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# A REMARK ON UNIQUE CONTINUATION FOR THE CAUCHY-RIEMANN OPERATOR

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Abstract. In this note we obtain a unique continuation result for the differential inequality $|\overline{\partial}u| \leq |Vu|$, where $\overline{\partial} = (i\partial_y + \partial_x)/2$ denotes the Cauchy-Riemann operator and $V(x, y)$ is a function in $L^2(\mathbb{R}^2)$.

## 1. Introduction

The unique continuation property is one of the most interesting properties of holomorphic functions $f \in H(\mathbb{C})$. This property says that if $f$ vanishes in a non-empty open subset of $\mathbb{C}$, then it must be identically zero. Note that $u \in C^1(\mathbb{R}^2)$ satisfies the Cauchy-Riemann equation $(i\partial_y + \partial_x)u = 0$ if and only if it defines a holomorphic function $f(x + iy) \equiv u(x, y)$ on $\mathbb{C}$. From this point of view, one can see that a $C^1$ function satisfying the equation has the unique continuation property.

In this note we consider a class of non-holomorphic functions $u$ which satisfy the differential inequality

$$|\overline{\partial}u| \leq |Vu|, \tag{1.1}$$

where $\overline{\partial} = (i\partial_y + \partial_x)/2$ denotes the Cauchy-Riemann operator and $V(x, y)$ is a function on $\mathbb{R}^2$.

The best positive result for (1.1) is due to Wolff [9] (see Theorem 4 there) who proved the property for $V \in L^p$ with $p > 2$. On the other hand, there is a counterexample [8] to unique continuation for (1.1) with $V \in L^p$ for $p < 2$. The remaining case $p = 2$ seems to be unknown for the differential inequality (1.1), and note that $L^2$ is a scale-invariant space of $V$ for the equation $\overline{\partial}u = Vu$. Here we shall handle this problem. Our unique continuation result is the following theorem which is based on bounds for a Fourier multiplier from $L^p$ to $L^q$.

**Theorem 1.1.** *Let $1 < p < 2 < q < \infty$ and $1/p - 1/q = 1/2$. Assume that $u \in L^p \cap L^q$ satisfies the inequality (1.1) with $V \in L^2$ and vanishes in a non-empty open subset of $\mathbb{R}^2$. Then it must be identically zero.*

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The unique continuation property also holds for harmonic functions, which satisfy the Laplace equation $\Delta u = 0$, since they are real parts of holomorphic functions. This was first extended by Carleman [1] to a class of non-harmonic functions satisfying the inequality $|\Delta u| \leq |Vu|$ with $V \in L^\infty(\mathbb{R}^2)$. There is an extensive literature on later developments in this subject. In particular, the problem of finding all the possible $L^p$ functions $V$, for which $|\Delta u| \leq |Vu|$ has the unique continuation, is completely solved (see [3, 5, 7]). See also the survey papers of Kenig [4] and Wolff [10] for more details, and the recent paper of Kenig and Wang [6] for a stronger result which gives a quantitative form of the unique continuation.

Throughout the paper, the letter $C$ stands for positive constants possibly different at each occurrence. Also, the notations $\widehat{f}$ and $\mathcal{F}^{-1}(f)$ denote the Fourier and the inverse Fourier transforms of $f$, respectively.

## 2. A preliminary lemma

The standard method to study the unique continuation property is to obtain a suitable Carleman inequality for relevant differential operator. This method originated from Carleman's classical work [1] for elliptic operators. In our case we need to obtain the following inequality for the Cauchy-Riemann operator $\overline{\partial} = (i\partial_y + \partial_x)/2$, which will be used in the next section for the proof of Theorem 1.1:

**Lemma 2.1.** *Let $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. For all $t > 0$, we have*

$$\big\| |z|^{-t} f \big\|_{L^q} \leq C \big\| |z|^{-t} \overline{\partial} f \big\|_{L^p} \tag{2.1}$$

*if $1 < p < 2 < q < \infty$ and $1/p - 1/q = 1/2$. Here, $z = x + iy \in \mathbb{C}$ and $C$ is a constant independent of $t$.*

*Proof.* First we note that

$$\overline{\partial}(z^{-t} f) = z^{-t} \overline{\partial} f + f \overline{\partial}(z^{-t}) = z^{-t} \overline{\partial} f$$

for $z \in \mathbb{C} \setminus \{0\}$. Then the inequality (2.1) is equivalent to

$$\big\| z^{-t} f \big\|_{L^q} \leq C \big\| \overline{\partial}(z^{-t} f) \big\|_{L^p}.$$

By setting $g = z^{-t} f$, we are reduced to showing that

$$\|g\|_{L^q} \leq C \|(i\partial_y + \partial_x) g\|_{L^p}$$

for $g \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. To show this, let us first set

$$(i\partial_y + \partial_x) g = h, \tag{2.2}$$

and let $\psi_\delta : \mathbb{R}^2 \to [0, 1]$ be a smooth function such that $\psi_\delta = 0$ in the ball $B(0, \delta)$ and $\psi_\delta = 1$ in $\mathbb{R}^2 \setminus B(0, 2\delta)$. Then, using the Fourier transform in (2.2), we see that

$$(-\eta + i\xi) \widehat{g}(\xi, \eta) = \widehat{h}(\xi, \eta).$$

Thus, by Fatou's lemma we are finally reduced to showing the following uniform boundedness for a multiplier operator having the multiplier $m(\xi,\eta) = \psi_\delta(\xi,\eta)/(-\eta+i\xi)$:

$$\left\| \mathcal{F}^{-1}\Big(\frac{\psi_\delta(\xi,\eta)\widehat{h}(\xi,\eta)}{-\eta+i\xi}\Big)\right\|_{L^q} \le C\|h\|_{L^p} \tag{2.3}$$

uniformly in $\delta > 0$.

From now on, we will show (2.3) using Young's inequality for convolutions and Littlewood-Paley theorem ([2]). Let us first set for $k \in \mathbb{Z}$

$$\widehat{Th}(\xi,\eta) = m(\xi,\eta)\widehat{h}(\xi,\eta) \quad \text{and} \quad \widehat{T_kh}(\xi,\eta) = m(\xi,\eta)\chi_k(\xi,\eta)\widehat{h}(\xi,\eta),$$

where $\chi_k(\cdot) = \chi(2^k\cdot)$ for $\chi \in C_0^\infty(\mathbb{R}^2)$ which is such that $\chi(\xi,\eta) = 1$ if $|(\xi,\eta)| \sim 1$, and zero otherwise. Also, $\sum_k \chi_k = 1$. Now we claim that

$$\|T_kh\|_{L^q} \le C\|h\|_{L^p} \tag{2.4}$$

uniformly in $k \in \mathbb{Z}$. Then, since $1 < p < 2 < q < \infty$, by the Littlewood-Paley theorem together with Minkowski's inequality, we get the desired inequality (2.3) as follows:

$$\begin{aligned}
\Big\|\sum_k T_kh\Big\|_{L^q} &\le C\Big\|\Big(\sum_k |T_kh|^2\Big)^{1/2}\Big\|_{L^q} \\
&\le C\Big(\sum_k \|T_kh\|_{L^q}^2\Big)^{1/2} \\
&\le C\Big(\sum_k \|h_k\|_{L^p}^2\Big)^{1/2} \\
&\le C\Big\|\Big(\sum_k |h_k|^2\Big)^{1/2}\Big\|_{L^p} \\
&\le C\Big\|\sum_k h_k\Big\|_{L^p},
\end{aligned}$$

where $h_k$ is given by $\widehat{h_k}(\xi,\eta) = \chi_k(\xi,\eta)\widehat{h}(\xi,\eta)$. Now it remains to show the claim (2.4). But, this follows easily from Young's inequality. Indeed, note that

$$T_kh = \mathcal{F}^{-1}\Big(\frac{\psi_\delta(\xi,\eta)\chi_k(\xi,\eta)}{-\eta+i\xi}\Big) * h$$

and by Plancherel's theorem

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\Big(\frac{\psi_\delta(\xi,\eta)\chi_k(\xi,\eta)}{-\eta+i\xi}\Big)\right\|_{L^2} &= \left\|\frac{\psi_\delta(\xi,\eta)\chi_k(\xi,\eta)}{-\eta+i\xi}\right\|_{L^2} \\
&\le C\bigg(\int_{|(\xi,\eta)|\sim 2^{-k}} \frac{1}{\eta^2+\xi^2}d\xi d\eta\bigg)^{1/2} \\
&\le C.
\end{aligned}$$

Since we are assuming the gap condition $1/p-1/q=1/2$, by Young's inequality for convolutions, this readily implies that

$$\|T_k h\|_{L^q} \le \left\|\mathcal{F}^{-1}\left(\frac{\psi_\delta(\xi,\eta)\chi_k(\xi,\eta)}{-\eta+i\xi}\right)\right\|_{L^2}\|h\|_{L^p} \le C\|h\|_{L^p}$$

as desired. □

## 3. Proof of Theorem 1.1

The proof is standard once one has the Carleman inequality (2.1) in Lemma 2.1.

Without loss of generality, we may show that $u$ must vanish identically if it vanishes in a sufficiently small neighborhood of zero. Then, since we are assuming that $u \in L^p \cap L^q$ vanishes near zero, from (2.1) with a standard limiting argument involving a $C_0^\infty$ approximate identity, it follows that

$$\big\||z|^{-t}u\big\|_{L^q} \le C\big\||z|^{-t}\overline{\partial}u\big\|_{L^p}.$$

Thus by (1.1) we see that

$$\big\||z|^{-t}u\big\|_{L^q(B(0,r))} \le C\big\||z|^{-t}Vu\big\|_{L^p(B(0,r))} + C\big\||z|^{-t}\overline{\partial}u\big\|_{L^p(\mathbb{R}^2\setminus B(0,r))},$$

where $B(0,r)$ is the ball of radius $r>0$ centered at 0. Then, using Hölder's inequality with $1/p-1/q=1/2$, the first term on the right-hand side in the above can be absorbed into the left-hand side as follows:

$$\begin{aligned} C\big\||z|^{-t}Vu\big\|_{L^p(B(0,r))} &\le C\|V\|_{L^2(B(0,r))}\big\||z|^{-t}u\big\|_{L^q(B(0,r))} \\ &\le \frac{1}{2}\big\||z|^{-t}u\big\|_{L^q(B(0,r))} \end{aligned}$$

if we choose $r$ small enough. Here, $\||z|^{-t}u\|_{L^q(B(0,r))}$ is finite since $u \in L^q$ vanishes near zero. Hence we get

$$\begin{aligned} \|(r/|z|)^t u\|_{L^q(B(0,r))} &\le 2C\|\overline{\partial}u\|_{L^p(\mathbb{R}^2\setminus B(0,r))} \\ &\le 2C\|V\|_{L^2}\|u\|_{L^q} \\ &< \infty. \end{aligned}$$

By letting $t\to\infty$, we now conclude that $u=0$ on $B(0,r)$. This implies $u\equiv 0$ by a standard connectedness argument.

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