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ON CANTOR SETS AND PACKING MEASURES

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Abstract. For every doubling gauge g, we prove that there is a Cantor set of positive finite $\mathcal{H}^g$-measure, $\mathcal{P}^g$-measure, and $\mathcal{P}_0^g$-premeasure. Also, we show that every compact metric space of infinite $\mathcal{P}_0^g$-premeasure has a compact countable subset of infinite $\mathcal{P}_0^g$-premeasure. In addition, we obtain a class of uniform Cantor sets and prove that, for every set E in this class, there exists a countable set F, with $\overline{F} = E \cup F$, and a doubling gauge g such that $E \cup F$ has different positive finite $\mathcal{P}^g$-measure and $\mathcal{P}_0^g$-premeasure.

1. Introduction

Let X be a metric space. Let g: $[0,\infty) \to [0,\infty)$ be a gauge, i.e., a nondecreasing continuous function, with $g(t) = 0$ if and only if $t = 0$. Let $E \subset X$ and $\delta > 0$. A δ-packing of E is a family of disjoint balls $\{B(x_i, r_i)\}$ with $2r_i \leq \delta$ and $x_i \in E$. For every δ-packing $\{B(x_i, r_i)\}$ of E one has a sum $\sum g(2r_i)$. Let $\mathcal{P}_\delta^g(E) = \sup \sum g(2r_i)$ be the supremum of such sums. The packing premeasure of E with respect to the gauge g is defined by

$$\mathcal{P}_0^g(E) = \lim_{\delta\to 0} \mathcal{P}_\delta^g(E).$$

The packing measure of E with respect to the gauge g is defined by

$$\mathcal{P}^g(E) = \inf \left\{ \sum_{i=1}^\infty \mathcal{P}_0^g(E_i) : E \subset \bigcup_{i=1}^\infty E_i \right\}.$$

When $g(t) = t^s$, where $s > 0$, the above definitions give us the ordinary s-dimensional packing premeasure and measure, which are denoted by $\mathcal{P}_0^s$ and $\mathcal{P}^s$, respectively. The packing dimension of a subset E of X is defined by

$$\dim_P E = \inf\{s > 0 : \mathcal{P}^s(E) = 0\} = \sup\{s > 0 : \mathcal{P}^s(E) = \infty\}.$$

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The packing measure and dimension were introduced by C. Tricot [15] in 1982. See also [14]. The packing measure on a given metric space is a metric outer measure. The packing premeasure is monotonic and finitely subadditive, but it is usually not an outer measure, because it is not countably subadditive. If E is a subset of X, then

$$\mathcal{P}_0^g(E) = \mathcal{P}_0^g(\overline{E}), \tag{1.1}$$

where $\overline{E}$ denotes the closure of the set E. If E and F are subsets of X with $\mathrm{dist}(E, F) > 0$, then

$$\mathcal{P}_0^g(E \cup F) = \mathcal{P}_0^g(E) + \mathcal{P}_0^g(F). \tag{1.2}$$

The s-dimensional packing premeasure can be used to give an equivalent definition for the upper box-counting dimension $\overline{\dim}_B E$ of a subset E of a metric space (see [3] for other definitions of $\overline{\dim}_B E$). That is,

$$\overline{\dim}_B E = \inf\{s > 0 : \mathcal{P}_0^s(E) = 0\} = \sup\{s > 0 : \mathcal{P}_0^s(E) = \infty\}.$$

The packing dimension is different from the upper box-counting dimension. The former is countably stable, but the latter is not. However, if E is a compact subset of a complete metric space X, such that $\overline{\dim}_B E = \overline{\dim}_B(E \cap U)$ for all open sets $U \subset X$ with $U \cap E \neq \emptyset$, then, by Baire's category theorem, one has

$$\dim_P E = \overline{\dim}_B E \text{ (cf. Falconer [3]).}$$

The packing dimension is also different from the Hausdorff dimension. Indeed, given $0 \leq s_1 \leq s_2 \leq 1$, one may construct a middle interval Cantor set E (see its definition in (1.5)) with $\dim_H E = s_1$ and $\dim_P E = s_2$.

The packing measure $\mathcal{P}^g$ is used in the study of fractals in a way dual to the Hausdorff measure $\mathcal{H}^g$. If X is a separable metric space and g is a gauge, then we have $\mathcal{H}^g \leq \mathcal{P}^g$ by a basic covering argument (cf. Mattila [9]). On the other hand, for the packing measure $\mathcal{P}^n$ and the Hausdorff measure $\mathcal{H}^n$ on $\mathbb{R}^n$ we have $\Omega_n \mathcal{H}^n = \Omega_n \mathcal{P}^n = \mathcal{L}^n$, where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$ and $\Omega_n = \mathcal{L}^n(B(0, 1/2))$. For further study on the relationship between packing measure and Hausdorff measure we refer to Feng [4] and Rajala [12].

We say that a gauge g is doubling, if there are constants $C, \delta > 0$ such that

$$g(2t) \leq C g(t)$$

for all $t \in (0, \delta)$. Clearly, a gauge g is doubling if and only if

$$\lambda_g = \lim_{x \uparrow 1} \liminf_{t \downarrow 0} \frac{g(xt)}{g(t)} > 0. \tag{1.3}$$

Wen and Wen [16] proved that for every doubling gauge g there is a compact metric space X such that $0 < \mathcal{P}^g(X) < +\infty$. This result is in some sense dual to a theorem of Devoretzky for the Hausdorff measure (cf. [2, 13]). In this paper we shall give a constructive proof for the following:

Theorem 1. *Let g be a doubling gauge. Then there is a positive integer n and a Cantor set E in the Euclidean n-space such that*

$$0 < \mathcal{H}^g(E) \le \mathcal{P}^g(E) \le \mathcal{P}_0^g(E) < \infty. \tag{1.4}$$

Self-similar sets with the strong separation condition are a subclass of Cantor sets. One may ask that if the Cantor set in Theorem 1 can be always chosen from self-similar sets. The answer to the question is "no". In fact, by [17], a self-similar set E of dimension s with the open set condition has the property (1.4) for some doubling gauge g if and only if

$$0 < \liminf_{t\to 0} \frac{g(t)}{t^s} \le \limsup_{t\to 0} \frac{g(t)}{t^s} < \infty.$$

It follows that, for $g(t) = t^s \log \frac{1}{t}$, all self-similar sets with the open set condition do not satisfy (1.4). On the other hand, we note that there are Cantor sets, for which there is no gauge satisfying (1.4) (cf. Peres [10, 11]).

Feng-Hua-Wen [5] proved that $\mathcal{P}_0^s(K) = \mathcal{P}^s(K)$, if $K \subset \mathbb{R}^n$ is a compact set of finite $\mathcal{P}_0^s$-premeasure, where $s > 0$. This result may be extended. Indeed, we have $\mathcal{P}^g(K) \asymp \mathcal{P}_0^g(K)$ for all compact subsets K of X, provided that g is a doubling gauge and that X is a metric space of locally finite $\mathcal{P}_0^g$-premeasure, where the comparability constant is independent of K (cf. [16]). Recall that a metric space X is of locally finite $\mathcal{P}_0^g$-premeasure, if for every $x \in X$ there is a $r > 0$ such that the ball $B(x, r)$ is of finite $\mathcal{P}_0^g$-premeasure. We shall prove that Feng-Hua-Wen's Theorem in the extended form can be inverted.

Theorem 2. *Let X be a metric space and g be a doubling gauge. Then the following statements are equivalent.*

(1) *X is of locally finite $\mathcal{P}_0^g$-premeasure.*

(2) *$\mathcal{P}^g(K) \asymp \mathcal{P}_0^g(K)$ for all compact subsets K of X.*

M. Csörnyei [1] constructed a Cantor set E with the following property: There is a countable set F, with $\overline{F} = E \cup F$, and a doubling gauge g, such that

$$\mathcal{P}^g(E \cup F) < \mathcal{P}_0^g(E \cup F) < \infty.$$

This shows that Feng-Hua-Wen's inequality can not be extended to doubling gauges. We shall prove that a class of regular Cantor sets are examples of sets with this property.

Now we define uniform Cantor sets. Let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers and $\{c_k\}_{k=1}^\infty$ a sequence of real numbers in $(0, 1)$ with $n_k c_k < 1$ for all $k \ge 1$. The uniform Cantor set $E(\{n_k\}, \{c_k\})$ is defined by

$$E(\{n_k\}, \{c_k\}) = \bigcap_{k=0}^{\infty} E_k, \tag{1.5}$$

where $\{E_k\}$ is a sequence of nested compact sets in $[0, 1]$, $E_0 = [0, 1]$, and E_k is obtained by deleting n_k open intervals of equal length $c_k|I|$ from every component I of E_{k-1}, such that the remaining $n_k + 1$ closed intervals of I are of equal length. In the case where $n_k = 1$ for all $k \geq 1$, we also say that E is a middle internal Cantor set.

For a uniform Cantor set $E = E(\{n_k\}, \{c_k\})$ we denote by $\mathcal{G}_k$ the collection of components of $E_{k-1} \setminus E_k$. A member in $\mathcal{G}_k$ is also called a gap of level k. Denote by N_k, δ_k, and ε_k, respectively, the number of components of E_k, the length of a component of E_k, and the length of a gap of level k. Then

$$N_k = \prod_{i=1}^{k}(n_i + 1), \quad \delta_k = \prod_{i=1}^{k} \frac{1 - n_i c_i}{n_i + 1}, \quad \text{and } \varepsilon_k = c_k \delta_{k-1}. \tag{1.6}$$

With the above notation, the cardinality $\mathrm{card}(\mathcal{G}_k)$ of $\mathcal{G}_k$ is $n_k N_{k-1}$. Feng-Wen-Wu [7] showed that the packing dimension of E can be determined by

$$\dim_P E = \limsup_{k \to \infty} \frac{\log N_k}{-\log \frac{\delta_{k-1}}{n_k + 1}}. \tag{1.7}$$

Theorem 3. *Let $E = E(\{n_k\}, \{c_k\})$ be a uniform Cantor set. Suppose that there exists a strictly increasing sequence of positive integers $\{j_k\}_{k=1}^{\infty}$ such that*

(a) $N := 1 + \max\{\sup_k n_{j_k}, \sup_k n_{j_k+1}\} < \infty$;
(b) $\sum_{k=1}^{\infty} N_{j_k}/N_{j_{k+1}} < \infty$;
(c) $1 - n_{j_k} c_{j_k} \leq N_{j_k}/N_{j_{k+1}}$ *for every* $k \geq 1$; *and*
(d) $1 - n_{j_k - 1} c_{j_k - 1} \leq 1/5$ *and* $1 - n_{j_k+1} c_{j_k+1} \leq 1/5$ *for every* $k \geq 1$.

Then there is a countable set $F \subset [0,1]$ with $\overline{F} = E \cup F$ and a doubling gauge g such that

$$0 < \mathcal{P}^g(E \cup F) < \mathcal{P}_0^g(E \cup F) < \infty.$$

As we shall see in Section 5, for every $s \in [0, 1/2]$, there exists a uniform Cantor set E of $\dim_P E = s$, which satisfies the conditions of Theorem 3.

Theorem 1, Theorem 2, and Theorem 3 will be proved in Sections 2, 3, and 4, respectively. The discussion of Theorem 3 is clearly not complete. Some remarks are included in Section 5.

2. Proof of Theorem 1

A cube $Q(x,r)$ in $\mathbb{R}^n$ is a subset of the form $Q(x,r) = \prod_{i=1}^{n}[x_i - r, x_i + r]$. For a cube Q we denote by $\ell(Q)$ its side length. Let g be a doubling gauge and E a subset of $\mathbb{R}^n$. Define

$$\widetilde{\mathcal{H}^g}(E) = \lim_{\delta \to 0} \widetilde{\mathcal{H}^g_\delta}(E),$$

where $\widetilde{\mathcal{H}^g_\delta}(E) = \inf \sum g(\ell(Q_i))$ with the infimum being taken over all coverings of E by cubes of side length $\leq \delta$. Define

$$\widetilde{\mathcal{P}^g_0}(E) = \lim_{\delta \to 0} \widetilde{\mathcal{P}^g_\delta}(E),$$

where $\widetilde{\mathcal{P}^g_\delta}(E) = \sup \sum g(\ell(Q_i))$ with the supremum being over all packings of E by cubes of side length $\leq \delta$ centered in E. Then we easily see that

$$C^{-1}\widetilde{\mathcal{H}^g}(E) \leq \mathcal{H}^g(E) \leq C\widetilde{\mathcal{H}^g}(E) \quad \text{and} \quad C^{-1}\widetilde{\mathcal{P}^g_0}(E) \leq \mathcal{P}^g_0(E) \leq C\widetilde{\mathcal{P}^g_0}(E), \tag{2.1}$$

where the comparability constant C depends only on g and n.

Now we are ready to prove Theorem 1. Since the gauge g is doubling, there exists a positive integer n such that $g(2t) < 2^n g(t)$ for sufficiently small $t > 0$. We are going to construct a Cantor set E in the Euclidean space $\mathbb{R}^n$ such that

$$0 < \mathcal{H}^g(E) \leq \mathcal{P}^g(E) \leq \mathcal{P}^g_0(E) < \infty. \tag{2.2}$$

For every integer $k \geq 0$ let δ_k be a positive number satisfying $g(\delta_k) = 2^{-nk}$. Without loss of generality, assume that $g(2t) < 2^n g(t)$ for all $t \in (0, \delta_0]$. Then for every $k \geq 1$ we have $2\delta_k < \delta_{k-1}$, because g is nondecreasing with $g(2\delta_k) < 2^n g(\delta_k) = g(\delta_{k-1})$. Let E be a Cantor dust-like set in the cube $[0, \delta_0]^n$ defined as

$$E = \bigcap_{k=0}^{\infty} E_k,$$

where $E_0 = [0, \delta_0]^n$ and E_k is obtained by replacing every component Q of E_{k-1} with its 2^n disjoint subcubes of side length δ_k. By the definition, E_k consists of 2^{nk} disjoint cubes of side length δ_k.

Let μ be the unique Borel probability measure on E such that for every component I of E_k

$$\mu(I) = 2^{-nk}.$$

To prove (2.2), in view of (2.1), it suffices to show $\widetilde{\mathcal{H}^g}(E) > 0$ and $\widetilde{\mathcal{P}^g_0}(E) < \infty$.

For every cube J with side length $\ell(J) < \delta_0$, let $k \geq 0$ be the integer such that $\delta_{k+1} \leq \ell(J) < \delta_k$, then J intersects at most 2^n components of E_k, and so

$$\mu(J) \leq 2^n 2^{-nk} = 4^n g(\delta_{k+1}) \leq 4^n g(\ell(J)),$$

which, combined with the mass distribution principle, yields $\widetilde{\mathcal{H}^g}(E) \geq 4^{-n}$.

For every cube $Q(x, r)$ with $r < \delta_0$ and $x \in E$, let k be the integer such that $\delta_{k+1} \leq r < \delta_k$, then $Q(x, r)$ contains at least one component of E_{k+1}, and so

$$\mu(Q(x, r)) \geq 2^{-n(k+1)} = 2^{-n} g(\delta_k) \geq 2^{-n} g(r) \geq 4^{-n} g(\ell(Q)),$$

which, together with the definition of $\mathcal{P}^g_\delta(E)$, yields $\mathcal{P}^g_\delta(E) \leq 4^n$. Letting $\delta \to 0$ we obtain $\widetilde{\mathcal{P}^g_0}(E) \leq 4^n$. This completes the proof. □

Corollary 1. *For every $s > 0$ there is a Cantor set E of $\dim_P E = s$ and of $\mathcal{P}^s(E) = 0$.*

Proof. Let g be a gauge with

$$g(t) = t^s \log \tfrac{1}{t}$$

for $0 < t < e^{-1/s}$ and let $n > s$ be an integer. Then we have $g(2t) < 2^n g(t)$ for $0 < t < e^{-1/s}/2$. By Theorem 1 we may choose a Cantor set E such that $0 < \mathcal{P}^g(E) < \infty$. Clearly, the set E is of $\dim_P E = s$ and of $\mathcal{P}^s(E) = 0$. □

Corollary 2. *For every $s > 0$ there is a Cantor set E of $\dim_P E = s$ and of non-σ-finite $\mathcal{P}^s$-measure.*

Proof. Let g be a gauge with

$$g(t) = t^s / \log \frac{1}{t}$$

for $0 < t < e^{1/s}$ and let $n > s$ be an integer. Then we have $g(2t) < 2^n g(t)$ for sufficiently small $t > 0$. By Theorem 1 we may choose a Cantor set E such that $0 < \mathcal{P}^g(E) < \infty$. It is clear that the set E is of $\dim_P E = s$ and of non-σ-finite $\mathcal{P}^s$-measure. □

3. Proof of Theorem 2

The implication "(1) ⇒ (2)" is the extended Feng, Hua, Wen's Theorem. A proof can be found in [16]. Now we prove the implication "(2) ⇒ (1)". Assume that X is not of locally finite $\mathcal{P}_0^g$-premeasure. Then there is a point $x \in X$ such that

$$\mathcal{P}_0^g(B(x, \varepsilon)) = \infty \quad \text{for all } \varepsilon > 0. \tag{3.1}$$

We are going to construct a compact countable subset C of X with $\mathcal{P}_0^g(C) = \infty$.

Let $\varepsilon_1 > 0$ be arbitrarily given. From (3.1) we have $\mathcal{P}_{\varepsilon_1}^g(B(x, \varepsilon_1)) = \infty$, so there is a finite ε_1-packing of $B(x, \varepsilon_1)$ by closed balls $\{B(x_i, r_i)\}_{i=1}^m$ such that

$$\sum_{i=1}^m g(2r_i) > g(2\varepsilon_1) + 1.$$

It is clear that, in the above ε_1-packing, there is at most one ball containing x. Let $\mathcal{B}_1 = \{B(x_i, r_i) : x \notin B(x_i, r_i),\ 1 \le i \le m\}$. Then $\mathcal{B}_1$ is a finite ε_1-packing of $B(x, \varepsilon_1)$ with

$$x \notin \bigcup_{B \in \mathcal{B}_1} B \ \text{ and } \ \sum_{B \in \mathcal{B}_1} g(2r_B) > 1,$$

where, for a ball $B(x, r)$, we write x_B for x and r_B for r.

Let $\varepsilon_2 = \min\{\varepsilon_1, \text{dist}(x, \cup_{B \in \mathcal{B}_1} B)\}/3 > 0$. Then $\mathcal{P}_{\varepsilon_2}^g(B(x, \varepsilon_2)) = \infty$. By the same argument as above, we have a finite ε_2-packing $\mathcal{B}_2$ of $B(x, \varepsilon_2)$ such that

$$x \notin \bigcup_{B \in \mathcal{B}_2} B \ \text{ and } \ \sum_{B \in \mathcal{B}_2} g(2r_B) > 1.$$

Note that $\mathcal{B}_1 \cup \mathcal{B}_2$ is still a family of disjoint closed balls.

Proceeding infinitely, we get a sequence of positive numbers $\{\varepsilon_i\}$ and a sequence of ball families $\{\mathcal{B}_i\}$ such that $\{\varepsilon_i\}$ decreases strictly to 0, each $\mathcal{B}_i$ is a finite ε_i-packing of $B(x, \varepsilon_i)$ with

$$x \notin \bigcup_{B \in \mathcal{B}_i} B \ \text{ and } \ \sum_{B \in \mathcal{B}_i} g(2r_B) > 1,$$

and that $\cup_{i=1}^\infty \mathcal{B}_i$ is a family of disjoint closed balls.

Let

$$C = \{x\} \cup \{x_B : B \in \cup_{i=1}^{\infty} \mathcal{B}_i\}.$$

Then C has just one accumulation point x, so C is a compact countable set in X. Next we show that C has infinite packing premeasure. In fact, for each $\delta > 0$ choose a sufficiently large integer n such that $2\varepsilon_i \le \delta$ for all $i \ge n$, then $\cup_{i=n}^{\infty} \mathcal{B}_i$ is a δ-packing of C, so

$$\mathcal{P}_\delta^g(C) \ge \sum_{i=n}^{\infty} \sum_{B \in \mathcal{B}_i} g(2r_B) \ge \sum_{i=n}^{\infty} 1 = \infty,$$

which yields $\mathcal{P}_0^g(C) = \infty$ by letting $\delta \to 0$.

Now, since C is countable, one has $\mathcal{P}^g(C) = 0$. This shows that there is a compact set C with $\mathcal{P}^g(C) = 0$ and $\mathcal{P}_0^g(C) = \infty$, contradicting the statement (2). This completes the proof. □

According to Joyce and Preiss [8], every compact metric space of infinite $\mathcal{P}^g$-measure has a compact subset of finite positive $\mathcal{P}^g$-measure. By contrast, from the proof of Theorem 2 we have the following:

Corollary 3. *Let X be a compact metric space and g be a doubling gauge. If $\mathcal{P}_0^g(X) = \infty$, then there is a compact countable subset C of X with $\mathcal{P}_0^g(C) = \infty$.*

4. Proof of Theorem 3

Let $E = E(\{n_k\}, \{c_k\})$ be a uniform Cantor set and let $N_k, \delta_k, \varepsilon_k$, and $\mathcal{G}_k$ be the related data defined as in (1.6). Let μ be the unique Borel probability measure on E such that $\mu(I) = 1/N_k$ for every component I of E_k and for every $k \ge 1$. Let $\{j_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers, for which E satisfies the conditions of Theorem 3. We are going to prove that there is a countable set F, with $\overline{F} = E \cup F$, and a doubling gauge g, such that

$$0 < \mathcal{P}^g(E \cup F) < \mathcal{P}_0^g(E \cup F) < \infty.$$

From the conditions (a), (b), and (c), we see that $\{n_{j_k+1}\}_{k=1}^{\infty}$ is bounded, $\lim_{k\to\infty} N_{j_k}/N_{j_{k+1}} = 0$, and $1 - n_{j_k} c_{j_k} \le N_{j_k}/N_{j_{k+1}}$, so there is an integer $k_0 \ge 1$ such that $j_{k+1} - j_k > 1$ and $\varepsilon_{j_k} > 4\delta_{j_k}$ for every $k \ge k_0$. Note also that the condition (d) implies $\varepsilon_{j_k-1} > 4\delta_{j_k-1}$ and $\varepsilon_{j_k+1} > 4\delta_{j_k+1}$ for every $k \ge 1$. Here and in what follows please do not confuse the indexes $j_k + 1$ and j_{k+1}.

Constructing a gauge g. For every $k \ge k_0$ let

$$h_{j_k} = \varepsilon_{j_k+1} + \delta_{j_k+1}, \quad f_{j_k} = \varepsilon_{j_k} + \delta_{j_k} + \delta_{j_k+1}, \quad \text{and } e_{j_k} = \varepsilon_{j_k} + \delta_{j_k} + 2\delta_{j_k+1}.$$

Clearly, $e_{j_k} > f_{j_k} > h_{j_k}$. Since $j_{k+1} > j_k + 1$, one has

$$h_{j_k} > \delta_{j_k+1} \ge \varepsilon_{j_{k+1}} + 2\delta_{j_{k+1}} > e_{j_{k+1}}.$$

We define a gauge g on $[0, 2e_{j_{k_0}}]$ as follows: For every $k \ge k_0$ let

$$g(2f_{j_k}) = \frac{1}{(N+1)N_{j_k}} \text{ and } g(x) = \frac{1}{N_{j_{k+1}}} \text{ for } 2e_{j_{k+1}} \le x \le 2h_{j_k},$$

and then we extend g to the intervals $[2h_{j_k}, 2f_{j_k}]$ and $[2f_{j_k}, 2e_{j_k}]$ linearly.

Constructing a countable set F. For each $k > k_0$ we define a finite set F_k by taking one point from every gap of level $j_k - 1$ as in the following

$$F_k = \{e^-(J) + 4\delta_{j_k+1} : J \in \mathcal{G}_{j_k-1}\},$$

where $e^-(J)$ denotes the left endpoint of the gap J. Let

$$F = \bigcup_{k>k_0} F_k.$$

Clearly, $\operatorname{card}(F_k) = \operatorname{card}(\mathcal{G}_{j_k-1}) = n_{j_k-1}N_{j_k-2}$, F is countable, and $\overline{F} = E \cup F$.

Claim 1. $\mathcal{P}_0^g(E) \le \frac{N}{N+1}$.

Proof. Put $l_0 = \min_{1\le j\le j_{k_0+1}} \min\{\varepsilon_j, \delta_j\}$. It suffices to prove

$$g(|P|) \le \frac{N}{N+1}\mu(P)$$

for every interval P of length $|P| \in (0, l_0)$ with its midpoint $x(P) \in E$.

Let P be such an interval. Let $k > k_0$ be the unique integer such that P meets exactly one component of $E_{j_{k-1}+1}$ and at least two of E_{j_k+1}. Then

$$2\varepsilon_{j_k+1} \le |P| \le 2h_{j_{k-1}}.$$

Let I be the component of E_{j_k+1} containing the midpoint of P. Since $|P| \ge 2\varepsilon_{j_k+1} \ge 8\delta_{j_k+1}$, one has $P \supseteq I$, so $\mu(P) \ge 1/N_{j_k+1}$. Therefore, for the case $2\varepsilon_{j_k+1} \le |P| \le 2f_{j_k}$ we have from the definition of g that

$$g(|P|) \le g(2f_{j_k}) = \frac{1}{(N+1)N_{j_k}} \le \frac{N}{N+1}\mu(P).$$

For another case $2f_{j_k} < |P| \le 2h_{j_{k-1}}$, we see from the choice of f_k that P contains at least $n_{j_k+1}+2$ components of E_{j_k+1}, so $\mu(P) \ge (n_{j_k+1}+2)/N_{j_k+1}$, and so

$$g(|P|) \le g(2h_{j_{k-1}}) = \frac{1}{N_{j_k}} = \frac{n_{j_k+1}+1}{N_{j_k+1}} \le \frac{N}{N+1}\mu(P).$$

This completes the proof of Claim 1. □

Claim 2. $\mathcal{P}_0^g(F) \ge 1$.

Proof. It suffices to show that for every $k > k_0$ there is a $2e_{j_k}$-packing $\mathcal{P}$ of F such that

$$\sum_{P\in\mathcal{P}} g(|P|) \ge \sum_{P\in\mathcal{P}} \mu(P) = 1.$$

To this end, we shall construct a $2e_{j_k}$-packing $\mathcal{P}$ of F such that $g(|P|) = \mu(P)$ for every $P \in \mathcal{P}$ and that $\mu(\cup_{P\in\mathcal{P}} P) = 1$.

Let $k > k_0$ be fixed. Let

$$\mathcal{P}_k = \{[x - e_{j_k}, x + e_{j_k}] : x \in F_k\}.$$

Then, by the definitions of e_{j_k} and F_k, intervals in $\mathcal{P}_k$ are of the form

$$[e^-(J) - \varepsilon_{j_k} - \delta_{j_k} + 2\delta_{j_k+1}, e^-(J) + \varepsilon_{j_k} + \delta_{j_k} + 6\delta_{j_k+1}],$$

where, as mentioned, $J \in \mathcal{G}_{j_k-1}$ and $e^-(J)$ is the left endpoint of J. Since $\varepsilon_{j_k-1} \geq 4\delta_{j_k-1} > \varepsilon_{j_k} + \delta_{j_k} + 6\delta_{j_k+1}$ and $0 < \varepsilon_{j_k} - 2\delta_{j_k+1} < \varepsilon_{j_k}$, we see that intervals in $\mathcal{P}_k$ are pairwise disjoint and that every interval in $\mathcal{P}_k$ contains exactly one component of E_{j_k} and meets no others. Therefore,

$$g(|P|) = g(2e_{j_k}) = \frac{1}{N_{j_k}} = \mu(P)$$

for every $P \in \mathcal{P}_k$ and

(4.1)

$$\mu\left(\bigcup_{P\in\mathcal{P}_k} P\right) = \frac{\operatorname{card}(F_k)}{N_{j_k}} = \frac{n_{j_k-1}N_{j_k-2}}{N_{j_k}} = \frac{n_{j_k-1}}{(n_{j_k}+1)(n_{j_k-1}+1)} \geq \frac{1}{2N}.$$

Now suppose that $\mathcal{P}_k, \mathcal{P}_{k+1}, \ldots, \mathcal{P}_{n-1}$ have been defined for some $n > k$. We define

$$\mathcal{P}_n = \left\{[x - e_{j_n}, x + e_{j_n}] : x \in F_n \setminus \bigcup_{i=k}^{n-1} \bigcup_{P\in\mathcal{P}_i} P\right\}.$$

Inductively, we may define a sequence of interval families $\{\mathcal{P}_n\}_{n=k}^{\infty}$.

Let $\mathcal{P} = \cup_{n=k}^{\infty}\mathcal{P}_n$. It is clear that the family $\mathcal{P}$ is pairwise disjoint. Given $n \geq k$, noting that every interval in $\mathcal{P}_n$ contains exactly one component of E_{j_n} and meets no others, we have $g(|P|) = 1/N_{j_n} = \mu(P)$ for every $P \in \mathcal{P}_n$. Next we show that

$$\mu\left(\bigcup_{P\in\mathcal{P}_n} P\right) \geq \frac{1}{2N}\mu\left(E_{j_k-1} \setminus \bigcup_{i=k}^{n-1} \bigcup_{P\in\mathcal{P}_i} P\right). \tag{4.2}$$

Let $E^*_{j_{n-1}} = \{I :\ I$ is a component of $E_{j_{n-1}}$ and $I \subseteq E_{j_k-1} \setminus \cup_{i=k}^{n-1} \cup_{P\in\mathcal{P}_i} P\}$. We easily see that

$$\mu\left(E_{j_k-1} \setminus \bigcup_{i=k}^{n-1} \bigcup_{P\in\mathcal{P}_i} P\right) = \mu\left(\bigcup_{I\in E^*_{j_{n-1}}} I\right).$$

Let I be an interval in $E^*_{j_{n-1}}$ and let $\mathcal{P}_{n,I} = \{[x - e_{j_n}, x + e_{j_n}] : x \in F_n \cap I\}$. Then we have

$$\mathcal{P}_n = \bigcup_{I\in E^*_{j_{n-1}}} \mathcal{P}_{n,I}.$$

Therefore, to prove (4.2), it reduces to show that

$$\mu\left(\bigcup_{P\in\mathcal{P}_{n,I}} P\right) \geq \frac{\mu(I)}{2N}$$

for every $I \in E^*_{j_{n-1}}$. This is true, in fact, the cardinality

$$\operatorname{card}(\mathcal{P}_{n,I}) = n_{j_n-1} \prod_{i=j_{n-1}+1}^{j_n-2} (n_i+1) = \frac{n_{j_n-1} N_{j_n-2}}{N_{j_{n-1}}}$$

and the measure $\mu(I) = 1/N_{j_{n-1}}$, which imply that

$$\mu\left(\bigcup_{P\in\mathcal{P}_{n,I}} P\right) = \frac{n_{j_n-1} N_{j_n-2}}{N_{j_{n-1}}} \cdot \frac{1}{N_{j_n}} = \frac{n_{j_n-1}}{(n_{j_n-1}+1)(n_{j_n}+1) N_{j_{n-1}}} \geq \frac{\mu(I)}{2N}.$$

This proves the inequality (4.2), and by which we have $\mu(\cup_{P\in\mathcal{P}} P) = \mu(E_{j_k-1})$ $= 1$. Therefore, $\mathcal{P}$ is a $2e_{j_k}$-packing of F with the desired properties. This completes the proof of Claim 2. □

Claim 3. $\mathcal{P}^g_0(F) < \infty$.

Proof. For every l_0-packing $\mathcal{P}$ of F let $\mathcal{P}_k = \{P \in \mathcal{P} : m(P) \in F_k\}$, where $m(P)$ denotes the midpoint of the interval P and l_0 is the same as that in the proof of Claim 1. We are going to show that

$$\sum_{k>k_0} \sum_{P\in\mathcal{P}_k} g(|P|)$$

is bounded by a constant independent of $\mathcal{P}$. Let $P \in \mathcal{P}_k$. Consider three cases.

Case 1. P does not contain any component of E_{j_k+1}. In this case, we have $|P| \leq 10\delta_{j_k+1} \leq 2\varepsilon_{j_k+1} + 2\delta_{j_k+1} = 2h_{j_k}$, so

$$g(|P|) \leq g(2h_{j_k}) = \frac{1}{N_{j_{k+1}}}.$$

Case 2. P contains a component of E_{j_k+1} and meets only one component of $E_{j_{k-1}+1}$. In this case, we have $\mu(P) \geq 1/N_{j_k+1}$ and $|P| < 2h_{j_{k-1}}$, so

$$g(|P|) \leq g(2h_{j_{k-1}}) = \frac{1}{N_{j_k}} = \frac{n_{j_k+1}+1}{N_{j_k+1}} \leq N\mu(P).$$

Case 3. P meets at least two components of $E_{j_{k-1}+1}$. In this case, there is an integer $n \leq k-1$ such that P meets only one component of $E_{j_{n-1}+1}$ and at least two of E_{j_n+1}. Arguing as we did in Claim 1, we have

$$g(|P|) \leq \frac{N}{N+1}\mu(P).$$

Together the above three cases, we have

$$\begin{aligned}\sum_{k>k_0} \sum_{P\in\mathcal{P}_k} g(|P|) &\leq \sum_{k>k_0} \sum_{P\in\mathcal{P}_k} \left(\frac{1}{N_{j_{k+1}}} + N\mu(P) + \frac{N}{N+1}\mu(P)\right) \\ &\leq N + \frac{N}{N+1} + \sum_{k>k_0} \frac{\operatorname{card}(\mathcal{P}_k)}{N_{j_{k+1}}}.\end{aligned}$$

Now, observing that $\operatorname{card}(\mathcal{P}_k) \le N_{j_k-2}n_{j_k-1}$, we have from the condition (b) that

$$\sum_{k>k_0} \frac{\operatorname{card}(\mathcal{P}_k)}{N_{j_{k+1}}} \le \sum_{k>k_0} \frac{N_{j_k-2}n_{j_k-1}}{N_{j_{k+1}}} < \frac{1}{2}\sum_{k=1}^{\infty} \frac{N_{j_k}}{N_{j_{k+1}}} < \infty,$$

and so $\sum_{k>k_0}\sum_{P\in\mathcal{P}_k} g(|P|)$ is bounded. This proves Claim 3. □

Claim 4. $g(2x) \le 2(N+1)g(x)$ for all $x \in (0, h_{j_{k_0}}]$.

Proof. Given $k \ge k_0$, since $g(2x) = 1/N_{j_k+1}$ for all $x \in [e_{j_{k+1}}, h_{j_k}]$, we have

$$\frac{g(2e_{j_{k+1}})}{e_{j_{k+1}}} > \frac{g(2h_{j_k})}{h_{j_k}}. \tag{4.3}$$

On the other hand, since

$$h_{j_k} = \varepsilon_{j_k+1} + \delta_{j_k+1} \le \delta_{j_k} \text{ and } f_{j_k} = \varepsilon_{j_k} + \delta_{j_k} + \delta_{j_k+1} \ge \frac{\delta_{j_k-1}}{N},$$

we have from the condition (c) that

$$\frac{g(2h_{j_k})}{h_{j_k}} \ge \frac{(N+1)N_{j_k}\delta_{j_k-1}}{N_{j_{k+1}}\delta_{j_k}N}\frac{g(2f_{j_k})}{f_{j_k}} > \frac{g(2f_{j_k})}{f_{j_k}}. \tag{4.4}$$

It follows from (4.3), (4.4), and the definition of g that the function $g(2x)/x$ is decreasing on $[e_{j_{k+1}}, f_{j_k}]$.

Now we are ready to prove $g(2x) \le 2(N+1)g(x)$ for all $x \in (0, h_{j_{k_0}}]$. Given $x \in (0, h_{j_{k_0}}]$, let $k > k_0$ be the integer such that $x \in (h_{j_k}, h_{j_{k-1}}]$. Since $h_{j_k} < f_{j_k} < e_{j_k} < 2f_{j_k} < 2e_{j_k} < h_{j_{k-1}}$ by the choices of these quantities, one may write

$$(h_{j_k}, h_{j_{k-1}}] = (h_{j_k}, f_{j_k}] \cup (f_{j_k}, e_{j_k}] \cup (e_{j_k}, 2f_{j_k}] \cup (2f_{j_k}, 2e_{j_k}] \cup (2e_{j_k}, h_{j_{k-1}}].$$

Case 1. $x \in (h_{j_k}, e_{j_k}] = (h_{j_k}, f_{j_k}] \cup (f_{j_k}, e_{j_k}]$. Since $2e_{j_{k+1}} < h_{j_k} < e_{j_k} < 2f_{j_k}$, we have $x/2 \in [e_{j_{k+1}}, f_{j_k}]$. When $x \in (h_{j_k}, f_{j_k}]$, it follows from the monotonicity of $g(2x)/x$ that

$$\frac{g(2x)}{g(x)} = 2\frac{g(2x)/x}{g(x)/(x/2)} < 2.$$

When $x \in [f_{j_k}, e_{j_k}]$, we have

$$\frac{g(2x)}{g(x)} \le \frac{g(2e_{j_k})}{g(f_{j_k})} = \frac{(N+1)g(2f_{j_k})}{g(f_{j_k})} < 2(N+1).$$

Case 2. $x \in [e_{j_k}, 2f_{j_k}]$. We have

$$\frac{g(2x)}{g(x)} \le \frac{g(4e_{j_k})}{g(f_{j_k})} \le \frac{g(2f_{j_k})}{g(f_{j_k})} \cdot \frac{g(2h_{j_{k-1}})}{g(2f_{j_k})} \le 2(N+1).$$

Case 3. $x \in [2f_{j_k}, 2e_{j_k}]$. We have

$$\frac{g(2x)}{g(x)} \le \frac{g(4e_{j_k})}{g(2f_{j_k})} \le \frac{g(2h_{j_{k-1}})}{g(2f_{j_k})} \le N+1.$$

Case 4. $x \in [2e_{j_k}, h_{j_{k-1}}]$. We have

$$\frac{g(2x)}{g(x)} \le \frac{g(2h_{j_{k-1}})}{g(2e_{j_k})} = 1.$$

This completes the proof of Claim 4, so g is doubling. □

Claim 5. $\mathcal{P}_0^g(E) \ge \frac{1}{2(N+1)^2}$.

Proof. Let $k > k_0$ be fixed. Let

$$\mathcal{P}_k = \{[e^-(J) - \varepsilon_{j_k} - \delta_{j_k}, e^-(J) + \varepsilon_{j_k} + \delta_{j_k}] : J \in \mathcal{G}_{j_k - 1}\}.$$

Then, since $\varepsilon_{j_k - 1} > \varepsilon_{j_k} + \delta_{j_k}$, intervals in $\mathcal{P}_k$ are pairwise disjoint. Also, one has

$$\mu\left(\bigcup_{P \in \mathcal{P}_k} P\right) = \frac{n_{j_k - 1} N_{j_k - 2}}{N_{j_k}} \ge \frac{1}{2N}. \tag{4.5}$$

On the other hand, we have from Claim 4 that

$$g(|P|) = g(2(\varepsilon_{j_k} + \delta_{j_k})) \ge g(f_{j_k}) \ge \frac{g(2f_{j_k})}{2(N+1)} \ge \frac{\mu(P)}{2(N+1)^2} \tag{4.6}$$

for every $P \in \mathcal{P}_k$.

By an analogous argument as we did in Claim 2, we may inductively define a sequence of interval families $\{\mathcal{P}_n\}_{n=k}^\infty$, such that every $\mathcal{P}_n$ has the properties (4.5) and (4.6). Let $\mathcal{P} = \cup_{n=k}^\infty \mathcal{P}_n$. Then it can be checked that $\mathcal{P}$ is a $(\varepsilon_{j_k} + \delta_{j_k})$-packing of E with $\mu(\cup_{P \in \mathcal{P}} P) = 1$ and $g(|P|) \ge \mu(P)/(2(N+1)^2)$ for every $P \in \mathcal{P}$. The desired inequality then follows. □

Finally, Claims 1-5, combined with Theorem 2 and Equality (1.1), gives the conclusion of Theorem 3.

5. Remarks on Theorem 3

Remark 1. At first, we prove that, for every $s \in [0, 1/2]$, there is a middle interval Cantor set E of $\dim_P E = s$ satisfying the conditions of Theorem 3.

Let $\{j_k\}_{k=1}^\infty$ be a sequence of positive integers defined by

$$j_1 = 2 \quad \text{and} \quad j_{k+1} = j_k + k + 2 \quad \text{for every } k \ge 1. \tag{5.1}$$

Clearly, $j_k = k(k+3)/2$ by (5.1), so one has $\lim_{k\to\infty} j_k/(k^2/2) = 1$. Let $E = E(\{c_k\})$ be a middle interval Cantor set with

$$1 - c_{j_k} = 2^{-k-2} \quad \text{and} \quad 1 - c_{j_k - 1} = 1 - c_{j_k + 1} = 1/8$$

for every $k \ge 1$. It can be checked that E satisfies the conditions of Theorem 3, no matter how $c_j \in (0,1)$ is chosen for $j \in \mathbb{N}_1$, where

$$\mathbb{N}_1 = \mathbb{N} \setminus \cup_{k \ge 1}\{j_k - 1, j_k, j_k + 1\}.$$

Now, given $s \in [0, 1/2]$, we are going to choose c_j for $j \in \mathbb{N}_1$ such that $\dim_P E = s$. We consider three cases.

Case 1. $0 < s < 1/2$. Let $1 - c_j = 2^{-\alpha}$ for every $j \in \mathbb{N}_1$, where $\alpha = 1/s - 2$. For every $k > j_1$ let $l = l(k)$ be the biggest integer such that $j_l < k$. Then $\lim_{k\to\infty} k/(l^2/2) = 1$. By the construction of E, the number of components of E_k is 2^k and the length of a component of E_k is

$$\delta_k = \prod_{i=1}^{k} \frac{1-c_i}{2} = 2^{-k} \cdot 2^{-(3+4+\cdots+(l+2))} \cdot 8^{-2l} \cdot 2^{-\alpha(k-3l)},$$

so

$$\log N_k = k \log 2 \quad \text{and} \quad \lim_{k\to\infty} (-\log \delta_k)/((2+\alpha)k \log 2) = 1.$$

It then follows from the formula (1.7) that

$$\dim_P E = \frac{1}{2+\alpha} = s.$$

Case 2. $s = 0$. Let $1 - c_j = 2^{-j}$ for every $j \in \mathbb{N}_1$. We easily see that $\dim_P E = 0$.

Case 3. $s = 1/2$. Let $1 - c_j = 2^{-1/j}$ for every $j \in \mathbb{N}_1$. We see that $\dim_P E = 1/2$.

Remark 2. Secondly, we show that every uniform Cantor set E satisfying the condition (c) of Theorem 3 is of $\dim_P E \le 1/2$.

Let $E = E(\{n_k\}, \{c_k\})$ be a uniform Cantor set. Suppose it satisfies the condition (c) of Theorem 3. By the packing dimension formula (1.7), we have

$$\dim_P E = \limsup_{k\to\infty} \frac{\log N_k}{-\log \frac{\delta_{k-1}}{n_k+1}} = \limsup \frac{\log N_k}{\log N_k - \log N_{k-1}\delta_{k-1}},$$

from which we see that, to prove $\dim_P E \le 1/2$, it suffices to show

$$\liminf \frac{-\log N_{k-1}\delta_{k-1}}{\log N_k} \ge 1. \tag{5.2}$$

Let $k > j_1$ be fixed. Let l_k be the biggest integer such that $j_{l_k} < k$ and let $\Gamma_k = \{j_1, j_2, \ldots, j_{l_k}\}$. Then it follows from the condition (c) of Theorem 3 that

$$N_{k-1}\delta_{k-1} = \prod_{i=1}^{k-1} (1 - n_i c_i) \le \prod_{i\in\Gamma_k} (1 - n_i c_i) \le \frac{N_{j_1}}{N_{j_{l_k+1}}},$$

which yields

$$\liminf \frac{-\log N_{k-1}\delta_{k-1}}{\log N_k} \ge \liminf \frac{\log N_{j_{l_k+1}} - \log N_{j_1}}{\log N_k} \ge 1.$$

This proves the inequality (5.2), so one has $\dim_P E \le 1/2$.

Problem 1. Is there a uniform Cantor set of $\dim_P E > 1/2$ such that the conclusion of Theorem 3 holds? The question is still open.

Remark 3. Let $E = E(\{n_k\}, \{c_k\})$ be a uniform Cantor set, without additional restrictions. Let μ be the unique Borel probability measure on E such that $\mu(I) = 1/N_k$ for every component I of E_k and for every $k \geq 1$. Let g be the distributive function of μ, i.e., $g(t) = \mu([0, t])$ for all $t \in [0, \infty)$. Then the function g is obviously a doubling gauge. We claim that

$$0 < \mathcal{P}^g(E) \leq \mathcal{P}_0^g(E) < \infty. \tag{5.3}$$

The fact is proved in [6]. Here we give a proof for the completeness. We first show that the inequality

$$\mu(P)/4 \leq g(|P|) \leq 4\mu(P) \tag{5.4}$$

holds for every interval P of length $|P| \in (0, \delta_1)$ centered in E. Let P be such an interval. Let $k \geq 1$ be an integer such that $\delta_k \leq |P|/2 < \delta_{k-1}$. Then we have an integer m_k, with $1 \leq m_k \leq n_k$, such that

$$m_k\delta_k + (m_k - 1)\varepsilon_k \leq |P|/2 < (m_k + 1)\delta_k + m_k\varepsilon_k.$$

Since the center of P is in E, we have

$$\frac{m_k}{N_k} \leq \mu(P) \leq \frac{4m_k}{N_k} \quad \text{and} \quad \frac{m_k}{N_k} \leq g(|P|) \leq \frac{4m_k}{N_k}.$$

This proves (5.4). The inequality on the right of (5.4) directly gives $\mathcal{P}_0^g(E) < 4$. Noting that μ is a doubling measure on E, the inequality on the left of (5.4), together with Vitali's covering lemma, yields $\mathcal{P}_0^g(E) > 1/4$. Now the desired inequality (5.3) follows from Theorem 2.

Problem 2. We do not know whether $\mathcal{P}^g(E) < \mathcal{P}_0^g(E)$ for the gauge g in Remark 3. The following question is also open: Is there a countable set F, with $\overline{F} = E \cup F$, such that $E \cup F$ has different positive finite $\mathcal{P}^g$-measure and $\mathcal{P}_0^g$-premeasure?

References

[1] M. Csörnyei, *An example illustrating $P^g(K) \neq P_0^g(K)$ for sets of finite pre-measure*, Real Anal. Exchange **27** (2001/02), no. 1, 65–70.

[2] A. Dvoretzky, *A note on Hausdorff dimension functions*, Math. Pro. Camb. Phil. Soc. **44** (1948), 13–16.

[3] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wily & Sons, 1990.

[4] D. J. Feng, *Comparing packing measures to Hausdorff measures on the line*, Math. Nachr. **241** (2002), 65–72.

[5] D. J. Feng, S. Hua, and Z. Y. Wen, *Some relations between packing pre-measure and packing measure*, Bull. London Math. Soc. **31** (1999), no. 6, 665–670.

[6] D. J. Feng, H. Rao, Z. Y. Wen, and J. Wu, *Dimensions and gauges for symmetric cantor sets*, Approx. Theory Appl. **11** (1995), no. 4, 108–110.

[7] D. J. Feng, W. Wen, and J. Wu, *Some dimensional results for homogeneous Moran sets*, Sci. China Ser. A **40** (1997), no. 5, 475–482.

[8] H. Joyce and D. Preiss, *On the existence of subsets of finite positive packing measure*, Mathematika **42** (1995), no. 1, 15–24.

[9] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[10] Y. Peres, *The packing measure of self-affine carpets*, Math. Proc. Cambridge Philos. Soc. **115** (1994), no. 3, 437–450.

[11] ———, *The self-affine carpets of McMullen and Bedford have infinite Hausdorff measure*, Math. Proc. Cambridge Philos. Soc. **116** (1994), no. 3, 513–526.

[12] T. Rajala, *Comparing the Hausdorff and packing measures of sets of small dimension in metric spaces*, Monatsh Math. **164** (2011), no. 3, 313–323.

[13] C. Rogers, *Hausdorff Measures*, Cambridge University Press, 1998.

[14] S. J. Taylor and C. Tricot, *Packing measure and its evaluation for a Brownian path*, Trans. Amer. Math. Soc. **288** (1985), no. 2, 679–699.

[15] C. Tricot, *Two definitions of fractional dimension*, Math. Pro. Cambridge Philos. Soc. **91** (1982), no. 1, 57–74.

[16] S. Y. Wen and Z. Y. Wen, *Some properties of packing measure with doubling gauge*, Studia Math. **165** (2004), no. 2, 125–134.

[17] S. Y. Wen, Z. X. Wen, and Z. Y. Wen, *Gauges for the self-similar sets*, Math. Nachr. **281** (2008), no. 8, 1205–1214.

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