) and (4.76), it is seen that the normalized terminal misses depend only on λ_1, λ_2, and ζ. It implies that the terminal miss-distance due to the system lag can be reduced by choosing an adequate pair of guidance coefficients of the generalized guidance law for given homing geometry. Especially, from (4.72)-(4.75), we have a condition for zero miss-distance and finite impact angle error as

$$c_2(\gamma_0, \gamma_f, a_0, \zeta, T, \lambda_1, \lambda_2) = 0. \tag{4.77}$$

From this result, it is noted that there exists the optimal initial geometry in the homing phase to minimize terminal misses due to the system lag.

Figure 8 to Figure 9 provide normalized terminal miss contours of k_1 versus k_2 for $\zeta = 20$, 15, 10, and 5. In the figures, the contrast between light and shade represents the magnitude of terminal misses; a brighter region stands for better performance. It is seen that small ζ yields bad performance, as evidenced by lack of maneuvering time. In a miss-distance point of view, while the guidance coefficients in the region between $\lambda_2 = 1$ and $\lambda_2 = \lambda_1$ provides good performance for large ζ, those in the region of $1 \le \lambda_2 \le 2$ are better selection for small ζ. It is noted that the guidance coefficients in F_3 provide bad performance even if the solutions of them for a lag-free system satisfy the terminal constraints. From Figure 9, it is noted that the guidance coefficients near to the line of $\lambda_2 = 2$ cause small impact angle errors for large

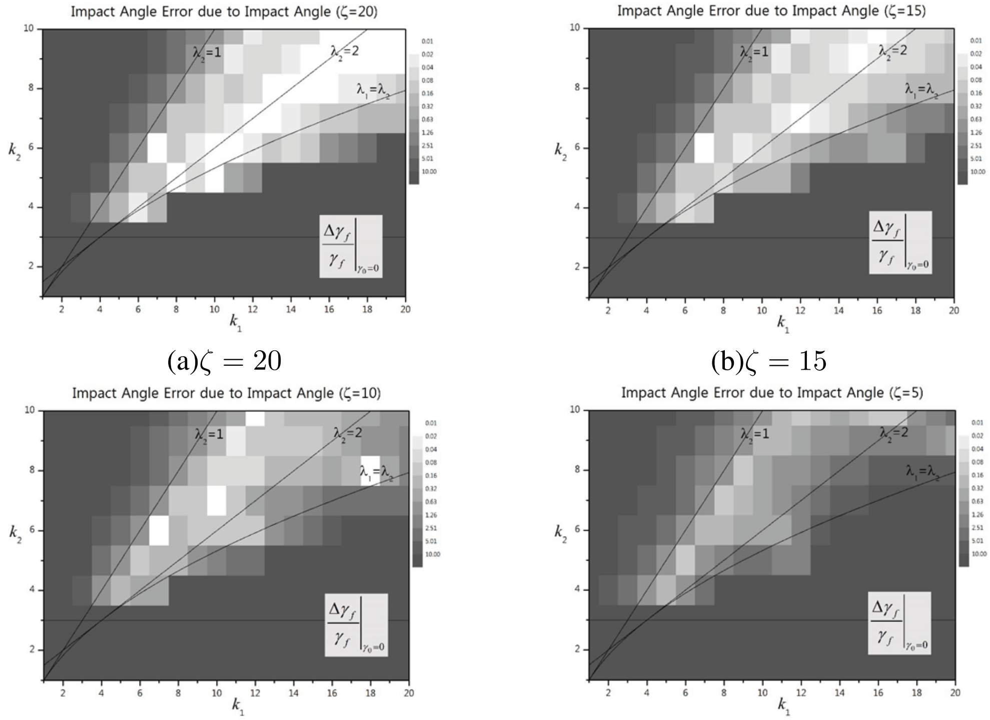
ζ while those in the region of $1 < \lambda_2 < 2$ provide better performance for small ζ. It is found from the results that λ_2 becomes a dominant factor in the terminal misses due to the system lag. Most of all, it is very important that $\lambda_2 \approx 2$ or less provides more robust performance for impact angle control against system lag. This fact will provide useful clues in designing an impact-angle-control guidance law.



(a)$\zeta = 20$ (b)$\zeta = 15$

(c)$\zeta = 10$ (d)$\zeta = 5$

FIGURE 8. Sensitivity of miss distance to impact angle.



(a)$\zeta = 20$ (b)$\zeta = 15$

(c)$\zeta = 10$ (d)$\zeta = 5$

FIGURE 9. Sensitivity of impact angle error to impact angle.

4.5. **Verification.** In order to examine the analytic solution of the generalized impact angle control guidance for a single lag system, a simple linear simulation is carried out. It is assumed that the flight time t_f is given as 10 sec and the missile is taken as a first-order system whose time constant T is 1 sec. In addition, the initial conditions, y_0 and v_0, are assumed to be 10 m and 10 m/sec, respectively. A pair of guidance coefficients which is one of TOGL coefficients is chosen for this examination. In Fig. 10 to Fig. 12, missile positions, velocities perpendicular to the predetermined collision course, and the normal accelerations of missile are illustrated. The analytic solutions for a first-order lag system shown in scattered symbols are compared with linear simulation results and the analytic solutions for lag-free system. In the figures, it is observed that the analytic solutions are in perfect agreement with the simulation results and show different time histories from those for a lag-free system. It is noted that the acceleration blows up near the final time even if TOGL is adopted. Especially, approximated solutions using nth order polynomials instead of the infinite series functions are compared with the exact solutions. From the results, polynomial approximation with a degree greater than 5 provides good agreement with the exact solution. Thus, it is expected that good approximated solutions which are easy to utilize is possible.

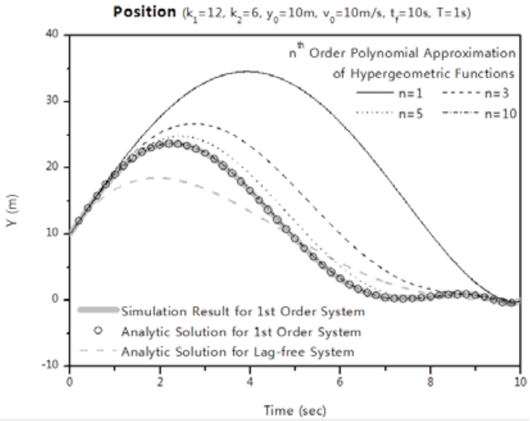


FIGURE 10. Comparisons between analytic solutions and simulation results(position).

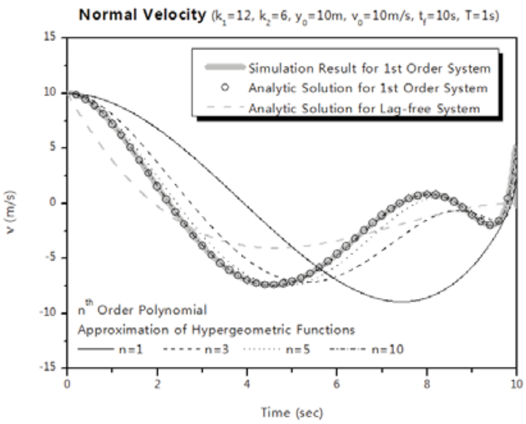


FIGURE 11. Comparisons between analytic solutions and simulation results(normal velocity).

4.6. **Closing Remarks.** In this chapter, analytic solutions of a generalized guidance law with arbitrary guidance constants under the assumption that the missile is given by a first-order lag system have been investigated by solving a third-order linear time-varying ordinary differential equation. It is noted that the solutions are represented by combinations of polynomial functions, a logarithmic function, and a lot of bounded infinite series functions. It is found that the guidance command, the acceleration of the missile, and the velocity component perpendicular to the collision course tend to diverge as the missile approaches the target mathematically. Terminal misses due to the system lag have been derived by using the analytic solutions and effects of guidance coefficients on the terminal misses are investigated. Moreover, effective homing geometries

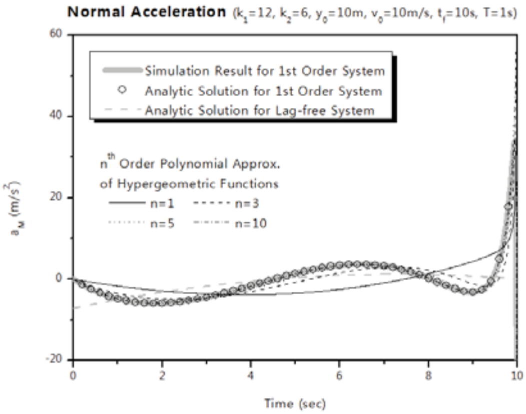


FIGURE 12. Comparisons between analytic solutions and simulation results(acceleration).

5. CONCLUSION

In this paper, the following three topics are mainly stated; a generalized impact-angle-control guidance law with a arbitrary pair of guidance coefficients, optimality of the generalized guidance law, and analytic solutions of the generalized guidance law for a single-lag system.

The generalized guidance law is based on a practical homing loop structure with two guidance coefficients. Explicit closed-form solutions for a lag-free system show three classes of trajectories depending on the guidance coefficients: one is the time-to-go polynomial trajectory, another is the trajectory combined with polynomial and logarithmic functions, and the other is the oscillatory trajectory combined with polynomial, logarithmic, and harmonic functions. The entire feasible sets of the guidance coefficients to satisfy the terminal constraints have been demonstrated. Based on the closed-form solutions, a practical and precise time-to-go calculation method is proposed as well.

Another important work is to prove the optimality of the generalized guidance law. By solving an inverse optimal problem, it is found that the guidance law with arbitrary guidance coefficients can minimize a corresponding performance index. In addition, the relationship between the guidance coefficients and the corresponding quadratic performance index has been discussed. The proposed results serve as a theoretical foundation to design the guidance law to improve certain performance.

Analytic solutions of the generalized guidance law for a single-lag system have been investigated by solving third-order linear time-varying ordinary differential equations. It is found that the solutions are represented by combinations of polynomial functions, a logarithmic function, and infinite power series functions. The analytic solutions provide an insight into the blowing-up behavior of the homing loop as the missile approaches the target. Terminal misses due to the system lag have been investigated by using the analytic solutions, and effects of guidance coefficients on the terminal misses have been discussed. Effective homing geometries which are advantageous in terms of the terminal misses are also proposed. Furthermore, good approximated solutions which are easy to utilize is possible from the results.

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