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A NOTE OF WEIGHTED COMPOSITION OPERATORS ON BLOCH-TYPE SPACES

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Abstract. We obtain a new criterion for the boundedness and compactness of the weighted composition operators ψC_φ from $\mathcal{B}^\alpha(0<\alpha<1)$ to $\mathcal{B}^\beta$ in terms of the sequence $\{\psi\varphi^n\}$. An estimate for the essential norm of ψC_φ is also given.

1. Introduction

Denote by $H(\mathbb{D})$ the space of all analytic functions on the unit disk $\mathbb{D}=\{z : |z|<1\}$ in the complex plane. Let $0<\alpha<\infty$. An $f\in H(\mathbb{D})$ is said to belong to Bloch-type spaces(or α-Bloch spaces), denoted by $\mathcal{B}^\alpha$, if

$$\|f\|_\alpha=\sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2)^\alpha<\infty.$$

The classical Bloch space $\mathcal{B}$ is just $\mathcal{B}^1$. It is clear that $\mathcal{B}^\alpha$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha}=|f(0)|+\|f\|_\alpha$. See [1, 15, 16] for the theory of Bloch-type spaces.

Let X and Y be Banach spaces of analytic functions on $\mathbb{D}$, $\psi\in H(\mathbb{D})$ and let φ be an analytic self mapping of $\mathbb{D}$. The weighted composition operator with symbols ψ and φ from X to Y is the operator ψC_φ defined by

$$\psi C_\varphi f=M_\psi C_\varphi f=\psi(f\circ\varphi)\quad\text{for } f\in X,$$

where M_ψ denotes the multiplication operator with symbol ψ and C_φ denotes the composition operator with symbol φ. A basic problem concerning composition operators on various Banach function spaces is to relate the operator theoretic properties of C_φ to the function theoretic properties of the symbol φ, which attracted a lots of attention recently, the reader can refer to [3].

Recall that the essential norm of a operator T between X and Y is the distance to the compact operators K, that is $\|T\|_e^{X\to Y}=\inf\{\|T-K\| : K \text{ is compact}\}$, where $\|\cdot\|$ is the operator norm. It is easy to see that $\|T\|_e^{X\to Y}=0$ if and only if T is compact.

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It is well known that the composition operator is automatically bounded on the Bloch space by Schwarz-Pick Lemma. The compactness of the composition operator on the Bloch space was characterized in [7]. In [13], Wulan, Zheng and Zhu obtained a new characterization for the compactness of the composition operator acting on the Bloch space as follows:

Theorem A. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is compact on the Bloch space if and only if*

$$\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} = 0.$$

The boundedness and the compactness of composition operators on Bloch type spaces was given in [5] by Lou. In [14], Zhao extended Theorem A to Bloch-type spaces. Among other results, he proved the following result.

Theorem B. *Let $0 < \alpha, \beta < \infty$, and φ be a self-map of $\mathbb{D}$. Then the essential norm of composition operator $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is*

$$\|C_\varphi\|_e^{\mathcal{B}^\alpha \to \mathcal{B}^\beta} = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n\to\infty} n^{\alpha-1} \|\varphi^n\|_\beta.$$

In [10], Ohno, Stroethoff and Zhao studied the boundedness and compactness of weighted composition operators on Bloch-type spaces. The essential norm of weighted composition operators on Bloch-type spaces are given in [6]. In [8], Manhas and Zhao gave an estimate for the essential norm of $\psi C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$. Especially, when $0 < \alpha < 1$, they obtained the following result.

Theorem C. *Suppose $0 < \alpha < 1$ and $0 < \beta < \infty$ and suppose that $\psi C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then the essential norm of composition operator $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is*

$$\|\psi C_\varphi\|_e^{\mathcal{B}^\alpha \to \mathcal{B}^\beta} = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n\to\infty} n^{\alpha-1} \|I_\psi(\varphi^n)\|_\beta,$$

where

$$I_\psi f(z) = \int_0^z f'(\zeta)\psi(\zeta) d\zeta.$$

Motivated by Theorems A, B and C, in this work we show that $\psi C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded (respectively, compact) if and only if the sequence $\left(\frac{\|\psi\varphi^n\|_\beta}{\|z^n\|_\alpha}\right)_{n=1}^\infty$ is bounded (respectively, convergent to 0 as $n \to \infty$) when $0 < \alpha < 1$. Moreover, we give the exact essential norm for the operators ψC_φ.

2. Boundedness of ψC_φ on Bloch-type spaces

In this section, we give a characterization for the boundedness of $\psi C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ when $0 < \alpha < 1$. For this purpose, we need the following result which is given in [10].

Lemma 2.1. *Let $0<\alpha<1$, $0<\beta<\infty$, $\psi\in H(\mathbb{D})$ and let φ be an analytic self-map of $\mathbb{D}$. Then $\psi C_\varphi:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded if and only if $\psi\in\mathcal{B}^\beta$ and*

$$\sup_{z\in\mathbb{D}}|\psi(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|<\infty.$$

The boundedness of $\psi C_\varphi:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ implies that $\psi\in\mathcal{B}^\beta$ if we choose $f=1\in\mathcal{B}^\alpha$. So we always assume that $\psi\in\mathcal{B}^\beta$. We are now ready to state and prove the main results in this section.

Theorem 2.2. *Let $0<\alpha<1$, $0<\beta<\infty$, $\psi\in H(\mathbb{D})$ and let φ be an analytic self-map of $\mathbb{D}$. Assume that $\psi\in\mathcal{B}^\beta$, then $\psi C_\varphi:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{n\in\mathbb{N}}\frac{\|\psi\varphi^n\|_\beta}{\|z^n\|_\alpha}<\infty. \tag{1}$$

Proof. Assume that $\psi C_\varphi:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded. Since for any nonnegative integer n, the sequence $f_n(z)=z^n/\|z^n\|_\alpha$ is bounded in $\mathcal{B}^\alpha$, we get

$$\frac{\|\psi\varphi^n\|_\beta}{\|z^n\|_\alpha}=\left\|\psi C_\varphi\left(\frac{z^n}{\|z^n\|_\alpha}\right)\right\|_\beta\le\|\psi C_\varphi\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta}<\infty.$$

The desired result follows.

Now we assume that (1) holds. Let $M:=\sup_{n\in\mathbb{N}}\frac{\|\psi\varphi^n\|_\beta}{\|z^n\|_\alpha}$. For $n\ge 2$, we define

$$\mathbb{D}_n=\{z\in\mathbb{D}: r_n\le|\varphi(z)|\le r_{n+1}\},$$

where $r_n=\sqrt{\frac{n-1}{n-1+2\alpha}}$. Fix an integer $N>2$. For $|\varphi(z)|\le r_N$, by the product rule, we have

$$\begin{aligned}\frac{(1-|z|^2)^\beta|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}&\le\frac{(1-|z|^2)^\beta(|(\psi\varphi)'(z)|+|\psi'(z)\varphi(z)|)}{(1-r_N^2)^\alpha}\\&\le\left(\frac{N-1+2\alpha}{2\alpha}\right)^\alpha(\|\psi\varphi\|_\beta+\|\psi\|_\beta)<\infty.\end{aligned}\tag{2}$$

Note that for $n\in\mathbb{N}$ and $0<\alpha<1$,

$$\begin{aligned}\|z^n\|_\alpha=\|z^n\|_{\mathcal{B}^\alpha}&=\max_{z\in\mathbb{D}}n|z^{n-1}|(1-|z|^2)^\alpha\\&=n\left(\frac{2\alpha}{n-1+2\alpha}\right)^\alpha\left(\frac{n-1}{n-1+2\alpha}\right)^{(n-1)/2}=n(1-r_n^2)^\alpha r_n^{n-1},\end{aligned}\tag{3}$$

the max is attained at any point on the circle with radius r_n.

For $|\varphi(z)|>r_N$, there exists $n\ge N$ such that $z\in\mathbb{D}_n$. So

$$\begin{aligned}\frac{(1-|z|^2)^\beta|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}&=\frac{(1-|z|^2)^\beta|\psi(z)\varphi(z)^{n-1}\varphi'(z)|\|z^n\|_\alpha}{(1-|\varphi(z)|^2)^\alpha|\varphi(z)|^{n-1}\|z^n\|_\alpha}\\&\le\frac{(1-|z|^2)^\beta|\psi(z)\varphi(z)^{n-1}\varphi'(z)|n(1-r_n^2)^\alpha r_n^{n-1}}{(1-r_{n+1}^2)^\alpha r_n^{n-1}\|z^n\|_\alpha}\end{aligned}$$

$$\le 2^{\alpha} \frac{(1-|z|^2)^{\beta}\Big[|(\psi\varphi^n)'(z)| + |\psi'(z)\varphi^n(z)|\Big]}{\|z^n\|_{\alpha}}$$

$$\le 2^{\alpha} \frac{\|\psi\varphi^n\|_{\beta} + \|\psi\|_{\beta}}{\|z^n\|_{\alpha}} \le 2^{\alpha}\left(M + \frac{\|\psi\|_{\beta}}{\|z^n\|_{\alpha}}\right) < \infty, \tag{4}$$

where we apply the inequality $1 - r_n^2 < 2(1 - r_{n+1}^2),\ n \ge 2$ and the fact that $\|z^n\|_{\alpha} \to \infty$ as $n \to \infty$ when $0 < \alpha < 1$. From (2), (4) and Lemma 2.1, we deduce that ψC_{φ} is bounded from $\mathcal{B}^{\alpha}$ into $\mathcal{B}^{\beta}$. □

Let $\psi \equiv 1$. We get the following result, which appeared in Theorem 2.1 of [14].

Corollary 2.3. *Let $0 < \alpha < 1$, $0 < \beta < \infty$ and let φ be an analytic self-map of $\mathbb{D}$. The $C_{\varphi} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is bounded if and only if*

$$\sup_{n \in \mathbb{N}} \frac{\|\varphi^n\|_{\beta}}{\|z^n\|_{\alpha}} < \infty.$$

3. Essential norm of ψC_{φ} on Bloch-type spaces

The following criterion for compactness follows by a standard argument similar, for example, to that outlined in Proposition 3.11 of [3].

Lemma 3.1. *Let $0 < \alpha < 1$, $0 < \beta < \infty$, $\psi \in H(\mathbb{D})$ and let φ be an analytic self-map of $\mathbb{D}$. The operator $\psi C_{\varphi} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is compact if and only if for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}^{\alpha}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|\psi C_{\varphi} f_n\|_{\mathcal{B}^{\beta}} \to 0$ as $n \to \infty$.*

Denote by $K_r f(z) = f(rz)$ for $r \in (0,1)$. Then K_r is a compact operator on the space $\mathcal{B}^{\alpha}$ for $\alpha > 0$. It is easy to see that $\|K_r\| \le 1$. Let I denote the identity operator. The following result can be found in [6].

Lemma 3.2. *Let $0 < \alpha < 1$. There is a sequence $\{r_k\}$, with $0 < r_k < 1$ tending to 1, such that the compact operator $L_n = \frac{1}{n}\sum_{k=1}^{n} K_{r_k}$ on $\mathcal{B}_0^{\alpha}$ satisfies*

(i) *For any $t \in (0,1)$,* $\lim_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^{\alpha}} \le 1} \sup_{|z| \le t} |((I - L_n)f)'(z)| = 0.$

(ii) $\lim_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^{\alpha}} \le 1} \sup_{z \in \mathbb{D}} |(I - L_n)f(z)| = 0.$

(iii) $\limsup_{n\to\infty} \|I - L_n\| \le 1.$

Furthermore, these statements holds as well for the sequence of biadjoints L_n^{**} (which is the same form as L_n) on $\mathcal{B}^{\alpha}$.

Theorem 3.3. *Let $0 < \alpha < 1$, $0 < \beta < \infty$, $\psi \in H(\mathbb{D})$ and let φ be an analytic self-map of $\mathbb{D}$. Suppose that $\psi C_{\varphi} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is bounded, then the essential norm of $\psi C_{\varphi} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is*

$$\|\psi C_{\varphi}\|_e^{\mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}} = \limsup_{n\to\infty} \frac{\|\psi\varphi^n\|_{\beta}}{\|z^n\|_{\alpha}}. \tag{5}$$

Proof. By the assumption that ψC_φ is bounded from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$, we easily get that $\|\psi\varphi\|_\beta < \infty$ and $\|\psi\|_\beta < \infty$. Moreover, $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi(z)\varphi'(z)| < \infty$.

First, we prove that (5) holds when $\sup_{z\in\mathbb{D}}|\varphi(z)| < 1$. Assume $\sup_{z\in\mathbb{D}}|\varphi(z)| < 1$, then there is a number $\delta \in (0,1)$ such that $\sup_{z\in\mathbb{D}}|\varphi(z)| < \delta$. Choose a bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}^\alpha$ which converges to zero uniformly on compact subset of $\mathbb{D}$. Then $\{f_n'\}_{n\in\mathbb{N}}$ also converges to zero on compact subsets of $\mathbb{D}$ as $n \to \infty$. It follows that

$$\begin{aligned}\lim_{n\to\infty}\|\psi C_\varphi f_n\|_{\mathcal{B}^\beta} &= \lim_{n\to\infty}\left(|\psi(0)f_n(\varphi(0))| + \|\psi C_\varphi f_n\|_\beta\right)\\ &= \lim_{n\to\infty}\sup_{z\in\mathbb{D}}|\psi'(z)f_n(\varphi(z)) + \psi(z)f_n'(\varphi(z))\varphi'(z)|(1-|z|^2)^\beta\\ &\le \lim_{n\to\infty}\left(\|\psi\|_\beta \sup_{|z|\le\delta}|f_n(z)| + \sup_{|z|\le\delta}|f_n'(z)|\sup_{z\in\mathbb{D}}|\psi\varphi'|(1-|z|^2)^\beta\right)\\ &= 0.\end{aligned}$$

Then the operator $\psi C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact by Lemma 3.1. This gives that $\|\psi C_\varphi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} = 0$. On the other hand,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{\|\psi\varphi^n\|_\beta}{\|z^n\|_\alpha}\\ =\ &\limsup_{n\to\infty}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\psi'(z)\varphi^n(z) + n\psi(z)\varphi^{n-1}(z)\varphi'(z)|}{\|z^n\|_\alpha}\\ \le\ &\limsup_{n\to\infty}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta(|\psi'(z)\varphi^n(z)| + n|\psi(z)\varphi^{n-1}(z)\varphi'(z)|)}{\|z^n\|_\alpha}\\ \le\ &\limsup_{n\to\infty}\left(\frac{\|\psi\|_\beta\delta^n}{\|z^n\|_\alpha} + \sup_{z\in\mathbb{D}}\frac{n|(\psi\varphi)'(z) - \psi'(z)\varphi(z)|\delta^{n-1}}{\|z^n\|_\alpha}(1-|z|^2)^\beta\right)\\ \le\ &\limsup_{n\to\infty}\frac{\|\psi\|_\beta\delta^n + n\|\psi\|_\beta\delta^n + n\|\psi\varphi\|_\beta\delta^{n-1}}{\|z^n\|_\alpha} = 0.\end{aligned}$$

Thus, when $\sup_{z\in\mathbb{D}}|\varphi(z)| < 1$, we have

$$\|\psi C_\varphi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} = \limsup_{n\to\infty}\frac{\|\psi\varphi^n\|_\beta}{\|z^n\|_\alpha} = 0.$$

Next, we prove that (5) holds when $\sup_{z\in\mathbb{D}}|\varphi(z)| = 1$.

We first give the lower estimate for the essential norm. Choose the sequence of function $f_n(z) = z^n/\|z^n\|_\alpha$, $n \in \mathbb{N}$. Then $\|f_n\|_{\mathcal{B}^\alpha} = 1$, and f_n converges to zero weakly on $\mathcal{B}^\alpha$ as $n \to \infty$. Thus we have $\lim_{n\to\infty}\|Kf_n\|_{\mathcal{B}^\alpha} = 0$ for any given compact operator K on $\mathcal{B}^\alpha$. The basic inequality gives that

$$\|\psi C_\varphi - K\|^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \ge \|(\psi C_\varphi - K)f_n\|_{\mathcal{B}^\beta} \ge \|\psi C_\varphi f_n\|_{\mathcal{B}^\beta} - \|Kf_n\|_{\mathcal{B}^\beta}.$$

Thus we obtain that

$$\|\psi C_\varphi - K\|^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \ge \limsup_{n\to\infty}\|\psi C_\varphi f_n\|_{\mathcal{B}^\beta} \ge \limsup_{n\to\infty}\|\psi C_\varphi f_n\|_\beta.$$

So we have

$$\|\psi C_\varphi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} = \inf_K \|\psi C_\varphi - K\| \geq \limsup_{n\to\infty} \frac{\|\psi\varphi^n\|_\beta}{\|z^n\|_\alpha}.$$

Next, still under the assumption that $\sup_{z\in\mathbb{D}}|\varphi(z)| = 1$, we give the upper estimate for the essential norm. Let L_n be the sequence of operators given in Lemma 3.2. Since L_n is compact on $\mathcal{B}^\alpha$ and $\psi C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, then $\psi C_\varphi L_n : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is also compact. Hence

$$\begin{aligned}
\|\psi C_\varphi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} &\leq \limsup_{n\to\infty} \|\psi C_\varphi - \psi C_\varphi L_n\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \\
&= \limsup_{n\to\infty} \|\psi C_\varphi(I - L_n)\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \\
&= \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \|\psi C_\varphi(I - L_n)f\|_{\mathcal{B}^\beta} \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} |\psi(0)((I - L_n)f)(\varphi(0))|$$

and

$$I_2 = \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \sup_{z\in\mathbb{D}} |(\psi(z)((I - L_n)f(\varphi(z))))'|(1 - |z|^2)^\beta.$$

It follows from Lemma 3.2 that $I_1 = 0$.

Let k be the smallest positive integer such that $\mathbb{D}_k \neq \emptyset$. Since $\sup_{z\in\mathbb{D}}|\varphi(z)| = 1$, $\mathbb{D}_n$ is not empty for every integer $n \geq k$ and $\mathbb{D} = \bigcup_{n=k}^\infty \mathbb{D}_n$. Then we have that $I_2 \leq I_{21} + I_{22} + I_{23}$, where

$$I_{21} = \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \sup_{z\in\mathbb{D}} |\psi'(z)||((I - L_n)f)(\varphi(z))|(1 - |z|^2)^\beta,$$

$$I_{22} = \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \sup_{N\leq i} \sup_{z\in\mathbb{D}_i} |\psi(z)((I - L_n)f)'(\varphi(z))\varphi'(z)|(1 - |z|^2)^\beta$$

and

$$I_{23} = \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \sup_{k\leq i\leq N-1} \sup_{z\in\mathbb{D}_i} |\psi(z)((I - L_n)f)'(\varphi(z))\varphi'(z)|(1 - |z|^2)^\beta.$$

Here N is a positive integer determined as follows.

From (3), for any given $\epsilon > 0$, there exists a positive integer N such that

$$\frac{\|\psi\|_\beta}{\|z^n\|_{\mathcal{B}^\alpha}} < \epsilon \text{ and } \frac{(1 - r_n^2)^\alpha}{(1 - r_{n+1}^2)^\alpha} < 1 + \epsilon, \tag{6}$$

when $n \geq N$.

For such N it follows by Lemma 3.2 that

$$\begin{aligned}
I_{22} &= \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \sup_{N\leq i} \sup_{z\in\mathbb{D}_i} |\psi(z)((I - L_n)f)'(\varphi(z))\varphi'(z)|(1 - |z|^2)^\beta \\
&\leq \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \|(I - L_n)f\|_\alpha \sup_{N\leq i} \sup_{z\in\mathbb{D}_i} \frac{|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha}(1 - |z|^2)^\beta
\end{aligned}$$

$$\le \limsup_{n\to\infty}\|I-L_n\|\sup_{N\le i}\sup_{z\in\mathbb{D}_i}\frac{|\psi(z)\varphi'(z)\varphi^{i-1}(z)|\|z^i\|_\alpha}{(1-|\varphi(z)|^2)^\alpha|\varphi(z)|^{i-1}\|z^i\|_\alpha}(1-|z|^2)^\beta$$

$$\le \sup_{N\le i}\sup_{z\in\mathbb{D}_i}\frac{i(1-r_i^2)^\alpha r_i^{i-1}|\psi(z)\varphi'(z)\varphi^{i-1}(z)|}{(1-r_{i+1}^2)^\alpha r_i^{i-1}\|z^i\|_\alpha}(1-|z|^2)^\beta$$

$$\le (1+\epsilon)\sup_{N\le i}\sup_{z\in\mathbb{D}_i}\frac{|(\psi\varphi^i)'(z)|+|\psi'(z)\varphi^i(z)|}{\|z^i\|_\alpha}(1-|z|^2)^\beta$$

$$\le (1+\epsilon)\sup_{N\le i}\frac{\|\psi\varphi^i\|_\beta+\|\psi\|_\beta}{\|z^i\|_\alpha}$$

$$\le (1+\epsilon)\left(\sup_{N\le i}\frac{\|\psi\varphi^i\|_\beta}{\|z^i\|_\alpha}+\epsilon\right). \tag{7}$$

Also it follows by Lemma 3.2 that

$$I_{23} = \limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{k\le i\le N-1}\sup_{z\in\mathbb{D}_i}|\psi(z)((I-L_n)f)'(\varphi(z))\varphi'(z)|(1-|z|^2)^\beta$$

$$= \limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{|\varphi(z)|\le r_N}|\psi(z)((I-L_n)f)'(\varphi(z))\varphi'(z)|(1-|z|^2)^\beta$$

$$\le \limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{|\varphi(z)|\le r_N}|((I-L_n)f)'(\varphi(z))|\cdot\sup_{z\in\mathbb{D}}|\psi(z)\varphi'(z)|(1-|z|^2)^\beta$$

$$= \limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{|w|\le r_N}|((I-L_n)f)'(w)|\cdot\sup_{z\in\mathbb{D}}|\psi(z)\varphi'(z)|(1-|z|^2)^\beta = 0. \tag{8}$$

By Lemma 3.2, we have

$$I_{21} \le \|\psi\|_\beta\limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{z\in\mathbb{D}}|((I-L_n)f)(\varphi(z))|$$

$$\le \|\psi\|_\beta\limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{w\in\mathbb{D}}|((I-L_n)f)(w)| = 0. \tag{9}$$

It follows from (7), (8) and (9) that

$$I_2 < (1+\epsilon)\left(\sup_{N\le i}\frac{\|\psi\varphi^i\|_\beta}{\|z^i\|_\alpha}+\epsilon\right). \tag{10}$$

From (10) we obtain that

$$\|\psi C_\varphi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \le I_1+I_2 < (1+\epsilon)\left(\sup_{N\le i}\frac{\|\psi\varphi^i\|_\beta}{\|z^i\|_\alpha}+\epsilon\right).$$

Since ϵ is arbitrary, it follows that

$$\|\psi C_\varphi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \le \limsup_{i\to\infty}\frac{\|\psi\varphi^i\|_\beta}{\|z^i\|_\alpha}.$$

The proof is complete. □

From Theorem 3.3, we obtain the following result.

Corollary 3.4. *Let $0 < \alpha < 1$, $0 < \beta < \infty$, $\psi \in H(\mathbb{D})$ and let φ be an analytic self-map of $\mathbb{D}$. Suppose that $\psi \in \mathcal{B}^\beta$. Then $\psi C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if*

$$\limsup_{n\to\infty} \frac{\|\psi\varphi^n\|_\beta}{\|z^n\|_\alpha} = 0.$$

Corollary 3.5. *Let $0 < \alpha < 1$, $0 < \beta < \infty$ and let φ be an analytic self-map of $\mathbb{D}$. Then the essential norm of $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is*

$$\|C_\varphi\|_e^{\mathcal{B}^\alpha \to \mathcal{B}^\beta} = \limsup_{n\to\infty} \frac{\|\varphi^n\|_\beta}{\|z^n\|_\alpha}.$$

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