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COSET OF A HYPERCOMPLEX NUMBER SYSTEM IN CLIFFORD ANALYSIS

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Abstract. We give certain properties of elements in a coset group with hypercomplex numbers and research a monogenic function and a Clifford regular function with values in a coset group by defining differential operators. We give properties of those functions and a power of elements in a coset group with hypercomplex numbers.

1. Introduction

Many kinds of quaternion, specially, split quaternions and dual quaternions, etc., have applications in physics and computer systems. There are conventional and mathematical constructions of quaternions by multiplication rules of each elements. Leo [10] formulated special relativity by a quaternionic algebra on reals and showed that a complexified quaternionic version of special relativity is not a necessity. Hasebe [3] constructed quantum Hall effect on split quaternions and analyzed that a wave function and membrane-like excitations are derived explicitly. Brody and Graefe [1] introduced quaternionic and coquaternionic (split signature analogue of quaternions) extensions of Hamiltonian mechanics and offered complexified classical and quantum mechanics. Hucks [4] introduced basic properties and definitions for the hyperbolic complex numbers, and applied the Dirac equation in 4 dimensions to special relativistic physics. Sobczyk [12] explored an underlying geometric framework in which matrix multiplication arises from the product of numbers in a geometric (Clifford) algebra. Jaglom [5] generated the mathematical operations and representations between complex numbers and geometry. We [6, 7, 8, 9] have researched corresponding Cauchy-Riemann systems and properties of a regularity of functions with values in special quaternions on Clifford analysis and gave a regular function with values in dual split quaternions and relations

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between a corresponding Cauchy-Riemann system and a regularity of functions with values in dual split quaternions.

In the conventional mathematical construction of complex and multicomplex numbers, multiplication rules are suggested instead of being derived from a general principle. Petrache [11] proposed a systematic approach based on the concept of a coset product from the group theory. He showed that extensions of real numbers in two or more dimensions follow from the closure property of finite coset groups with the utility of multidimensional number systems expressed by elements of small group symmetries.

In this paper, we give the form of elements in a coset group with special unit matrix bases and the multiplication of those elements. Also, we consider certain properties of elements in a coset group with hypercomplex numbers and then investigate a monogenic function and a Clifford regular function with values in a coset group by defining differential operators. We give properties of those functions and a power of elements in a coset group with hypercomplex numbers.

2. Preliminaries

Throughout this paper, let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ be the sets of real and complex numbers, and positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Referring Petrache [11], we consider the coset group $G = \{\mathbb{R},\ g_1\mathbb{R},\ g_2\mathbb{R},\ g_3\mathbb{R}\}$, where g_m is an element of the set outside of $\mathbb{R}$ but compatible with operations in $\mathbb{R}$ and g_m $(m = 1, 2, 3)$. Then we obtain the following numbers by generating a set within cosets:

$$A = \{\zeta = p + gq \mid p, q \in \mathbb{C}\},$$

where g is an element of the set outside of $\mathbb{C}$ for which addition and multiplication rules follow from the properties of g: For any $\zeta, \eta \in A$,

$$\zeta + \eta = (p_1 + p_2) + g(q_1 + q_2)$$

and

$$\zeta\eta = (p_1p_2 + \alpha q_1q_2) + g(p_1q_2 + p_2q_1),$$

where $\alpha = g^2$ is a complex number. From the above multiplication rule, the product $\zeta\eta$ can be written in a matrix form:

$$\begin{pmatrix} p_1p_2 + \alpha q_1q_2 & p_1q_2 + p_2q_1 \end{pmatrix} = \begin{pmatrix} p_1 & q_1 \end{pmatrix} \begin{pmatrix} p_2 & q_2 \\ \alpha q_2 & p_2 \end{pmatrix},$$

which gives the following matrix form:

$$\zeta = \begin{pmatrix} p & q \\ \alpha q & p \end{pmatrix} \qquad (p,\ q \in \mathbb{C}).$$

Since complex numbers p and q are also obtained by the above process, we obtain

$$\zeta = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ \alpha x_1 & x_0 & \alpha x_3 & x_2 \\ \alpha x_2 & \alpha x_3 & x_0 & x_1 \\ \alpha^2 x_3 & \alpha x_2 & \alpha x_1 & x_0 \end{pmatrix} = x_0\mathbf{1} + x_1 I + x_2 J + x_3 K,$$

where $\mathbf{1}$ is the unit matrix,

$$I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \end{pmatrix}, \ J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}, \ K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ \alpha^2 & 0 & 0 & 0 \end{pmatrix}.$$

We consider properties of $\mathbf{1}, I, J$ and K. By the multiplication of matrices, we obtain

$$I^2 = J^2 = \alpha\mathbf{1},\ K^2 = \alpha^2\ \mathbf{1},$$
$$IJ = JI = K,\ JK = KJ = \alpha I,\ KI = IK = \alpha J.$$

If $\alpha = -1 + i0$, where $i = \sqrt{-1}$ is the imaginary unit in $\mathbb{C}$, then

$$I^2 = J^2 = -\mathbf{1},\ K^2 = \mathbf{1},$$
$$IJ = JI = K,\ JK = KJ = -I,\ KI = IK = -J.$$

Let $\mathcal{C}$ be a set of ζ with $\mathbf{1}, I, J$ and K as follows:

$$\mathcal{C} = \{z = z_1 + z_2 J \mid z_1 = x_0 + x_1 I,\ z_2 = x_2 + x_3 I,\ x_r \in \mathbb{R}\ (r = 0, 1, 2, 3)\}$$

and the elements of $\mathcal{C}$ be said to be pseudo split quaternions.

We give the commutative multiplication of elements of $\mathcal{C}$: For any $z, w \in \mathcal{C}$,

$$\begin{aligned} zw &= (z_1w_1 - z_2w_2) + (z_1w_2 + z_2w_1)J \\ &= (x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3) + (x_0y_1 + x_1y_0 - x_2y_3 - x_3y_2)I \\ &\quad + (x_0y_2 - x_1y_3 + x_2y_0 - x_3y_1)J + (x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0)K. \end{aligned}$$

When z_1 is a scalar multiplication of z_2, we have the conjugate number $\overline{z} = \overline{z_1} - \overline{z_2}J$, the norm of z is $|z|^2 = z\overline{z} = \sum_{r=0}^{3} x_r^2$ and the inverse number of z is $z^{-1} = \frac{\overline{z}}{|z|^2}$.

Also, when z_1 satisfies the equation $z_1 = \mathfrak{K} z_2 I$, where $\mathfrak{K}$ is a scalar number, we have the conjugate number $z^* = \overline{z_1} + \overline{z_2}J$, the modulus of z is $N(z) = zz^* = x_0^2 + x_1^2 - x_2^2 - x_3^2$ and the inverse number of z is $z^{-1} = \frac{z^*}{N(z)}$.

Let Ω be an open set in $\mathbb{C}^2$. We give a function $f : \mathbb{C}^2 \to \mathcal{C}$ such that

$$f(z) = f(z_1, z_2) = f_1(z_1, z_2) + f_2(z_1, z_2)J,$$

where $f_1 = u_0 + u_1 I$ and $f_2 = u_2 + u_3 I$ with u_r $(r = 0, 1, 2, 3)$ are real valued functions. We give differential operators as follows:

$$D := \frac{1}{2}\Big(\frac{\partial}{\partial x_0} - I\frac{\partial}{\partial x_1} - J\frac{\partial}{\partial x_2} + K\frac{\partial}{\partial x_3}\Big) = \frac{\partial}{\partial z_1} - J\frac{\partial}{\partial z_2},$$

$$\overline{D} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + I\frac{\partial}{\partial x_1} + J\frac{\partial}{\partial x_2} + K\frac{\partial}{\partial x_3}\Big) = \frac{\partial}{\partial \overline{z_1}} + J\frac{\partial}{\partial \overline{z_2}}.$$

When $\frac{\partial f}{\partial z_1}$ is a scalar multiplication of $\frac{\partial f}{\partial z_2}$, there is a Laplacian operator such that

$$D\overline{D}f = \overline{D}Df = \frac{\partial^2 f}{\partial z_1 \partial \overline{z_1}} + \frac{\partial^2 f}{\partial z_2 \partial \overline{z_2}} = \frac{1}{4}\sum_{r=0}^{3}\frac{\partial f}{\partial x_r}.$$

Also, we have

$$D^* = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + I\frac{\partial}{\partial x_1} - J\frac{\partial}{\partial x_2} - K\frac{\partial}{\partial x_3}\Big) = \frac{\partial}{\partial \overline{z_1}} - J\frac{\partial}{\partial \overline{z_2}}.$$

When $\frac{\partial f}{\partial z_1}$ satisfies the equation $\frac{\partial f}{\partial z_1} = \mathfrak{K} I \frac{\partial f}{\partial z_2}$, where $\mathfrak{K}$ is a scalar number, there is a Coulomb operator [2] such that

$$DD^*f = D^*Df = \frac{\partial^2 f}{\partial z_1 \partial \overline{z_1}} - \frac{\partial^2 f}{\partial z_2 \partial \overline{z_2}} = \frac{1}{4}\Big(\frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_3^2}\Big).$$

Remark 2.1. By the definition of differential operators, we have

$$\overline{D}f = \Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}}\Big) + \Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\Big)J$$

and

$$D^*f = \Big(\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}}\Big) + \Big(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial f_1}{\partial \overline{z_2}}\Big)J.$$

3. Properties of functions with values in $\mathcal{C}$

Definition. Let Ω be an open set in $\mathbb{C}^2$. A function $f(z) = f_1(z) + f_2(z)J$ is said to be L(R)-monogenic in Ω if the following two conditions are satisfied:

(i) $f_1(z)$ and $f_2(z)$ are continuously differential functions on Ω, and

(ii) $\overline{D}f(z) = 0$ (resp. $f(z)\overline{D} = 0$) on Ω.

Definition. Let Ω be an open set in $\mathbb{C}^2$. A function $f(z) = f_1(z) + f_2(z)J$ is said to be L(R)-Clifford regular in Ω if the following two conditions are satisfied:

(i) $f_1(z)$ and $f_2(z)$ are continuously differential functions on Ω, and

(ii) $D^*f(z) = 0$ (resp. $f(z)D^* = 0$) on Ω.

Since the equation $\overline{D}f = 0$ (resp. $D^*f = 0$) is equivalent to the equation $f\overline{D} = 0$ (resp. $fD^* = 0$), we don't need to distinguish between left and right monogenic (resp. Clifford regular).

Remark 3.1. The equation $\overline{D}f(z) = 0$ is equivalent to the following equations:

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial f_2}{\partial \overline{z_2}} \quad\text{and}\quad \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial f_1}{\partial \overline{z_2}}. \tag{3.1}$$

Also, the equation $D^*f(z) = 0$ is equivalent to the following equations:

$$\frac{\partial f_1}{\partial \overline{z_1}} = -\frac{\partial f_2}{\partial \overline{z_2}} \quad\text{and}\quad \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial f_1}{\partial \overline{z_2}}. \tag{3.2}$$

The Equations (3.1) and (3.2) are the analogue of the Cauchy-Riemann systems in $\mathcal{C}$.

Remark 3.2. Let Ω be an open set in $\mathbb{C}^2$. If a function $f(z) = f_1(z) + f_2(z)J$ is monogenic in Ω, then it satisfies

$$D^* f = 2\frac{\partial f}{\partial \overline{z_1}} = -2J\frac{\partial f}{\partial \overline{z_2}}.$$

Also, if a function $f(z) = f_1(z)+f_2(z)J$ is Clifford regular in Ω, then it satisfies

$$\overline{D} f = 2\frac{\partial f}{\partial \overline{z_1}} = 2J\frac{\partial f}{\partial \overline{z_2}}.$$

Proposition 3.3. *Let Ω be an open set in $\mathbb{C}^2$. For $n \in \mathbb{N}_0$, a function*

$$f(z) = z^n = (z_1 + z_2 J)^n = f_1 + f_2 J,$$

where

$$f_1 = \sum_{\substack{k=0\\ k:even}}^{n} (-1)^{\frac{k}{2}} \binom{n}{k} z_1^{n-k} z_2^k \quad and \quad f_2 = \sum_{\substack{k=1\\ k:odd}}^{n} (-1)^{\frac{k-1}{2}} \binom{n}{k} z_1^{n-k} z_2^k,$$

is monogenic and Clifford regular in Ω.

Proof. By the definition of differential operators, we have

$$\frac{\partial}{\partial \overline{z_t}} z_p^m = 0,$$

where $m \in \mathbb{N}_0$ and $t,\ p = 1,\ 2$. Hence, $\overline{D} z^n = 0$ and $D^* z^n = 0$. Therefore, we obtain z^n is monogenic and Clifford regular in Ω. □

Proposition 3.4. *Let Ω be an open set in $\mathbb{C}^2$. A function $f(z) = z^{-1}$ $(z \neq 0)$ is monogenic and Clifford regular in Ω.*

Proof. Since a function $f(z) = z^{-1}$ is defined by one of two cases as follows:
(i) $z_1 = \mathfrak{K} z_2$, where $\mathfrak{K}$ is scalar.
(ii) $z_1 = \mathfrak{K} z_2 I$, where $\mathfrak{K}$ is scalar.
If z satisfies the case (i), then

$$\overline{D} z^{-1} = \overline{D}\Big(\frac{\overline{z}}{z\overline{z}}\Big) = \Big(\frac{\partial}{\partial \overline{z_1}} + J\frac{\partial}{\partial \overline{z_2}}\Big)\Big(\frac{\overline{z_1} - \overline{z_2} J}{z_1\overline{z_1} + z_2\overline{z_2}}\Big) = 0.$$

Also, if z satisfies the case (ii), then

$$D^* z^{-1} = D^*\Big(\frac{z^*}{zz^*}\Big) = \Big(\frac{\partial}{\partial \overline{z_1}} - J\frac{\partial}{\partial \overline{z_2}}\Big)\Big(\frac{\overline{z_1} + \overline{z_2} J}{z_1\overline{z_1} - z_2\overline{z_2}}\Big) = 0.$$

Therefore, we obtain the result. Furthermore, we have

$$\overline{D}\Big(\frac{z^*}{zz^*}\Big) = 0 \quad \text{and} \quad D^*\Big(\frac{\overline{z}}{z\overline{z}}\Big) = 0.$$

Therefore, a function of the inverse form is monogenic (resp. Clifford regular) in Ω, regardless of calculating operators. □

Theorem 3.5. *Let Ω be an open set in $\mathbb{C}^2$ and a function f be monogenic in Ω. Then we have*

$$Df = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_3}K = -\frac{\partial f}{\partial x_1}I - \frac{\partial f}{\partial x_2}J.$$

Proof. From Remark 1 and the system (3.1), we have

$$\begin{aligned} Df &= \Big(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\Big) + \Big(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\Big)J \\ &= \Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}}\Big) + \Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\Big)J \\ &\quad + \frac{\partial}{\partial x_1}(u_1 - u_0 I + u_3 J - u_2 K) + \frac{\partial}{\partial x_3}(u_2 + u_3 I - u_0 J - u_1 K) \\ &= -\frac{\partial f}{\partial x_1}I - \frac{\partial f}{\partial x_2}J; \end{aligned}$$

or

$$\begin{aligned} Df &= -\Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}}\Big) - \Big(\frac{\partial f_1}{\partial \overline{z_2}} + \frac{\partial f_2}{\partial \overline{z_1}}\Big)J \\ &\quad + \frac{\partial}{\partial x_0}(u_0 + u_1 I + u_2 J + u_3 K) + \frac{\partial}{\partial x_3}(u_3 - u_2 I - u_1 J + u_0 K) \\ &= \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_3}K. \end{aligned}$$

Therefore, we obtain the result. □

Corollary 3.6. *Let Ω be an open set in $\mathbb{C}^2$ and a function f be Clifford regular in Ω. Then we have*

$$Df = \frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial x_2}J = -\frac{\partial f}{\partial x_1}I + \frac{\partial f}{\partial x_3}K.$$

Proof. Following the process of proof of Theorem 3.5, we also obtain the result. □

Proposition 3.7. *Let Ω be an open set in $\mathbb{C}^2$ and let f and g be monogenic (resp. Clifford regular) in Ω. Then the following properties are satisfied:*

(i) *$f\alpha$ and αf are monogenic (resp. Clifford regular) in Ω, where α is a constant in $\mathcal{C}$.*

(ii) *fg is monogenic (resp. Clifford regular) in Ω.*

Proof. From the property of multiplication in $\mathcal{C}$ and the definition of monogenic (resp. Clifford regular) in Ω, the results are obtained. □

We let

$$\omega := dz_1 \wedge dz_2 \wedge d\overline{z_2} + J dz_1 \wedge d\overline{z_1} \wedge dz_2.$$

Theorem 3.8. *Let Ω be a domain in $\mathbb{C}^2$ and U be any domain in Ω with a smooth boundary bU such that $\overline{U} \subset \Omega$. If a function f is monogenic in Ω, then*

$$\int_{bU} \omega f = 0,$$

where ωf is the product on $\mathcal{C}$ of the form ω on the function $f(z)$.

Proof. Since the function $f = f_1 + f_2 J$ has the equation

$$\omega f = f_1 dz_1 \wedge dz_2 \wedge d\overline{z_2} - f_2 dz_1 \wedge d\overline{z_1} \wedge dz_2 + J(f_1 dz_1 \wedge d\overline{z_1} \wedge dz_2 + f_2 dz_1 \wedge dz_2 \wedge d\overline{z_2}),$$

we have

$$\begin{aligned} d(\omega f) &= \frac{\partial f_1}{\partial \overline{z_1}} d\overline{z_1} \wedge dz_1 \wedge dz_2 \wedge d\overline{z_2} + J \frac{\partial f_2}{\partial \overline{z_1}} d\overline{z_1} \wedge dz_1 \wedge dz_2 \wedge d\overline{z_2} \\ &\quad - \frac{\partial f_2}{\partial \overline{z_2}} d\overline{z_2} \wedge dz_1 \wedge d\overline{z_1} \wedge dz_2 + J \frac{\partial f_1}{\partial \overline{z_2}} d\overline{z_2} \wedge dz_1 \wedge d\overline{z_1} \wedge dz_2 \\ &= \Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}} \Big) dV + J \Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}} \Big) dV, \end{aligned}$$

where $dV = dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$. Since f is monogenic in Ω, f satisfies the equation (3.1). Hence, we have $d(\omega f) = 0$. Therefore, by Stokes' theorem, we obtain the result. □

Corollary 3.9. *Let Ω be a domain in $\mathbb{C}^2$ and U be any domain in Ω with a smooth boundary bU such that $\overline{U} \subset \Omega$. Suppose*

$$\omega = dz_1 \wedge dz_2 \wedge d\overline{z_2} - J dz_1 \wedge d\overline{z_1} \wedge dz_2$$

and a function f is Clifford regular in Ω. Then

$$\int_{bU} \omega f = 0,$$

where ωf is the product on $\mathcal{C}$ of the form ω on the function $f(z)$.

Proof. Using the process of proof of Theorem 3.8, we have

$$d(\omega f) = \Big(\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}} \Big) dV + J \Big(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial f_1}{\partial \overline{z_2}} \Big) dV,$$

where $dV = dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$. Since f is Clifford regular in Ω, f satisfies the equation (3.2). Therefore, we have $d(\omega f) = 0$ and by Stokes' theorem, we obtain the result. □

Example 3.10. Let Ω be a domain in $\mathbb{C}^2$ and U be any domain in Ω with a smooth boundary bU such that $\overline{U} \subset \Omega$ and let $\omega = dz_1 \wedge dz_2 \wedge d\overline{z_2} + J dz_1 \wedge d\overline{z_1} \wedge dz_2$. Suppose $f(z) = z^n$ $(n \in \mathbb{N}_0)$ be monogenic in Ω. Then

$$\begin{aligned} \int_{bU} \omega f &= \int_U d(\omega f) \\ &= \int_U \Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}} \Big) dV + \int_U J \Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}} \Big) dV = 0, \end{aligned}$$

where ωf is the product on $\mathcal{C}$ of the form ω on the function $f(z)$ and $dV = dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$.

Example 3.11. Let Ω be a domain in $\mathbb{C}^2$ and U be any domain in Ω with a smooth boundary bU such that $\overline{U} \subset \Omega$ and let $\omega = dz_1 \wedge dz_2 \wedge d\overline{z_2} + J dz_1 \wedge d\overline{z_1} \wedge dz_2$. If $f(z) = z^n$ $(n \in \mathbb{N}_0)$ is Clifford regular in Ω, then

$$\begin{aligned} \int_{bU} \omega f &= \int_U d(\omega f) \\ &= \int_U \Big(\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}} \Big) dV + \int_U J \Big(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial f_1}{\partial \overline{z_2}} \Big) dV = 0, \end{aligned}$$

where ωf is the product on $\mathcal{C}$ of the form ω on the function $f(z)$ and $dV = dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$.

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