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ON THE LIE DERIVATIVE OF REAL HYPERSURFACES IN $\mathbb{C}P^2$ AND $\mathbb{C}H^2$ WITH RESPECT TO THE GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. In this paper the notion of Lie derivative of a tensor field T of type (1,1) of real hypersurfaces in complex space forms with respect to the generalized Tanaka-Webster connection is introduced and is called *generalized Tanaka-Webster Lie derivative*. Furthermore, three dimensional real hypersurfaces in non-flat complex space forms whose generalized Tanaka-Webster Lie derivative of 1) shape operator, 2) structure Jacobi operator coincides with the covariant derivative of them with respect to any vector field X orthogonal to ξ are studied.

1. Introduction

A *complex space form* is an n-dimensional Kähler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is analytically isometric to a complex projective space $\mathbb{C}P^n$ if $c > 0$, or to a complex Euclidean space $\mathbb{C}^n$ if $c = 0$, or to a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. The complex projective and complex hyperbolic spaces are called *non-flat complex space forms*, since $c \neq 0$ and the symbol $M_n(c)$ is used to denote them when it is not necessary to distinguish them.

A real hypersurface M is an immersed submanifold with real co-dimension one in $M_n(c)$. The Kähler structure (J, G), where J is the complex structure and G is the Kähler metric of $M_n(c)$, induces on M an almost contact metric structure (φ, ξ, η, g). The vector field ξ is called *structure vector field* and when it is an eigenvector of the shape operator A of M the real hypersurface is called *Hopf hypersurface* and the corresponding eigenvalue is $\alpha = g(A\xi, \xi)$.

The study of real hypersurfaces M in $M_n(c)$ was initiated by Takagi, who classified homogeneous real hypersurfaces in $\mathbb{C}P^n$ and divided them into six types, namely (A_1), (A_2), (B), (C), (D) and (E) in [13]. These real hypersurfaces are Hopf ones with constant principal curvatures. In case of $\mathbb{C}H^n$,

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the study of real hypersurfaces with constant principal curvatures, was started by Montiel in [5] and completed by Berndt in [1]. They are divided into two types, namely (A) and (B), depending on the number of constant principal curvatures. The real hypersurfaces found by them are homogeneous and Hopf ones.

Last years many geometers have studied real hypersurfaces in $M_n(c)$ when they satisfy certain geometric conditions. More precisely, the *structure Jacobi operator* of them plays an important role in their study. Generally, the *Jacobi operator* with respect to a vector field X on a manifold is defined by $R(\cdot, X)X$, where R is the Riemannian curvature of the manifold. In case of real hypersurfaces for $X = \xi$ the Jacobi operator is called structure Jacobi operator and is denoted by $l = R_\xi = R(\cdot, \xi)\xi$.

Another topic that has been of great importance is the study of real hypersurfaces in $M_n(c)$ in terms of their generalized Tanaka-Webster connection. The notion of *generalized Tanaka-Webster connection* was first introduced by Tanno in [14] in case of contact metric manifolds in the following way

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\varphi Y.$$

In [2] Cho extended Tanno's work by defining the notion of generalized Tanaka-Webster connection for real hypersurfaces M in $M_n(c)$ in the following way

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y,$$

where X, Y are tangent to M and k is a non-null real number. The notion of *k-th Cho operator* associated to a vector field X was introduced by the second author in [11] and is given by $F^{(k)}_X Y = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y$. If X is any vector field orthogonal to ξ the k-th Cho operator associated to X becomes $F_X Y = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX$ and is called *Cho operator* associated to X. Furthermore, the relation of generalized Tanaka-Webster connection due to the above becomes

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + F^{(k)}_X Y. \tag{1.1}$$

In [12] the second author and Suh studied commuting conditions of k-th Cho operator associated to 1) any vector X orthogonal to ξ, 2) structure vector field ξ with the shape operator. More precisely, they classified real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose k-th Cho operator associated to ξ commutes with the shape operator and they also proved that the shape operator only of ruled hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, commutes with Cho operator associated to any vector X orthogonal to ξ. In [11] the second author studied the commuting conditions of k-th Cho operator associated to vector field X with the structure Jacobi operator. More precisely, he proved that the k-th Cho operator associated to ξ commutes with the structure Jacobi operator only in case of real hypersurfaces of type (A) or of real hypersurfaces with $\alpha = 0$ in $\mathbb{C}P^n$, $n \geq 3$. Furthermore, he proved that ruled hypersurfaces are the only real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose Cho operator associated to any vector X orthogonal to ξ commutes with the structure Jacobi operator.

The Lie derivative of a tensor field T of type (1,1) with respect to the generalized Tanaka-Webster connection is denoted by $\hat{\mathcal{L}}_X^{(k)}T$, called *generalized Tanaka-Webster Lie derivative with respect to* X and is given by

$$(\hat{\mathcal{L}}_X^{(k)}T)Y = \hat{\nabla}_X^{(k)}TY - \hat{\nabla}_{TY}^{(k)}X - T\hat{\nabla}_X^{(k)}Y + T\hat{\nabla}_Y^{(k)}X,$$

where X, Y are tangent to M. Taking into account relation (1.1) the above relation implies

$$(\hat{\mathcal{L}}_X^{(k)}T)Y = \nabla_X TY + F_X^{(k)}TY - \nabla_{TY}X - F_{TY}^{(k)}X - T\nabla_X Y - TF_X^{(k)}Y + T\nabla_Y X + TF_Y^{(k)}X. \tag{1.2}$$

In [10] the second author began the study of real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose Lie derivative with respect to ξ of 1) shape operator, 2) structure Jacobi operator coincides with the covariant derivative of them with respect to ξ and the generalized Tanaka-Webster connection, i.e $(\mathcal{L}_\xi T)Y = (\hat{\nabla}_\xi^{(k)}T)Y$, where T is either the shape operator or the structure Jacobi operator.

Motivated by the work that so far has been done the following question raised

Question. Are there real hypersurfaces in $M_n(c)$, $n \geq 2$, whose generalized Lie derivative of a tensor field T of type $(1,1)$ with respect to a vector field X coincides with the covariant derivative of them, i.e., $(\hat{\mathcal{L}}_X^{(k)}T)Y = (\nabla_X T)Y$?

The aim of the present paper is to answer the above question in case of three dimensional real hypersurfaces in $M_2(c)$ in case of 1) shape operator and 2) structure Jacobi operator. Let M be a three dimensional real hypersurface in $M_2(c)$, whose shape or structure Jacobi operator satisfy the relation

$$(\hat{\mathcal{L}}_\xi^{(k)}T)Y = (\nabla_\xi T)Y.$$

The above relation due to (1.1) and since $\hat{\nabla}^{(k)}\xi = 0$ implies $F_\xi^{(k)}TY = TF_\xi^{(k)}Y$, where T is either the shape or the structure Jacobi operator. Real hypersurfaces satisfying this condition have been studied by the authors in [7] and [8] respectively.

So in this paper we are focused on the study of three dimensional real hypersurfaces in $M_2(c)$, whose shape or structure Jacobi operator, denoted when it is not necessary to be distinguished by T satisfy relation

$$(\hat{\mathcal{L}}_X^{(k)}T)Y = (\nabla_X T)Y, \tag{1.3}$$

where X is any vector field orthogonal to ξ and Y is tangent vector to M. More precisely, the following Theorems are proved.

Theorem 1.1. *There do not exist real hypersurfaces in $M_2(c)$ whose generalized Tanaka-Webster Lie derivative of shape operator coincides with the covariant derivative of it with respect to any vector field X orthogonal to ξ.*

Theorem 1.2. *There do not exist real hypersurfaces in $M_2(c)$ whose generalized Tanaka-Webster Lie derivative of structure Jacobi operator coincides with the covariant derivative of it with respect to any vector field X orthogonal to ξ.*

This paper is organized as follows: In Section 2 basic relations and results about real hypersurfaces in $M_n(c)$, $n \geq 2$, are given. In Section 3 the proof of Theorem 1.1 is provided. Finally, in Section 4 the proof of Theorem 1.2 is given.

2. Preliminaries

Throughout this paper all manifolds, vector fields etc are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces M are supposed to be without boundary.

Let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c. In case of $\mathbb{C}P^n$ we have $c = 4$ and in case of $\mathbb{C}H^n$ we have $c = -4$.

Let N be a locally defined unit normal vector field on M and $\xi = -JN$ be the structure vector field of M. For a vector field X tangent to M relation

$$JX = \varphi X + \eta(X)N$$

holds, where φX and $\eta(X)N$ are respectively the tangential and the normal component of JX. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G.

The *shape operator* A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (φ, ξ, η, g) induced from J of $M_n(c)$, where φ is the *structure tensor*, which is a tensor field of type (1,1) and η is an 1-form such that

$$g(\varphi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Moreover, the following relations hold

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y).$$

The fact that J is parallel implies $\overline{\nabla} J = 0$ and this leads to

$$\nabla_X \xi = \varphi AX, \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.1}$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in the Gauss and Codazzi equations are respectively given by

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \tag{2.2}$$

$$-g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z] + g(AY, Z)AX - g(AX, Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi],$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M.

Relation (2.2) implies that the structure Jacobi operator l is given by

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi, \tag{2.3}$$

for any vector field X tangent to M, where $\alpha = \eta(A\xi) = g(A\xi, \xi)$.

The tangent space $T_P M$ at every point $P \in M$ can be decomposed as

$$T_P M = span\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ and is called (*maximal*) *holomorphic distribution* (*if* $n \geq 3$). Due to the above decomposition the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U,$$

where $\beta = |\varphi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi\xi \in \ker(\eta)$ is a unit vector field, provided that $\beta \neq 0$.

Next, the following results concern any non-Hopf real hypersurface M in $M_2(c)$ with local orthonormal basis $\{U, \varphi U, \xi\}$ at a point P of M.

Lemma 2.1. *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold on M*

$$\begin{aligned} &AU = \gamma U + \delta\varphi U + \beta\xi, && A\varphi U = \delta U + \mu\varphi U, && A\xi = \alpha\xi + \beta U, \\ &\nabla_U\xi = -\delta U + \gamma\varphi U, && \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, && \nabla_\xi\xi = \beta\varphi U, \\ &\nabla_U U = \kappa_1\varphi U + \delta\xi, && \nabla_{\varphi U}U = \kappa_2\varphi U + \mu\xi, && \nabla_\xi U = \kappa_3\varphi U, \\ &\nabla_U\varphi U = -\kappa_1 U - \gamma\xi, && \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, && \nabla_\xi\varphi U = -\kappa_3 U - \beta\xi, \end{aligned} \tag{2.4}$$

where $\alpha, \beta, \gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\beta \neq 0$.

Remark 2.2. The proof of Lemma 2.1 is included in [9].

The structure Jacobi operator for $X = U$, $X = \varphi U$ and $X = \xi$ due to (2.4) is given by

$$lU = (\frac{c}{4} + \alpha\gamma - \beta^2)U + \alpha\delta\varphi U, \;\; l\varphi U = \alpha\delta U + (\frac{c}{4} + \alpha\mu)\varphi U \;\text{ and }\; l\xi = 0. \tag{2.5}$$

The Codazzi equation for $X \in \{U, \varphi U\}$ and $Y = \xi$ because of Lemma 2.1 implies the following relations

$$\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2, \tag{2.6}$$

$$\xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3, \tag{2.7}$$

$$(\varphi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu, \tag{2.8}$$

$$(\varphi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu, \tag{2.9}$$

and for $X = U$ and $Y = \varphi U$

$$U\delta - (\varphi U)\gamma \;=\; \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu. \tag{2.10}$$

Furthermore, combination of the Gauss equation (2.2) with the formula of Riemannian curvature $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, taking into account relations of Lemma 2.1, implies

$$U\kappa_2 - (\varphi U)\kappa_1 = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c, \tag{2.11}$$

$$(\varphi U)\kappa_3 - \xi\kappa_2 = 2\beta\mu - \mu\kappa_1 + \delta\kappa_2 + \kappa_3\kappa_1 + \beta\kappa_3. \tag{2.12}$$

Finally, the following Theorem which in case of $\mathbb{C}P^n$ is owed to Maeda [4] and in case of $\mathbb{C}H^n$ is owed to Montiel [5] (also Corollary 2.3 in [6]).

Theorem 2.3. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then*
i) *α is constant.*
ii) *If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then*

$$(\lambda - \frac{\alpha}{2})A\varphi W = (\frac{\lambda\alpha}{2} + \frac{c}{4})\varphi W.$$

iii) *If the vector field W satisfies $AW = \lambda W$ and $A\varphi W = \nu\varphi W$, then*

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{2.13}$$

Remark 2.4. In case of real hypersurfaces of dimension greater to three the third case of Theorem 2.3 occurs when $\alpha^2 + c \neq 0$, since in this case relation $\lambda \neq \frac{\alpha}{2}$ holds. Furthermore, the first of (2.1) and (2.3) for $X = W$ and $X = \varphi W$ respectively implies

$$\nabla_W\xi = \lambda\varphi W \;\text{ and }\; \nabla_{\varphi W}\xi = -\nu W, \tag{2.14}$$

$$lW = (\frac{c}{4} + \alpha\lambda)W \;\text{ and }\; l\varphi W = (\frac{c}{4} + \alpha\nu)\varphi W. \tag{2.15}$$

Remark 2.5. In case of three dimensional Hopf hypersurfaces we can always consider a local orthonormal basis $\{W, \varphi W, \xi\}$ at some point $P \in M$ such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$. So relations (2.13), (2.14) and (2.15) hold.

3. Proof of Theorem 1.1

Let M be a real hypersurface in $M_2(c)$ whose shape operator satisfies relation (1.3) for $T = A$ with respect to any vector field $X \in \mathbb{D}$. The latter due to $F_X^{(k)}Y = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y$ and $X \in \mathbb{D}$ becomes

$$\begin{aligned} &g(\varphi AX, AY)\xi + g(AY, A\varphi X)\xi + k\eta(AY)\varphi X + \eta(Y)A\varphi AX \\ &\quad + A\nabla_Y X + g(\varphi AY, X)A\xi \\ &= \eta(AY)\varphi AX + \nabla_{AY}X + g(\varphi AX, Y)A\xi + k\eta(Y)A\varphi X, \end{aligned} \tag{3.1}$$

where $X \in \mathbb{D}$ and $Y \in TM$.

We consider the open subset $\mathcal{N}$ of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

On $\mathcal{N}$ the inner product of (3.1) for $X = U$ and $Y = \xi$ with ξ due to (2.4) and relations of Lemma 2.1 implies $\delta = 0$ and relation (2.4) becomes

$$AU = \gamma U + \beta\xi, \quad A\varphi U = \mu\varphi U \text{ and } A\xi = \alpha\xi + \beta U. \tag{3.2}$$

The inner product of relation (3.1) for $X = U$ and $Y = \varphi U$ with U because of (3.2) and relations of Lemma 2.1 results in $\gamma = 0$. Moreover, the inner product of relation (3.1) for $X = \varphi U$ and $Y = \xi$ with ξ and for $X = \varphi U$ and $Y = U$ with ξ due to (3.2), relations of Lemma 2.1 and $\gamma = 0$ respectively implies

$$k = 2\mu + \kappa_3 \text{ and } \kappa_1\beta = \alpha\mu. \tag{3.3}$$

The inner product of relation (3.1) for $X = U$ and $Y = \xi$ with φU because of (3.3) yields

$$\mu(2\mu - \alpha) = 0.$$

If $\alpha \neq 2\mu$ the above relation implies $\mu = 0$ and the second of (3.3) yields $\kappa_1 = 0$. Relation (2.6) because of $\delta = \gamma = \mu = \kappa_1 = 0$ yields $\beta^2 = \frac{c}{4}$. Differentiating the latter with respect to φU and taking into account relation (2.9) and $\mu = \gamma = \delta = \kappa_1 = 0$ we obtain $c = 0$, which is a contradiction.

So on $\mathcal{N}$ relations $\delta = \gamma = 0$, $\alpha = 2\mu$ and (3.3) hold. Relation (2.10) due to $\gamma = 0$ implies $\mu(\kappa_1 - 2\beta) = 0$. If $\kappa_1 \neq 2\beta$, then $\mu = 0$ and following similar steps to those in the previous case leads to a contradiction. Thus, $\kappa_1 = 2\beta$ and the second of (3.3) taking into account also that $\alpha = 2\mu$ implies $\mu^2 = \beta^2$. Relation (2.6) because of the first of (3.3) and the relations for μ and κ_1 implies $\beta^2 = k\mu + \frac{c}{4}$. Differentiation of the latter with respect to φU and taking into account relations (2.8), (2.9), $(\varphi U)\alpha = 2(\varphi U)\mu$, $\kappa_1 = 2\beta$, $\beta^2 = \mu^2$, $\kappa_3 = k - 2\mu$, $\alpha = 2\mu$ and $\beta^2 = k\mu + \frac{c}{4}$ results in $11\beta^2 = k^2 - \frac{c}{4}$. Differentiating the latter with respect to φU yields $(\varphi U)\beta = 0$ and because of (2.9) results in $4\beta^2 + \frac{c}{2} = 0$.

Finally, the inner product of relation (3.1) for $X = Y = \varphi U$ with ξ implies $\kappa_2 = 0$. So relation (2.11) due to $\kappa_2 = 0$, $\kappa_1 = 2\beta$, $\mu^2 = \beta^2$, the first of (3.3), $(\varphi U)\kappa_1 = 2\beta(\varphi U)\beta = 0$ and $4\beta^2 + \frac{c}{2}$ yields $c = 0$, which is a contradiction.

Therefore, $\mathcal{N}$ is empty and the following proposition is proved:

Proposition 3.1. *Every real hypersurface in $M_2(c)$ whose shape operator satisfies relation* (1.3) *is Hopf.*

Because of the above proposition, Theorem 2.3 and Remark 2.4 hold. The inner product of (3.1) for $X = \varphi W$ and $Y = W$ with ξ implies

$$\nu(\lambda - \alpha) = 0.$$

Let $\nu \neq 0$ then the above relation results in $\lambda = \alpha$. The inner product of (3.1) for $X = W$ and $Y = \varphi W$ with ξ due to the previous relation yields $\alpha(\nu - \alpha) = 0$. If $\alpha \neq 0$, then $\nu = \alpha$ and M is umbilical, which is impossible since such real hypersurfaces do not exist. So $\alpha = \lambda = 0$ and substitution of this in (2.13) leads to $c = 0$, which is a contradiction.

So $\nu = 0$ and the inner product of (3.1) for $X = W$ and $Y = \varphi W$ with ξ implies $\alpha\lambda = 0$. Substitution of the latter in (2.13) results in $c = 0$, which is a contradiction and this completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Let M be a real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relation (1.3) for $T = l$ with respect to any vector field $X \in \mathbb{D}$. The latter due to $F_X^{(k)}Y = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y$ and $X \in \mathbb{D}$ becomes

$$(4.1)\quad g(\varphi AX, lY)\xi + \eta(Y)l\varphi AX + l\nabla_Y X = g(\varphi AlY, X)\xi + k\eta(Y)l\varphi X + \nabla_{lY}X,$$

where $X \in \mathbb{D}$ and $Y \in TM$.

We consider the open subset $\mathcal{N}$ of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

On $\mathcal{N}$ the inner product of (4.1) for $X = Y = \varphi U$ with ξ because of (2.4) and (2.5) yields $\delta = 0$. So relation (2.4) becomes

$$AU = \gamma U + \beta\xi, \quad A\varphi U = \mu\varphi U \quad \text{and} \quad A\xi = \alpha\xi + \beta U.$$

The inner product of (4.1) for $X = U$ and $Y = \varphi U$ with ξ and for $X = \varphi U$ and $Y = U$ with ξ owing to the above relation and (2.5) yields respectively

$$(4.2)\qquad \gamma(\frac{c}{4} + \alpha\mu) = 0 \quad \text{and} \quad \mu(\frac{c}{4} + \alpha\gamma - \beta^2) = 0.$$

Suppose that $\gamma \neq 0$ then the first of (4.2) implies $\frac{c}{4} + \alpha\mu = 0$. Thus, $\mu \neq 0$ and the second results in $\frac{c}{4} + \alpha\gamma = \beta^2$. The last two relations due to (2.5) yields $lU = l\varphi U = 0$, which implies that the structure Jacobi operator vanishes identically, which is impossible due to Proposition 8 [3].

So on $\mathcal{N}$ relation $\gamma = 0$ holds and the second of (4.2) becomes

$$\mu(\frac{c}{4} - \beta^2) = 0.$$

Let $\beta^2 \neq \frac{c}{4}$ then we have $\mu = 0$ and the first of (2.5) implies that $lU \neq 0$. Relation (4.1) for $X = \varphi U$ and $Y = \xi$ because of $A\varphi U = 0$ and Lemma 2.1 results in $(k - \kappa_3)lU = 0$, which implies that $\kappa_3 = k$. Since $\delta = \mu = 0$ relation (2.7) implies that $\kappa_2 = 0$. Thus relation (2.12) due to $\kappa_3 = k$ gives $\kappa_1 = -\beta$. Therefore, relation (2.6) due to $\gamma = \delta = \mu = 0$ and the latter implies $\beta^2 = \frac{c}{2}$. Differentiation of the latter with respect to φU yields $(\varphi U)\beta = 0$ and due to $\kappa_1 = -\beta$ we obtain $(\varphi U)\kappa_1 = 0$. So relation (2.11) leads to $\beta^2 = -c$. Combination of the last one with $\beta^2 = \frac{c}{2}$ results in $c = 0$, which is a contradiction.

Therefore, on $\mathcal{N}$ we have

$$\gamma = 0 \quad \text{and} \quad \beta^2 = \frac{c}{4},$$

which implies that $lU = 0$. Relation (4.1) for $X = U$ and $Y = \xi$ due to $AU = \beta\xi$ and Lemma 2.1 implies $(k - \kappa_3)l\varphi U = 0$. The last one yields that

$\kappa_3 = k$, since if $l\varphi U = 0$, then the structure Jacobi operator vanishes and this is impossible due to Proposition 8 in [3].

Relation (2.10) due to all the above relations implies $\mu(\kappa_1 - 2\beta) = 0$. Suppose that $\kappa_1 \neq 2\beta$ then $\mu = 0$ and following similar steps as above we lead to a contradiction. So $\kappa_1 = 2\beta$. Relation (2.6) because of the last one, $\gamma = 0$, $\beta^2 = \frac{c}{4}$ and $\kappa_3 = k$ implies $2\beta^2 + k\mu = 0$. Differentiating the latter with respect to ξ and taking into account that $\xi\beta = 0$, since $\beta^2 = \frac{c}{4}$ results in $\xi\mu = 0$. The last one because of (2.7) gives $\kappa_2 = 0$. Thus, relation (2.11) due to all the previous relations implies $6\beta^2 + c = 0$ which in combination to $\beta^2 = \frac{c}{4}$ leads to a contradiction.

Thus, $\mathcal{N}$ is empty and the following proposition is proved:

Proposition 4.1. *Every real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relation* (1.3) *is Hopf.*

Due to the above proposition, relations in Theorem 2.3 and Remark 2.4 hold. The inner product of (4.1) for $X = W$ and $Y = \varphi W$ and for $X = \varphi W$ and $Y = W$ with ξ due to (2.14) and (2.15) implies respectively

$$\lambda(\frac{c}{4} + \alpha\nu) = 0 \text{ and } \nu(\frac{c}{4} + \alpha\lambda) = 0.$$

Suppose that $\lambda \neq 0$ then above two relations implies $\frac{c}{4} + \alpha\nu = 0$. Thus, $\nu \neq 0$ and relation $\frac{c}{4} + \alpha\lambda = 0$ holds. Substitution of the above relations in (2.13) yields $\lambda\nu = 0$, which is a contradiction.

Therefore, on M relation $\lambda = 0$ holds and the second relation results in $\nu = 0$. Substitution of the previous in (2.13) leads to $c = 0$, which is a contradiction and this completes the proof of Theorem 1.2.

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