Bull. Korean Math. Soc. **52** (2015), No. 5, pp. 1631–1647
http://dx.doi.org/10.4134/BKMS.2015.52.5.1631

OPTIMAL INEQUALITIES FOR THE CASORATI CURVATURES OF SUBMANIFOLDS OF GENERALIZED SPACE FORMS ENDOWED WITH SEMI-SYMMETRIC METRIC CONNECTIONS

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Abstract. In this paper, we prove two optimal inequalities involving the intrinsic scalar curvature and extrinsic Casorati curvature of submanifolds of generalized space forms endowed with a semi-symmetric metric connection. Moreover, we also characterize those submanifolds for which the equality cases hold.

1. Introduction

After Friedmann and Schouten [14] introduced the notion of a semi-symmetric linear connection on a differentiable manifold, Hayden [16] defined the notion of a semi-symmetric metric connection on a Riemannian manifold. Later, Yano [41] proved that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes if and only if the Riemannian manifold is conformally flat. In [18, 19], Imai found several interesting properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. Nakao showed in [30] that a metric semi-symmetric linear connection on a Riemannian manifold (N, g) induces a similar connection on a submanifold of (N, g), and he derives equations similar to those of Gauss and Codazzi-Mainardi, generalizing the work of Imai for hypersurfaces. In 2008, Tripathi [37] showed there a complete theory of connections which unifies the concepts of various metric connections such as a semi-symmetric metric connection and a quarter-symmetric metric connection and various non-metric connection such as the Weyl connection and different kind of semi-symmetric non-metric connections.

On the other hand, the theory of Chen invariants or δ-invariants, initiated by B.-Y. Chen [7] in a seminal paper published in 1993, is presently one of the most interesting research topic in differential geometry of submanifolds.

Received October 30, 2014; Revised January 5, 2015.
2010 *Mathematics Subject Classification.* 53C40, 49K35.
Key words and phrases. Casorati curvature, real space form, semi-symmetric metric connection.

Chen established optimal inequalities between the main intrinsic invariants and the main extrinsic invariants of a submanifold in real space forms with any codimension in [8]. Many interesting results concerned Chen invariants and inequalities were later obtained for different classes of submanifolds in various ambient spaces, like complex space forms [3, 9, 10, 11, 17, 22, 27, 31, 36, 38, 42]. Recently, in [28, 29], C. Özgür and A. Mihai proved Chen inequalities for submanifolds of real, complex, and Sasakian space forms endowed with semi-symmetric metric connections. Moreover, P. Zhang, L. Zhang and W. Song [43] obtained Chen-like inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature endowed with a semi-symmetric metric connection by using an algebraic approach.

Instead of concentrating on the sectional curvature with the extrinsic squared mean curvature, the Casorati curvature of a submanifold in a Riemannian manifold was considered as an extrinsic invariant defined as the normalized square of the length of the second fundamental form. The notion of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold. Several geometers in [1, 6, 15, 39, 40] found geometrical meaning and the importance of the Casorati curvature. Therefore, it is of great interest to obtain optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces. Recently, some optimal inequalities involving Casorati curvatures were proved in [12, 13, 23, 34] for submanifolds in ambient space forms. As a natural prolongation of our research, in this paper we will study these inequalities for submanifolds in generalized space forms, endowed with semi-symmetric metric connections.

2. Preliminaries

Let N^m be an m-dimensional Riemannian manifold and $\widetilde{\nabla}$ a linear connection on N^m. If the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$, defined by

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}],$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^m, satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \phi(\widetilde{Y})\widetilde{X} - \phi(\widetilde{X})\widetilde{Y}$$

for a 1-form ϕ, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection.*

Let g be a Riemannian metric on N^m. If $\widetilde{\nabla}g = 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* on N^m. Following [41], a semi-symmetric metric connection $\widetilde{\nabla}$ on N^m is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \mathring{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \phi(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})P,$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^m, where $\mathring{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and P is a vector field defined by $g(P,\widetilde{X}) = \phi(\widetilde{X})$, for any vector field $\tilde{X}$.

We will consider a Riemannian manifold N^m endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\mathring{\widetilde{\nabla}}$. Let M^n be an n-dimensional submanifold of an m-dimensional Riemannian manifold N^m. On the submanifold M^n, we consider the induced semi-symmetric metric connection, denoted by ∇ and the induced Levi-Civita connection, denoted by $\mathring{\nabla}$. Let $\widetilde{R}$ be the curvature tensor of N^m with respect to $\widetilde{\nabla}$ and $\mathring{\widetilde{R}}$ the curvature tensor of N^m with respect to $\mathring{\widetilde{\nabla}}$. We also denote by R and $\mathring{R}$ the curvature tensors of ∇ and $\mathring{\nabla}$, respectively, on M^n. The Gauss formulae with respect to ∇ and $\mathring{\nabla}$, respectively, can be written as

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y),\ X,\ Y \in \chi(M^n),$$

$$\mathring{\widetilde{\nabla}}_X Y = \mathring{\nabla}_X Y + \mathring{h}(X,Y),\ X,\ Y \in \chi(M^n),$$

where $\mathring{h}$ is the second fundamental form of M^n in N^m and h is a $(0,2)$-tensor on M^n. According to the formula (7) from [30], h is also symmetric. One denotes by $\mathring{H}$ the mean curvature vector of M^n in N^m. Let $N^m(c)$ be a real space form of constant sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$.

From [30], the Gauss equation for the submanifold M^n into the real space form $N^m(c)$ is

$$\begin{aligned}\mathring{\widetilde{R}}(X,Y,Z,W) = \mathring{R}(X,Y,Z,W) &+ g\left(\mathring{h}(X,Z),\mathring{h}(Y,W)\right)\\ &- g\left(\mathring{h}(X,W),\mathring{h}(Y,Z)\right).\end{aligned} \tag{2.1}$$

The curvature tensor $\widetilde{R}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on N^m can be written as (see [19])

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = \mathring{\widetilde{R}}(X,Y,Z,W) &- \alpha(Y,Z)g(X,W)\\ &+ \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z)\\ &+ \alpha(Y,W)g(X,Z)\end{aligned} \tag{2.2}$$

for any vector fields $X,\ Y,\ Z,\ W \in \chi(M^n)$, where α is a $(0,2)$-tensor field defined by

$$\alpha(X,Y) = \left(\mathring{\widetilde{\nabla}}_X \phi\right) Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X,Y),\ \forall\ X,\ Y \in \chi(M^n).$$

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection ∇. For any orthonormal basis $\{e_1,\dots,e_n\}$ of the tangent space T_xM^n and $\{e_{n+1},\dots,e_m\}$ an orthonormal basis of the normal space $T_x^{\perp}M$, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1\le i<j\le n} K\left(e_i \wedge e_j\right),$$

and the normalized scalar curvature ρ of M is defined by

$$\rho = \frac{2\tau}{n(n-1)}.$$

We denote by H the mean curvature vector, that is,

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

and we also set

$$h_{ij}^{\alpha} = g\left(h(e_i, e_j), e_\alpha\right), \quad i, j \in \{1, \ldots, n\}, \quad \alpha \in \{n+1, \ldots, m\}.$$

Then, the squared mean curvature of the submanifold M in N is defined by

$$||H||^2 = \frac{1}{n^2}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2,$$

and the squared norm of h over dimension n is denoted by $\mathcal{C}$ and is called the *Casorati curvature* of the submanifold M. Therefore, we have

$$\mathcal{C} = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{\alpha}\right)^2.$$

The submanifold M is called *invariantly quasi-umbilical* if there exist $m-n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \ldots, \xi_m$ such that the shape operators with respect to all directions ξ_α have an eigenvalue of multiplicity $n-1$ and that for each ξ_α the distinguished eigendirection is the same [5].

Suppose now that L is an r-dimensional subspace of T_xM, $r \geq 2$, and $\{e_1, \ldots, e_r\}$ is an orthonormal basis of L. Then, the scalar curvature $\tau(L)$ of the r-plane section L is given by

$$\tau(L) = \sum_{1\leq\alpha<\beta\leq r} K\left(e_\alpha \wedge e_\beta\right),$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{r}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{r}\left(h_{ij}^{\alpha}\right)^2.$$

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ are given by

$$[\delta_c(n-1)]_x = \frac{1}{2}\mathcal{C}_x + \frac{n+1}{2n}\inf\{\mathcal{C}(L) \mid L : \text{a hyperplane of } T_xM\},$$

and

$$[\hat{\delta}_c(n-1)]_x = 2\mathcal{C}_x - \frac{2n-1}{2n}\sup\{\mathcal{C}(L) \mid L : \text{a hyperplane of } T_xM\}.$$

3. Casorati curvatures of submanifolds in generalized complex space forms with semi-symmetric metric connections

The concept of generalized complex space form has been introduced by F. Tricerri and L. Vanhecke [35] as a natural generalization of the notion of complex space form. An almost Hermitian manifold (N, J, g) is said to be a *generalized complex space form* if there exist two functions f_1 and f_2 on N such that

$$\overset{\circ}{\widetilde{R}}(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\} \tag{3.1}$$

for any vector fields X, Y, Z on N, where $\overset{\circ}{\widetilde{R}}$ denotes the curvature tensor of N (see [35]). In such a case, we will write $N(f_1, f_2)$. Many authors have studied these manifolds and their submanifolds. For example, one main reference concerning these spaces is [35], in which F. Tricerri and L. Vanhecke established an important obstruction for their existence in dimensions greater than or equal to 6. In fact, in these dimensions, a generalized complex space form reduces to a real space form or a complex space form. Moreover, as proved by R. Lemence [24], the result of F. Tricerri and L. Vanhecke is partially extendable to 4-dimensional case under compactness hypothesis. Nevertheless, Olszak provided some interesting examples of 4-dimensional generalized complex space forms with non-constant functions in [32].

If $N(f_1, f_2)$ is a generalized complex space form with a semi-symmetric metric connection $\widetilde{\nabla}$, then from (2.2) and (3.1), the curvature tensor $\widetilde{R}$ of $N(f_1, f_2)$ can be expressed as

$$\begin{aligned} &\widetilde{R}(X,Y,Z,W) \\ &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + f_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\} \\ &\quad + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z). \end{aligned} \tag{3.2}$$

Let M^n $(n \geq 3)$ be an n-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(f_1, f_2)$. For any tangent vector field X to M, we put

$$JX = PX + FX,$$

where PX and FX are the tangential and normal components of JX, respectively. We define

$$||P||^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j).$$

Theorem 3.1. *Let M^n be a submanifold of a generalized complex space form $N(f_1, f_2)$ with a semi-symmetric metric connection. Then*

(i) *The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies*

$$\rho \leq \delta_C(n-1) + f_1 + \frac{3f_2}{n(n-1)}||P||^2 - \frac{2}{n}\mathrm{trace}(\alpha).$$

Moreover, the equality sign holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $N(f_1, f_2)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \ldots, \xi_n\}$ and normal orthonormal frame $\{\xi_{n+1}, \ldots, \xi_m\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \ldots, m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2a \end{pmatrix}, \ A_{n+2} = \cdots = A_m = 0.$$

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_C(n-1)$ satisfies*

$$\rho \leq \widehat{\delta}_C(n-1) + f_1 + \frac{3f_2}{n(n-1)}||P||^2 - \frac{2}{n}\mathrm{trace}(\alpha).$$

Moreover, the equality sign holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $N(f_1, f_2)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \ldots, \xi_n\}$ and normal orthonormal frame $\{\xi_{n+1}, \ldots, \xi_m\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \ldots, m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2a & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & 0 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}, \ A_{n+2} = \cdots = A_m = 0.$$

Proof. (i) Let $x \in M^n$ and $\{e_1, e_2, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_m\}$ be orthonormal bases of T_xM^n and $T_x^{\perp}M^n$, respectively. For $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, from the equation (3.2), it follows that

$$\widetilde{R}(e_i, e_j, e_j, e_i) = f_1 + 3f_2g^2(Je_i, e_j) - \alpha(e_i, e_i) - \alpha(e_j, e_j). \tag{3.3}$$

From (3.3) and the Gauss equation with respect to the semi-symmetric metric connection, we get

$$\begin{aligned} &f_1 + 3f_2g^2(Je_i, e_j) - \alpha(e_i, e_i) - \alpha(e_j, e_j) \\ &= R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)). \end{aligned}$$

By summation over $1 \leq i, j \leq n$, it follows from the previous relation that

$$2\tau = n^2||H||^2 - n\mathcal{C} + (n^2 - n)f_1 + 3f_2 \sum_{i,j=1}^{n} g^2(Je_i, e_j) - 2(n-1)\mathrm{trace}(\alpha). \tag{3.4}$$

We define now the following function, denoted by $\mathcal{P}$, which is a quadratic polynomial in the components of the second fundamental form:

$$\begin{aligned}\mathcal{P} = {} & \frac{1}{2}n(n-1)\mathcal{C} + \frac{1}{2}(n+1)(n-1)\mathcal{C}(L) - 2\tau + (n^2 - n)f_1 \\ & + 3f_2||P||^2 - 2(n-1)\mathrm{trace}(\alpha).\end{aligned}$$

Without loss of generality, by assuming that L is spanned by $e_1, \ldots, e_{n-1}$, it derives that

$$\begin{aligned}\mathcal{P} = {} & \frac{n+1}{2} \sum_{\alpha=n+1}^{m} \left(\sum_{i,j=1}^{n} \left(h_{i,j}^{\alpha}\right)^2 \right) + \frac{n+1}{2} \sum_{\alpha=n+1}^{m} \left(\sum_{i,j=1}^{n-1} \left(h_{i,j}^{\alpha}\right)^2 \right) \\ & - \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2\end{aligned}$$

and now we obtain easily that

$$\begin{aligned}\mathcal{P} = {} & \sum_{\alpha=n+1}^{m} \sum_{i=1}^{n-1} \left[n \left(h_{ii}^{\alpha}\right)^2 + (n+1)\left(h_{in}^{\alpha}\right)^2 \right] \\ & + \sum_{\alpha=n+1}^{m} \left[2(n+1) \sum_{i<j=1}^{n-1} \left(h_{ij}^{\alpha}\right)^2 - 2 \sum_{i<j=1}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} + \frac{n-1}{2} \left(h_{nn}^{\alpha}\right)^2 \right].\end{aligned} \tag{3.5}$$

From (3.5), it follows that the critical points

$$h^c = \left(h_{11}^{n+1}, h_{12}^{n+1}, \ldots, h_{nn}^{n+1}, \ldots, h_{11}^{m}, \ldots, h_{nn}^{m}\right)$$

of $\mathcal{P}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{cases} \dfrac{\partial \mathcal{P}}{\partial h_{ii}^{\alpha}} = 2(n+1)h_{ii}^{\alpha} - 2\displaystyle\sum_{k=1}^{n} h_{kk}^{\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial h_{nn}^{\alpha}} = (n-1)h_{nn}^{\alpha} - 2\displaystyle\sum_{k=1}^{n-1} h_{kk}^{\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial h_{ij}^{\alpha}} = 4(n+1)h_{ij}^{\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial h_{in}^{\alpha}} = 2(n+1)h_{in}^{\alpha} = 0, \end{cases} \tag{3.6}$$

with $i, j \in \{1, \ldots, n-1\}$, $i \neq j$ and $\alpha \in \{n+1, \ldots, m\}$. Thus, every solution h^c has $h_{ij}^{\alpha} = 0$ for $i \neq j$, and the determinant which corresponds to the first two sets of equations of the above system is zero (there exist solutions for non-totally geodesic submanifolds). Moreover, it is easy to see that the Hessian matrix of $\mathcal{P}$ has the form

$$\mathcal{H}(\mathcal{P})=\begin{pmatrix} H_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & H_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & H_3\end{pmatrix},$$

where

$$H_1=\begin{pmatrix} 2n & -2 & \cdots & -2 & -2\\ -2 & 2n & \cdots & -2 & -2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -2 & -2 & \cdots & 2n & -2\\ -2 & -2 & \cdots & -2 & n-1\end{pmatrix},$$

$\mathbf{0}$ denotes the null matrix of corresponding dimensions and H_2, H_3 are the next diagonal matrices

$$H_2=\operatorname{diag}\left(4(n+1),4(n+1),\ldots,4(n+1)\right),$$

$$H_3=\operatorname{diag}\left(2(n+1),2(n+1),\ldots,2(n+1)\right).$$

Therefore, we find that $\mathcal{H}(\mathcal{P})$ has the following eigenvalues:

$$\lambda_{11}=0,\ \lambda_{22}=n+3,\ \lambda_{33}=\cdots=\lambda_{nn}=2(n+1),$$

$$\lambda_{ij}=4(n+1),\ \lambda_{in}=2(n+1),\ \forall i,j\in\{1,\ldots,n-1\},\ i\neq j.$$

Therefore, $\mathcal{P}$ is parabolic and reaches a minimum $\mathcal{P}(h^c)=0$ for the solution h^c of the system (3.6). It follows $\mathcal{P}\geq 0$, and hence,

$$2\tau\leq\frac{1}{2}n(n-1)\mathcal{C}+\frac{1}{2}(n+1)(n-1)\mathcal{C}(L)+(n^2-n)f_1+3f_2||P||^2-2(n-1)\operatorname{trace}(\alpha).$$

Hence, we deduce that

$$\rho\leq\frac{1}{2}\mathcal{C}+\frac{n+1}{2n}\mathcal{C}(L)+f_1+\frac{3f_2}{n(n-1)}||P||^2-\frac{2}{n}\operatorname{trace}(\alpha)$$

for every tangent hyperplane L of M. Taking the infimum over all tangent hyperplanes L, the conclusion trivially follows.

Moreover, we can easily check that the equality sign holds in the theorem if and only if

$$h^\alpha_{ij}=0,\quad \forall i,j\in\{1,\ldots,n\},\ i\neq j \text{ and } \alpha\in\{n+1,\ldots,m\} \tag{3.7}$$

and

$$h^\alpha_{nn}=2h^\alpha_{11}=\cdots=2h^\alpha_{n-1\ n-1},\ \forall\alpha\in\{n+1,\ldots,m\}. \tag{3.8}$$

From (3.7) and (3.8), we conclude that the equality holds if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in N, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the forms below.

$$A_{n+1}=\begin{pmatrix} 2a & 0 & 0 & \cdots & 0 & 0\\ 0 & 2a & 0 & \cdots & 0 & 0\\ 0 & 0 & 2a & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & a & \cdots & 2a & 0\\ 0 & 0 & a & \cdots & 0 & a \end{pmatrix},\quad A_{n+2}=\cdots=A_m=0. \tag{3.9}$$

(ii) can be proved in a similar way, considering the following quadratic polynomial in the components of the second fundamental form:

$$\begin{aligned}\mathcal{Q} &= 2n(n-1)\mathcal{C}-\frac{1}{2}(2n-1)(n-1)\mathcal{C}(L)-2\tau+(n^2-n)f_1+3f_2||P||^2\\ &\quad -2(n-1)\mathrm{trace}(\alpha),\end{aligned}$$

where L is a hyperplane of T_pM.

Similarly, as in the proof of (i), it follows that $\mathcal{Q}\geq 0$, and hence,

$$\begin{aligned}2\tau &\leq 2n(n-1)\mathcal{C}-\frac{1}{2}(2n-1)(n-1)\mathcal{C}(L)+\ (n^2-n)f_1+3f_2||P||^2\\ &\quad -2(n-1)\mathrm{trace}(\alpha).\end{aligned}$$

Therefore, we deduce that

$$\rho\ \leq\ 2\mathcal{C}-\frac{2n-1}{2n}\mathcal{C}(L)+f_1+\frac{3f_2}{n(n-1)}||P||^2-\frac{2}{n}\mathrm{trace}(\alpha)$$

for every tangent hyperplane L of M. Taking the supremum over all tangent hyperplanes L, the conclusion trivially follows. □

Corollary 3.2. *Let M^n be a submanifold of a complex space form $N(c)$ with a semi-symmetric metric connection. Then*

(i) *The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies*

$$\rho\leq\delta_C(n-1)+\frac{c}{4}+\frac{3c}{4n(n-1)}||P||^2-\frac{2}{n}\mathrm{trace}(\alpha).$$

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_C(n-1)$ satisfies*

$$\rho\leq\widehat{\delta}_C(n-1)+\frac{c}{4}+\frac{3c}{4n(n-1)}||P||^2-\frac{2}{n}\mathrm{trace}(\alpha).$$

4. Casorati curvatures of submanifolds in generalized Sasakian space forms with semi-symmetric metric connections

A $(2m+1)$-dimensional Riemannian manifold $\left(N^{2m+1},g\right)$ has an almost contact metric structure if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying:

$$\varphi^2X=-X+\eta(X)\xi,\ \eta(\xi)=1$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$g(X,\xi) = \eta(X),$$

for any vector fields X, Y over N. Let Φ denotes the fundamental 2-form in N, given by $\Phi(X,Y) = g(X,\varphi Y)$ for all X, Y over N. If $\Phi = d\eta$, then N is called a *contact metric manifold*. The structure of N is called *normal* if

$$[\varphi,\varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. A *Sasakian manifold* is a normal contact metric manifold.

In [33], J. A. Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold $\tilde{M}$ is a *trans-Sasakian manifold* if there exist two functions α and β on $\tilde{M}$ such that

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{4.1}$$

for any vector fields X, Y on $\tilde{M}$. In particular, from (2.2) it is easy to see that the following equations hold for a trans-Sasakian manifold:

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \quad d\eta = \alpha\Phi. \tag{4.2}$$

In particular, if $\beta = 0$, $\tilde{M}$ is said to be an *α-Sasakian manifold*. Sasakian manifolds appear as examples of α-Sasakian manifolds with $\alpha = 1$.

Another important kind of trans-Sasakian manifolds is that of *cosymplectic manifolds*, obtained for $\alpha = \beta = 0$. In fact, it can be proved that this definition is equivalent to $\tilde{M}$ being normal with η and Φ closed forms; cosymplectic manifolds were defined this way in [4]. From (4.2), we have $\nabla_X \xi = 0$.

On the other hand, if $\alpha = 0$, $\tilde{M}$ is said to be a *β-Kenmotsu manifold*. Kenmotsu manifolds, defined in [21], are particular examples with $\beta = 1$. Actually, in [26] J. C. Marrero showed that a trans-Sasakian manifold of dimension greater or equal than 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.

Given an almost contact metric manifold (N,ϕ,ξ,η,g), a ϕ-section of N is section $\pi \subseteq T_pN$ spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\tilde{K}(\pi)$ with respect to a ϕ-section π is called a ϕ-sectional curvature. If a Sasakian manifold $\tilde{M}$ has constant ϕ-sectional curvature c, N is called a *Sasakian space form* and is denoted by $N(c)$.

On the other hand, the curvature tensor $\mathring{\tilde{R}}$ of a Sasakian space form $N(c)$ is given by

$$\begin{aligned}\mathring{\tilde{R}}(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & + \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & \qquad + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\end{aligned} \tag{4.3}$$

for any tangent vector fields X, Y, Z to $N(c)$. More generally, if the curvature tensor of an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ satisfies

$$\begin{aligned}(4.4)\qquad \mathring{\tilde{R}}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\}\\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\\ & \quad + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},\end{aligned}$$

f_1, f_2, f_3 being differential functions on N, then $\tilde{M}$ is said to be a *generalized Sasakian space form* (see [2]). In such a case, we will write $N(f_1, f_2, f_3)$.

This kind of a manifold appears as a natural generalization of the well-know Sasakian space forms $N(c)$, which can be obtained as particular cases of generalized Sasakian space forms, by taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$. Moreover, we can also find some other trivial examples:

Example 4.1. A cosymplectic space form (i.e., a cosymplectic manifold with constant ϕ-sectional curvature c) is a generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$ (see [25]).

Example 4.2. A Kenmotsu space form, i.e., a Kenmotsu manifold with constant ϕ-sectional curvature c, is a generalized Sasakian space form with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$ (see e.g. [21]).

Example 4.3. An almost contact metric manifold is said to be an *almost $C(\alpha)$-manifold* ([20]) if its Riemannian curvature tensor satisfies

$$\begin{aligned}\mathring{\tilde{R}}(X,Y,Z,W) = {} & \mathring{\tilde{R}}(X,Y,\phi Z,\phi W) + \alpha\{g(X,W)g(Y,Z)\\ & - g(X,Z)g(Y,W) + g(X,\phi Z)g(Y,\phi W) - g(X,\phi W)g(Y,\phi Z)\}\end{aligned}$$

for any vector fields X, Y, Z, W on N, where α is a real number. Moreover, if such a manifold has constant ϕ-sectional curvature equal to c, then its curvature tensor is given by

$$\begin{aligned}(4.5)\qquad \mathring{\tilde{R}}(X,Y)Z = {} & \frac{c+3\alpha^2}{4}\{g(Y,Z)X - g(X,Z)Y\}\\ & + \frac{c-\alpha^2}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\\ & + \frac{c-\alpha^2}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\\ & \quad + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},\end{aligned}$$

and hence, it is a generalized Sasakian space form with $f_1 = \frac{c+3\alpha^2}{4}$ and $f_2 = f_3 = \frac{c-\alpha^2}{4}$.

As we have seen from the previous examples, we found generalized Sasakian space forms with very different structures.

If $N(f_1, f_2, f_3)$ is a $(2m+1)$-dimensional generalized Sasakian space form of the constant φ-sectional curvature c with a semi-symmetric metric connection $\widetilde{\nabla}$, then from (2.2) and (4.1), the curvature tensor $\mathring{\widetilde{R}}$ of $N(f_1, f_2, f_3)$ can be expressed as

$$
\begin{aligned}
\mathring{\widetilde{R}}(X,Y,Z,W) = {} & f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\
& + f_2\{g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\
& + 2g(X,\phi Y)g(\phi Z,W)\} \\
& + f_3\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\
& + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)\} \\
& - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) \\
& - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).
\end{aligned} \tag{4.6}
$$

Let M^n $(n \geq 3)$ be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ of constant φ-sectional curvature c. For any tangent vector field X to M, we put

$$\varphi X = PX + FX,$$

where PX and FX are tangential and normal components of φX, respectively, and we decompose

$$\xi = \xi^\top + \xi^\perp,$$

where $\xi^\top$ and $\xi^\perp$ denote the tangential part and the normal part of ξ, respectively.

Theorem 4.1. *Let M^n be a submanifold of a generalized Sasakian space form $N(f_1, f_2, f_3)$ with a semi-symmetric metric connection. Then*

(i) *The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies*

$$\rho \leq \delta_C(n-1) + f_1 + \frac{3f_2}{n(n-1)}||P||^2 - \frac{2f_3}{n}||\xi^\top||^2 - \frac{2}{n}\mathrm{trace}(\alpha).$$

Moreover, the equality sign holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $N(f_1, f_2, f_3)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \ldots, \xi_n\}$ and normal orthonormal frame $\{\xi_{n+1}, \ldots, \xi_{2m+1}\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \ldots, 2m+1\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2a \end{pmatrix}, \ A_{n+2} = \cdots = A_{2m+1} = 0.$$

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_C(n-1)$ satisfies*

$$\rho \leq \widehat{\delta}_C(n-1) + f_1 + \frac{3f_2}{n(n-1)}||P||^2 - \frac{2f_3}{n}||\xi^\top||^2 - \frac{2}{n}\mathrm{trace}(\alpha).$$

Moreover, the equality sign holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $N(f_1, f_2, f_3)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \ldots, \xi_n\}$ and normal orthonormal frame $\{\xi_{n+1}, \ \ldots, \xi_{2m+1}\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \ldots, 2m+1\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2a & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & 0 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}, \ A_{n+2} = \cdots = A_{2m+1} = 0.$$

Proof. (i) Let $x \in M^n$ and $\{e_1, e_2, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ be orthonormal bases of T_xM^n and $T_x^\perp M^n$, respectively. For $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, from the equation (4.2), it follows that

$$\widetilde{R}(e_i, e_j, e_j, e_i) = f_1 + 3f_2 g^2(Pe_j, e_i) - f_3\left(\eta(e_i)^2 + \eta(e_j)^2\right) - \alpha(e_i, e_i) - \alpha(e_j, e_j). \tag{4.7}$$

From (4.3) and the Gauss equation with respect to the semi-symmetric metric connection, we get

$$\begin{aligned} &f_1 + 3f_2 g^2(Pe_j, e_i) - f_3\left(\eta(e_i)^2 + \eta(e_j)^2\right) - \alpha(e_i, e_i) - \alpha(e_j, e_j) \\ &= R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)). \end{aligned}$$

By summation over $1 \leq i, j \leq n$, it follows from the previous relation that

$$\begin{aligned} 2\tau = {} & n^2||H||^2 - n\mathcal{C} + n(n-1)f_1 \\ & + 3f_2||P||^2 - 2(n-1)f_3||\xi^\top||^2 - 2(n-1)\mathrm{trace}(\alpha). \end{aligned} \tag{4.8}$$

We define now the following function, denoted by $\mathcal{P}$, which is a quadratic polynomial in the components of the second fundamental form:

$$\begin{aligned} \mathcal{P} = {} & \frac{1}{2}n(n-1)\mathcal{C} + \frac{1}{2}(n+1)(n-1)\mathcal{C}(L) - 2\tau + n(n-1)f_1 \\ & + 3f_2||P||^2 - 2(n-1)f_3||\xi^\top||^2 - 2(n-1)\mathrm{trace}(\alpha). \end{aligned}$$

Similarly, as in the proof of Theorem 3.1 it follows that $\mathcal{P} \geq 0$, and hence,

$$\begin{aligned} 2\tau \leq {} & \frac{1}{2}n(n-1)\mathcal{C} + \frac{1}{2}(n+1)(n-1)\mathcal{C}(L) + n(n-1)f_1 \\ & + 3f_2||P||^2 - 2(n-1)f_3||\xi^\top||^2 - 2(n-1)\mathrm{trace}(\alpha). \end{aligned}$$

Therefore, we deduce that

$$\rho \ \leq \ \frac{1}{2}\mathcal{C} + \frac{n+1}{2n}\mathcal{C}(L) + f_1 + \frac{3f_2}{n(n-1)}||P||^2 - \frac{2f_3}{n}||\xi^\top||^2 - \frac{2}{n}\mathrm{trace}(\alpha)$$

for every tangent hyperplane L of M. Taking the infimum over all tangent hyperplanes L, the conclusion trivially follows. Moreover, with the same argument as in the proof of Theorem 3.1, we conclude that the equality holds if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in N, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the desired forms.

(ii) can be proved in a similar way, considering the following quadratic polynomial in the components of the second fundamental form:

$$\begin{aligned}\mathcal{Q} = 2n(n-1)\mathcal{C} - \frac{1}{2}(2n-1)(n-1)\mathcal{C}(L) - 2\tau + n(n-1)f_1 \\ + 3f_2||P||^2 - 2(n-1)f_3||\xi^\top||^2 - 2(n-1)\mathrm{trace}(\alpha),\end{aligned}$$

where L is a hyperplane of T_pM. □

Corollary 4.2. *We have the following table*:

Manifold	**Inequalities for Casorati curvatures of submanifold** M^n
$N(c)$	$\rho \leq \delta_C(n-1) + \frac{c+3}{4} + \frac{3(c-1)}{4n(n-1)}\|\|P\|\|^2 - \frac{c-1}{2n}\|\|\xi^\top\|\|^2 - \frac{2}{n}\mathrm{trace}(\alpha)$
	$\rho \leq \widehat{\delta}_C(n-1) + \frac{c+3}{4} + \frac{3(c-1)}{4n(n-1)}\|\|P\|\|^2 - \frac{c-1}{2n}\|\|\xi^\top\|\|^2 - \frac{2}{n}\mathrm{trace}(\alpha)$
$N_{cosy}(c)$	$\rho \leq \delta_C(n-1) + \frac{c}{4} + \frac{3c}{4n(n-1)}\|\|P\|\|^2 - \frac{c}{2n}\|\|\xi^\top\|\|^2 - \frac{2}{n}\mathrm{trace}(\alpha)$
	$\rho \leq \widehat{\delta}_C(n-1) + \frac{c}{4} + \frac{3c}{4n(n-1)}\|\|P\|\|^2 - \frac{c}{2n}\|\|\xi^\top\|\|^2 - \frac{2}{n}\mathrm{trace}(\alpha)$
$N_{Ken}(c)$	$\rho \leq \delta_C(n-1) + \frac{c-3}{4} + \frac{3(c+1)}{4n(n-1)}\|\|P\|\|^2 - \frac{c+1}{2n}\|\|\xi^\top\|\|^2 - \frac{2}{n}\mathrm{trace}(\alpha)$
	$\rho \leq \widehat{\delta}_C(n-1) + \frac{c-3}{4} + \frac{3(c+1)}{4n(n-1)}\|\|P\|\|^2 - \frac{c+1}{2n}\|\|\xi^\top\|\|^2 - \frac{2}{n}\mathrm{trace}(\alpha)$
$N_{C(\alpha)}(c)$	$\rho \leq \delta_C(n-1) + \frac{c+3\alpha^2}{4} + \frac{3(c-\alpha^2)}{4n(n-1)}\|\|P\|\|^2 - \frac{c-\alpha^2}{2n}\|\|\xi^\top\|\|^2 - \frac{2}{n}\mathrm{trace}(\alpha)$
	$\rho \leq \widehat{\delta}_C(n-1) + \frac{c+3\alpha^2}{4} + \frac{3(c-\alpha^2)}{4n(n-1)}\|\|P\|\|^2 - \frac{c-\alpha^2}{2n}\|\|\xi^\top\|\|^2 - \frac{2}{n}\mathrm{trace}(\alpha)$

where $N(c), N_{cosy}(c), N_{Ken}(c)$ *and* $N_{C(\alpha)}(c)$ *are the Sasakian space form, cosymplectic space form, Kenmotsu space form and almost* $C(\alpha)$*-space form, respectively.*

Acknowledgement. D. W. Yoon was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A2003994). G.-E. Vîlcu was supported by National Research Council - Executive Agency for Higher Education Research and Innovation Funding (CNCS-UEFISCDI), project number PN-II-ID-PCE-2011-3-0118.

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