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NORM CONVERGENT PARTIAL SUMS OF TAYLOR SERIES

Jongho Yang

Abstract. It is known that the partial sum of the Taylor series of an holomorphic function of one complex variable converges in norm on $H^p(\mathbb{D})$ for $1 < p < \infty$. In this paper, we consider various type of partial sums of a holomorphic function of several variables which also converge in norm on $H^p(\mathbb{B}_n)$ for $1 < p < \infty$. For the partial sums in several variable cases, some variables could be chosen slowly (fastly) relative to other variables. We prove that in any cases the partial sum converges to the original function, regardlessly how slowly (fastly) some variables are taken.

1. Introduction

Let $\mathbb{C}^n$ denote the Euclidean space of complex dimension n. The inner product on $\mathbb{C}^n$ given by

$$\langle z, w\rangle := z_1\overline{w}_1 + \cdots + z_n\overline{w}_n,$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$, so the associated norm is $|z| = \sqrt{\langle z, z\rangle}$. The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$$

and its boundary is the unit sphere

$$\mathbb{S}_n := \{z \in \mathbb{C}^n : |z| = 1\}.$$

In the case $n = 1$, we denote $\mathbb{D}$ in place of $\mathbb{B}_1$. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{B}_n)$ consists of all holomorphic functions on $\mathbb{B}_n$ with

$$\|f\|_p := \left(\sup_{0<r<1} \int_{\mathbb{S}_n} |f(r\zeta)|^p \, d\sigma_n(\zeta)\right)^{1/p} < \infty,$$

where $d\sigma_n$ is the normalized surface measure on $\mathbb{S}_n$. It is well known that each space $H^p(\mathbb{B}_n)$ is a Banach space and contains polynomials as a dense subset with respect to the above norm. Moreover we can use Talyor polynomials to approximate the original function in the norm on $H^p(\mathbb{D})$ when $1 < p < \infty$. See [2, Corollary 3].

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However there is only one trivial way in taking partial sum of Taylor series for one variable analytic function. In this paper we consider various type of convergent partial sums of Taylor series for a holomorphic function with several variables. For a holomorphic function f on $\mathbb{B}_n$, the Taylor series is given by

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha.$$

Here the summation is over all multi-indexes

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

with nonnegative integers α_i and

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

For a positive integer N we define the partial sum $S_N[f]$ as

$$S_N[f](z) := \sum_{|\alpha| \le N} c_\alpha z^\alpha,$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$. And we consider more partial sums of Taylor series in several variable case as

$$U_{\mathbf{N}}[f](z) := \sum_{\alpha_i \le N_i} c_\alpha z^\alpha, \qquad V_{\mathbf{N}}[f](z) := \sum_{\alpha_i > N_i} c_\alpha z^\alpha,$$

where $\mathbf{N} := (N_1, \dots, N_n) \in \mathbb{N}_0^n$. Here, $\mathbb{N}_0^n$ is the product set of nonnegative integer $\mathbb{N}_0$. And we call a sequence $\{\mathbf{N}^{(k)}\}_k$ in $\mathbb{N}_0^n$ *converges to* ∞ if

$$\lim_{k \to \infty} N_i^{(k)} = \infty$$

for each i. In the convergence of $\mathbf{N}^{(k)}$, each $N_i^{(k)}$ need not be growth regularly. Some $N_i^{(k)}$'s are could increase slowly(fastly) relative to others. Here is our main theorem.

Theorem 1.1. *For $1 < p < \infty$, we have*

(1) *$S_N[f]$ converges to f in the norm on $H^p(\mathbb{B}_n)$;*
(2) *$U_{\mathbf{N}^{(k)}}[f]$ converge to f in the norm on $H^p(\mathbb{B}_n)$;*
(3) *$V_{\mathbf{N}^{(k)}}[f]$ converge to the zero function in the norm on $H^p(\mathbb{B}_n)$;*

whenever N and a sequence $\{\mathbf{N}^{(k)}\}_k$ converges to ∞.

2. Proof of the theorem

We use the following lemma to prove the convergence of $S_N[f]$ on $H^p(\mathbb{B}_n)$. The proof is similar to [2, Proposition 1] but we include for completeness.

Lemma 2.1. *Suppose X is a Banach space of holomorphic functions on $\mathbb{B}_n$ with the properties that the polynomials are dense in X and S_N is a bounded operator on X for each N. Then $\|S_N[f] - f\|_X \to 0$ for each $f \in X$ if and only if there is a positive constant $C > 0$ such that $\|S_N\| \le C$ for all $N \ge 1$.*

Proof. First we assume that $\|S_N[f]-f\|_X \to 0$ for each $f \in X$. Then we have $S_N[f]$ converges to f the norm on X. Since each S_N is a bounded operator, by applying the uniformly boundedness principle we have $\sup_N \|S_N\|$ is bounded.

Conversely, fix $f \in X$ and $\epsilon > 0$. Since polynomials are dense in X, we can find a polynomial g such that $\|f-g\|_X < \epsilon$. So we have

$$\begin{aligned}\|S_N[f]-f\|_X &\le \|S_N[f-g]\|_X + \|S_N[g]-g\|_X + \|g-f\|_X \\ &\le C\|f-g\|_X + \|S_N[g]-g\|_X + \|g-f\|_X \\ &\le (C+1)\epsilon + \|g-S_N[g]\|_X.\end{aligned}$$

Since $S_N[g]=g$ for large k, we prove that

$$\lim_{k\to\infty} \|S_N[f]-f\|_X = 0.$$

□

In the proof of [2, Proposition 1], the author used the uniform boundedness principle; for example, see the [1, Theorem 5.8]. However to apply the principle, we have to know that each S_N is a bounded operator. In the Hardy space it is well-known that S_N is a bounded operator, but we cannot guarantee for a general Banach space of analytic function. It is an interesting problem whether there exists a Banach space with analytic function in $\mathbb{D}$ such that the partial sum converges but the operator norms $\|S_N\|$ are unbounded for some N.

We also use the well-known formulas

$$\int_{\mathbb{S}_n} f \, d\sigma_n = \frac{1}{2\pi}\int_{\mathbb{S}_n}\int_0^{2\pi} f(e^{i\theta}\zeta)\, d\theta\, d\sigma_n(\zeta) \tag{2.1}$$

and

$$\begin{aligned}&\int_{\mathbb{S}_n} f\, d\sigma_n \\ &= \binom{n-1}{m}\int_{\mathbb{B}_m}(1-|z|^2)^{n-m-1}\int_{\mathbb{S}_{n-m}} f(z,\sqrt{1-|z|^2}\eta)\, d\sigma_{n-m}(\eta)\, dv_m(z)\end{aligned} \tag{2.2}$$

for $1 \le m < n$ and $f \in L^1(\mathbb{S}_n)$; we refer to [3, Lemma 1.10]. Now we prove the Theorem 1.1(1).

Theorem 2.2. *For $1 < p < \infty$, the partial sum $S_N[f]$ converges to f on $H^p(\mathbb{B}_n)$.*

Proof. Let $f(z) = \sum_\alpha c_\alpha z^\alpha$. If we let

$$f_k(z) := \sum_{|\alpha|=k} c_\alpha z^\alpha$$

for each $k \ge 0$, then the Talyor series of f can be expressed to the homogeneous expansion as

$$f(z) = \sum_{k=0}^{\infty} f_k(z).$$

Then by the equation (2.1) we have

$$\begin{aligned}\int_{\mathbb{S}_n}\left|\sum_{k=0}^{N} f_k(z)\right|^p \, d\sigma_n(z) &= \int_{\mathbb{S}_n}\int_{\mathbb{S}_1}\left|\sum_{k=0}^{N} f_k(\lambda z))\right|^p \, d\sigma_1(\lambda)\; d\sigma_n(z)\\ &= \int_{\mathbb{S}_n}\int_{\mathbb{S}_1}\left|\sum_{k=0}^{N} \lambda^k f_k(z)\right|^p \, d\sigma_1(\lambda)\; d\sigma_n(z),\end{aligned}$$

where we used the homogeneous property of f_k. Considered as the analytic function in λ, we know that the partial sum converges. By [2, Proposition 1], we have

$$\begin{aligned}&\le C\int_{\mathbb{S}_n}\int_{\mathbb{S}_1}\left|\sum_{k=0}^{\infty} \lambda^k f_k(z)\right|^p \, d\sigma_1(\lambda)\; d\sigma_n(z)\\ &= C\int_{\mathbb{S}_n}\int_{\mathbb{S}_1}|f(\lambda z)|^p \, d\sigma_1(\lambda)\; d\sigma_n(z)\\ &= C\int_{\mathbb{S}_n}|f(z)|^p \, d\sigma_n(z).\end{aligned}$$

Thus $\sup_N \|S_N\|$ is bounded. By applying Lemma 2.1 we prove the theorem. □

Now we consider the partial sum $U_{\mathbf{N}}[f]$ and $V_{\mathbf{N}}[f]$. Recall that

$$U_{\mathbf{N}}[f](z) := \sum_{\alpha_i \le N_i} c_\alpha z^\alpha, \qquad V_{\mathbf{N}}[f](z) := \sum_{\alpha_i > N_i} c_\alpha z^\alpha$$

for $\mathbf{N} := (N_1, \dots, N_n) \in \mathbb{N}_0^n$. The operator $U_{\mathbf{N}}$ is the partial sum of Taylor series by collecting all polynomials with the power α_i of z^α is not greater than N_i for each i. We note that

$$U_{\mathbf{N}} = U_{N_1\mathbf{e}_1} \circ \cdots \circ U_{N_n\mathbf{e}_n}, \qquad V_{\mathbf{N}} = V_{N_1\mathbf{e}_1} \circ \cdots \circ V_{N_n\mathbf{e}_n},$$

independently with the composition order. Here

$$\mathbf{e}_1 := (1, 0, \dots, 0), \dots, \mathbf{e}_n := (0, \dots, 1),$$

be the standard unit vectors in $\mathbb{C}^n$. Though

$$U_{N\mathbf{e}_i} = I - V_{N\mathbf{e}_i}$$

for any positive integer N, in general

$$U_{\mathbf{N}} \neq I - V_{\mathbf{N}},$$

where I is the identity operator.

Lemma 2.3. *For $1 < p < \infty$ and $\mathbf{N} \in \mathbb{N}_0^n$, there exists a constant C depending only n such that*

$$\|U_{\mathbf{N}}[f]\|_p \le C\|f\|_p.$$

Proof. In case of $n = 1$, it was proved in [2, Theorem 2]. We use mathematical induction argument. Suppose that $n = m$ holds. By the equation (2.2) we have

$$\begin{aligned}
&\int_{\mathbb{S}_{m+1}} |U_{(N_1,\dots,N_{m+1})}[f]|^p d\sigma_{m+1} \\
&= m\int_{\mathbb{B}_1} (1-|z|^2)^{m-1} \int_{\mathbb{S}_m} \left| U_{(N_1,\dots,N_{m+1})}[f](z, \sqrt{1-|z|^2}\eta) \right|^p d\sigma_m(\eta)\ dv_1(z).
\end{aligned}$$

For a fixed $z \in \mathbb{B}_1$, we let $g(\eta) = U_{N_1\mathbf{e}_1}[f](z, \sqrt{1-|z|^2}\eta)$ and consider g as a function on $\mathbb{S}_m$. Clearly $g \in p(\mathbb{S}_n)$ and $V_{(N_1,N_2,\dots,N_{m+1})} = U_{(0,N_2,\dots,N_{m+1})} \circ U_{N_1 e_1}$,

$$U_{(N_1,N_2,\dots,N_{m+1})}[f](z, \sqrt{1-|z|^2}\eta) = U_{(N_2,\dots,N_{m+1})}[g](\eta).$$

By assumption, we get

$$\|U_{(N_2,\dots,N_{m+1})}[g]\|_p \le C_n \|g\|_p.$$

Thus we have

$$\begin{aligned}
&m\int_{\mathbb{B}_1} (1-|z|^2)^{m-1} \int_{\mathbb{S}_m} \left| U_{(N_1,\dots,N_{m+1})}[f](z, \sqrt{1-|z|^2}\eta) \right|^p d\sigma_m(\eta)\ dv_1(z) \\
&\le m\int_{\mathbb{B}_1} (1-|z|^2)^{m-1} \int_{\mathbb{S}_m} \left| U_{N_1\mathbf{e}_1}[f](z, \sqrt{1-|z|^2}\eta) \right|^p d\sigma_m(\eta)\ dv_1(z) \\
&= \int_{\mathbb{S}_{m+1}} |U_{N_1\mathbf{e}_1}[f]|^p\ d\sigma_{m+1}.
\end{aligned}$$

We apply again the equation (2.2), we have

$$\begin{aligned}
&\int_{\mathbb{S}_{m+1}} |U_{N_1\mathbf{e}_1}[f]|^p\ d\sigma_{m+1} \\
&= \int_{\mathbb{B}_m} \int_{\mathbb{S}_1} \left| U_{N_1\mathbf{e}_1}[f](\sqrt{1-|z|^2}\eta, z) \right|^p\ d\sigma_1(\eta)\ dv_m(z) \\
&\le \int_{\mathbb{B}_m} \int_{\mathbb{S}_1} \left| f(\sqrt{1-|z|^2}\eta, z) \right|^p\ d\sigma_1(\eta)\ dv_m(z) \\
&= \int_{\mathbb{S}_{m+1}} |f|^p\ d\sigma_{m+1}.
\end{aligned}$$

□

Lemma 2.4. *For $1 < p < \infty$ and $\mathbf{N} \in \mathbb{N}_0^n$, there exists a constant C depending only n such that*

$$\|V_{\mathbf{N}}[f]\|_p \le C\|f\|_p.$$

Proof. Let $\mathbf{N} = (N_1, \dots, N_n)$. Since $V_{N_i\mathbf{e}_i} = I - U_{N_i\mathbf{e}_i}$, we get

$$\begin{aligned}
V_{\mathbf{N}} &= V_{N_1\mathbf{e}_1} \circ \cdots \circ V_{N_n\mathbf{e}_n} \\
&= (I - U_{N_1\mathbf{e}_1}) \circ \cdots \circ (I - U_{N_n\mathbf{e}_n}).
\end{aligned}$$

By Lemma 2.3, we have $U_{N_i\mathbf{e}_i}$ is uniformly bounded of N. So is $I - U_{N_i\mathbf{e}_i}$. Since we compose n operators of uniformly bounded ones, we prove that $V_{\mathbf{N}}$ is a also uniformly bounded operators. □

Now we are ready to prove Theorem 1.1(2) and (3).

Theorem 2.5. *For $f \in H^p(\mathbb{B}_n)$ and $1 < p < \infty$ and a sequence $\{\mathbf{N}^{(k)}\}_k$ in $\mathbb{N}_0^n$ which converges to ∞, we have*

$$\lim_{k\to\infty} \|f - U_{\mathbf{N}^{(k)}}[f]\|_p = 0$$

and

$$\lim_{k\to\infty} \|V_{\mathbf{N}^{(k)}}[f]\|_p = 0.$$

Proof. Note that since the sequence $\{\mathbf{N}^{(k)}\}_k$ converges to ∞, for any polynomial g, we have

$$\lim_{k\to\infty} U_{\mathbf{N}^{(k)}}[g] = g, \qquad \lim_{k\to\infty} V_{\mathbf{N}^{(k)}}[g] = 0.$$

Fix $f \in p(\mathbb{B}_n)$ and $\epsilon > 0$. Since polynomials are dense in $H^p(\mathbb{B}_n)$, we can find a polynomial g such that $\|f - g\|_p < \epsilon$. It follows that

$$\begin{aligned} \|f - U_{\mathbf{N}^{(k)}}[f]\|_p &\le \|f - g\|_p + \|g - U_{\mathbf{N}^{(k)}}[g]\|_p + \|U_{\mathbf{N}^{(k)}}[g - f]\|_p \\ &\le \|f - g\|_p + \|g - U_{\mathbf{N}^{(k)}}[g]\|_p + C\|f - g\|_p \\ &\le (C+1)\epsilon + \|g - U_{\mathbf{N}^{(k)}}[g]\|_p, \end{aligned}$$

where we used Lemma 2.3 in the last inequality. Since $U_{\mathbf{N}^{(k)}}[g] = g$ for large k, we prove that

$$\lim_{k\to\infty} \|f - U_{\mathbf{N}^{(k)}}[f]\|_p = 0.$$

Similarly, by Lemma 2.4 we have

$$\begin{aligned} \|V_{\mathbf{N}^{(k)}}[f]\|_p &\le \|V_{\mathbf{N}^{(k)}}[f - g]\|_p + \|V_{\mathbf{N}^{(k)}}[g]\|_p \\ &\le C\|f - g\|_p + \|V_{\mathbf{N}^{(k)}}[g]\|_p. \end{aligned}$$

Since $\|V_{\mathbf{N}^{(k)}}[g]\|_p = 0$ for large k, we prove that

$$\lim_{k\to\infty} \|V_{\mathbf{N}^{(k)}}[f]\|_p = 0.$$

□

Remark. We proved the convergence of various partial sums on the Hardy spaces. However there are more classical analytic function space like the Bergman space and Fock space. These spaces have the radial weighted functions on symmetric domain. So by using polar coordinate integration, we can also show that the Theorem 1.1 is also hold for the Bergman or Fock space.

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Department of Mathematics
Korea University
Seoul 136-713, Korea
E-mail address: cachya@korea.ac.kr