

REAL COVERING OF THE GENERALIZED HANKEL-CLIFFORD TRANSFORM OF FOX KERNEL TYPE OF A CLASS OF BOEHMIANS

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ABSTRACT. We investigate some generalization of a class of Hankel-Clifford transformations having Fox H -function as part of its kernel on a class of Boehmians. The generalized transform is a one-to-one and onto mapping compatible with the classical transform. The inverse Hankel-Clifford transforms are also considered in the sense of Boehmians.

1. Introduction and preliminaries

One of the most youngest generalizations of functions, and, more particularly, of distributions, is the theory of Boehmians. The idea of construction of Boehmians was initiated by the concept of regular operators introduced by Boehme [17]. Regular operators form a subalgebra of the field of Mikusinski operators and they include only such functions whose supports are bounded from the left. In a concrete case, the space of Boehmians contains all regular operators, all distributions and some objects which are neither regular operators nor distributions. On the other hand, the construction is possible where there are zero divisors, such as the space of continuous functions with the operations of pointwise addition and convolution. For a somehow much more detailed account of the abstract construction of Boehmians and its extended integral transforms, we refer to the references [1]~[16], [19]~[23], [27]~[36].

H -functions, introduced by Fox [18] as symmetrical Fourier kernels, are an extreme generalization of the generalized hypergeometric function ${}_pF_q$, beyond Meijer functions. H -functions have recently found applications in a large variety of problems connected with reaction, diffusion, reaction diffusion, engineering, communication, fractional differential and integral equations and many areas of theoretical physics and statistical distribution theory as well.

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The utility and importance of the Fox H -function has been realized in recent years due to its occurrence as kernel of certain integral transforms. Most recently, H -functions, being related to the Mellin transforms, have been recognized to play a fundamental role in fractional calculus as well as their applications.

According to a standard notation, the Fox's H -function is defined as

$$(1) \quad H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathfrak{L}} j_{p,q}^{m,n}(\varsigma) z^\varsigma d\varsigma,$$

where \mathfrak{L} is a suitable path in the complex plane, $z^\varsigma = \exp\{\varsigma(\log|z| + i \arg z)\}$ and

$$j_{p,q}^{m,n}(\varsigma) = \frac{\mathbf{a}(\varsigma) \mathbf{b}(\varsigma)}{\mathbf{c}(\varsigma) \mathbf{d}(\varsigma)},$$

where

$$\mathbf{a}(\varsigma) := \prod_{j=1}^m \Gamma(b_j - \beta_j \varsigma), \quad \mathbf{b}(\varsigma) := \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \varsigma),$$

$$\mathbf{c}(\varsigma) := \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j \varsigma) \quad \text{and} \quad \mathbf{d}(\varsigma) := \prod_{j=n+1}^p \Gamma(a_j + \alpha_j \varsigma)$$

with $a_j, b_j \in \mathbb{C}$, $\alpha_j, \beta_j \in \mathbb{R}^+$ and $m, p, q \in \mathbb{N}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ satisfying $0 \leq n < p$ and $1 \leq m < q$. Here \mathbb{C} , \mathbb{R} , \mathbb{R}^+ and \mathbb{N} denote, respectively, the sets of complex numbers, real numbers, positive real numbers and positive integers.

The integral representation of (1) of the H -functions, by involving products and notations of gamma functions, is known to be of Mellin-Barnes integral type. A compact notation is usually adopted for

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{j=1,2,\dots,p} \\ (b_j, \beta_j)_{j=1,2,\dots,q} \end{matrix} \right. \right].$$

A useful and important formula for the H -function is that

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{j=1,2,\dots,p} \\ (b_j, \beta_j)_{j=1,2,\dots,q} \end{matrix} \right. \right] = H_{q,p}^{n,m} \left[\frac{1}{z} \left| \begin{matrix} (1 - b_j, \beta_j)_{j=1,2,\dots,q} \\ (1 - a_j, \alpha_j)_{j=1,2,\dots,p} \end{matrix} \right. \right],$$

which transforms the H -function with argument z to one with argument $1/z$.

Other important properties of the Fox H -function, which can be easily derived from its definition, are included in the list below:

- (a) The H -function is symmetric in the set of pairs $(a_1, \alpha_1), (a_2, \alpha_2), \dots, (a_p, \alpha_p)$ and $(b_1, \beta_1), (b_2, \beta_2), \dots, (b_q, \beta_q)$.
- (b) If one of the (a_j, α_j) ($j = 1, \dots, n$) is equal to one of the (b_j, β_j) ($j = m + 1, \dots, q$), or, one of the pairs (a_j, α_j) ($j = n + 1, \dots, p$) is equal to one of the (b_j, β_j) ($j = 1, \dots, m$), then the H -function reduces to one of the lower order, that is, p, q and n (or m) decrease by a unity. In fact, if $n > 1$ and $q > m$, we have

(i) $(a_1, \alpha_1) = (b_q, \beta_q)$

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{j=1,2,\dots,p} \\ (b_j, \beta_j)_{j=1,2,\dots,q-1} \end{matrix} \right. (a_1, \alpha_1) \right] = H_{p-1,q-1}^{m,n-1} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{j=2,\dots,p} \\ (b_j, \beta_j)_{j=1,2,\dots,q-1} \end{matrix} \right. \right];$$

(ii) $(a_p, \alpha_p) = (b_1, \beta_1)$

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{j=1,2,\dots,p-1} (b_1, \beta_1) \\ (b_1, \beta_1) (b_j, \beta_j)_{j=2,\dots,q} \end{matrix} \right. \right] = H_{p-1,q-1}^{m-1,n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{j=1,2,\dots,p-1} \\ (b_j, \beta_j)_{j=2,\dots,q-1} \end{matrix} \right. \right];$$

(iii) $a_j \rightarrow a_j + \sigma \alpha_j$ ($j = 1, \dots, p$) and $b_j \rightarrow b_j + \sigma \beta_j$ ($j = 1, \dots, q$)

$$z^\sigma H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{j=1,2,\dots,p} \\ (b_j, \beta_j)_{j=1,2,\dots,q} \end{matrix} \right. \right] = H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j + \sigma \alpha_j, \alpha_j)_{j=1,2,\dots,p} \\ (b_j + \sigma \beta_j, \beta_j)_{j=1,2,\dots,q} \end{matrix} \right. \right];$$

(iv) $\alpha_j \rightarrow c \alpha_j$ ($j = 1, \dots, p$) and $\beta_j \rightarrow c \beta_j$ ($j = 1, \dots, q$)

$$\frac{1}{c} H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j)_{j=1,2,\dots,p} \\ (b_j, \beta_j)_{j=1,2,\dots,q} \end{matrix} \right. \right] = H_{p,q}^{m,n} \left[z^c \left| \begin{matrix} (a_j, c \alpha_j)_{j=1,2,\dots,p} \\ (b_j, c \beta_j)_{j=1,2,\dots,q} \end{matrix} \right. \right].$$

A few interesting special cases of the H -function, which may be useful for researchers on integral transforms, fractional calculus, special functions, applied statistics, physical and engineering sciences, and astrophysics, are given here:

(i) The relation connecting Whittaker function and MacRobert's E -function is given as

$$H_{q+1,p}^{p,1} \left[z \left| \begin{matrix} (1, 1) (\beta_1, 1), \dots, (\beta_q, 1) \\ (\alpha_1, 1), \dots, (\alpha_p, 1) \end{matrix} \right. \right] = E(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z).$$

(ii) The relation connecting Whittaker function and the H -function is given as

$$H_{q+1,p}^{p,1} \left[\frac{z^2}{4} \left| \begin{matrix} (\rho - k + 1, 1) \\ (\rho + m + \frac{1}{2}, 1), (\rho - m + \frac{1}{2}, 1) \end{matrix} \right. \right] = z^\rho e^{-\frac{1}{2}z} W_{k,m}(z).$$

(iii) The relation connecting Whittaker function and Mittag-Liffler function is given as

$$H_{1,2}^{1,1} \left[-z \left| \begin{matrix} (0, 1) \\ (0, 1), (1 - \beta, \alpha) \end{matrix} \right. \right] = E_{\alpha,\beta}(z).$$

For further properties of H -functions, see [26].

The generalized Hankel-Clifford transform of a function f of one variable is defined by (see [24, Eq. (1.3)])

$$\mathcal{L}_\alpha^\beta(f(\tau))(y) = J(y) = y^{-\alpha-\beta} \int_0^\infty (y\tau)^{\frac{(\alpha+\beta)}{2}} J_{\alpha-\beta}(2\sqrt{y\tau}) f(\tau) d\tau,$$

where $J_{\alpha-\beta}$ is the Bessel function of first kind of order $\alpha-\beta \geq -\frac{1}{2}$ for $\alpha, \beta \in \mathbb{R}$.

It is useful for readers to take notice that the transform under consideration as an extension of both Hankel-Clifford and Hankel transforms was extended to a certain class of generalized functions by Malgonde and Bandewar in [25].

In view of [24, Eq. (1.7)], the generalized Hankel-Clifford transformation can be rectified in terms of Fox H -function as in the following definition.

Definition 1. Let f be a function defined on \mathbb{R}^+ . Then, the generalized Hankel-Clifford transformation of $f(\tau)$ is defined as

$$(2) \quad \mathcal{L}_\alpha^\beta(f(\tau))(y) = \int_0^\infty y^{-\alpha-\beta} H_{0,2}^{1,0} \left[y\tau \left| \begin{matrix} - \\ (\alpha, 1), (\beta, 1) \end{matrix} \right. \right] f(\tau) d\tau,$$

where the H -function is displayed here as

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n} \left[z \left| \begin{matrix} - \\ (\alpha, 1), (\beta, 1) \end{matrix} \right. \right].$$

Definition 2 (see [24, Eq. (3.3)]). For $c, d \in \mathbb{R}$, the space $M_{c,d}$ denotes the set of all smooth functions $\psi(\tau)$ ($\tau \in \mathbb{R}^+$) such that

$$\xi_{c,d,k}(\psi(\tau)) = \sup_{\tau \in \mathbb{R}^+} \left| \varrho_{c,d}(-\log \tau) (-\tau D_\tau)^k (\tau \psi(\tau)) \right|$$

is finite for $k \in \mathbb{N}_0$, where

$$\varrho_{c,d}(-\log \tau) = \begin{cases} \tau^{-c} & (0 < \tau \leq 1), \\ \tau^{-d} & (1 < \tau \leq \infty). \end{cases}$$

For simple notations, we write $\xi_{c,d,k}$ as

$$(3) \quad \xi_{c,d,k}(\psi(\tau)) = \sup_{\tau \in \mathbb{R}^+} \left| \varrho_{c,d}^{\tau,k}(\tau \psi(\tau)) \right|,$$

where $\varrho_{c,d}^{\tau,k} = \varrho_{c,d}(-\log \tau) (-\tau D_\tau)^k$.

The topology of $M_{c,d}$ is generalized by the set of multinorms $\{\xi_{c,d,k}\}_{k=0}^\infty$. Hence, as a consequence, $M_{c,d}$ is a complete countable multinormed space. The dual space of $M_{c,d}$ is denoted by $M'_{c,d}$.

Let $D(\mathbb{R}^+)$ denote the set of test functions of compact supports. Then $D(\mathbb{R}^+) \subset M_{c,d}$ and, therefore, the restriction of $t \in M'_{c,d}$ to $D(\mathbb{R}^+)$ belongs to $D'(\mathbb{R}^+)$, the space of distributions.

A function φ is said to belong to $M_{c,d}^{\text{loc}}$ if $\varphi \in M_{c,d}$ and φ is locally integrable on \mathbb{R}^+ .

We need the following constructive definition.

Definition 3. Let φ and ψ be integrable functions defined on \mathbb{R}^+ . Then we define a convolution product between φ and ψ as follows:

$$(4) \quad (\varphi \times_\alpha^\beta \psi)(y) = \int_0^\infty \varphi(yx) \Delta_\alpha^\beta \psi(x) dx,$$

where $\Delta_\alpha^\beta \psi(z) = z^{\alpha+\beta} \psi(z)$.

We also recall the Mellin type convolution product Υ of two integrable functions φ and ψ defined by

$$(5) \quad (\varphi \Upsilon \psi)(\tau) = \int_0^\infty \varphi(\tau x^{-1}) \Delta_{-1} \psi(x) dx,$$

where $\Delta_{-1}\psi(z) = z^{-1}\psi(z)$.

There is a relationship between the two types of products given in (4) and (5) asserted by the following theorem.

Theorem 1. For two integrable functions φ and ψ defined on \mathbb{R}^+ , we have the following relation:

$$(6) \quad \mathcal{L}_\alpha^\beta(\varphi \Upsilon \psi)(y) = (\mathcal{L}_\alpha^\beta \varphi \times_\alpha^\beta \psi)(y).$$

Proof. From (2) and (5), we have

$$\begin{aligned} & \mathcal{L}_\alpha^\beta(\varphi \Upsilon \psi)(y) \\ &= y^{-\alpha-\beta} \int_0^\infty H_{0,2}^{1,0} \left[y\tau \left| \begin{matrix} - \\ (\alpha, 1), (\beta, 1) \end{matrix} \right. \right] \int_0^\infty \varphi(\tau x^{-1}) \Delta_{-1}\psi(x) dx d\tau. \end{aligned}$$

Setting the variables with $xz = \tau$ and changing the order of integrations, which may be guaranteed by Fubini's theorem, we obtain

$$\begin{aligned} \mathcal{L}_\alpha^\beta(\varphi \Upsilon \psi)(y) &= \int_0^\infty \Delta_\alpha^\beta \psi(x) \left((yx)^{-\alpha-\beta} \int_0^\infty H_{0,2}^{1,0} \left[yxz \left| \begin{matrix} (\alpha, 1) \\ (\beta, 1) \end{matrix} \right. \right] \varphi(z) dz \right) dx \\ (7) \quad &= \int_0^\infty \mathcal{L}_\alpha^\beta(\varphi)(yx) \Delta_\alpha^\beta \psi(x) dx. \end{aligned}$$

By appealing to (4), the last expression of (7) is easily seen to be equal to the right-hand side of (6). This completes the proof. \square

2. Generalized spaces of Boehmians

In what follows we shall make a free use of some properties of the Mellin type product and therefore we find it worthwhile to describe them briefly as follows (see [35]):

- (i) $\varphi \Upsilon \psi = \psi \Upsilon \varphi$;
- (ii) $(\varphi \Upsilon \psi) \Upsilon \psi_1 = \varphi \Upsilon (\psi \Upsilon \psi_1)$;
- (iii) $(\varphi \Upsilon \psi) \Upsilon \psi_1 = (\varphi \Upsilon \psi_1) \Upsilon \varphi$;
- (iv) $\varphi \Upsilon (\psi + \psi_1) = \varphi \Upsilon \psi + \varphi \Upsilon \psi_1$;
- (v) $(\alpha\varphi) \Upsilon \psi_1 = \alpha(\varphi \Upsilon \psi_1)$,

where φ, ψ_1 and ψ are integrable functions defined on \mathbb{R}^+ , and α is a constant.

Now we establish the Boehmian space $\beta(M_{c,d}^{loc}, \times_\alpha^\beta)$ where Δ is a set of sequences $\{\delta_n\}$ of $D(\mathbb{R}^+)$ satisfying following properties:

$$(Prop. 1) \quad \int_0^\infty \delta_n(x) dx = 1 \quad \text{for all } n \in \mathbb{N};$$

$$(Prop. 2) \quad |\delta_n(x)| < M \quad \text{for some } M \in \mathbb{R}^+;$$

$$(Prop. 3) \quad \text{supp } \delta_n(x) \subseteq (a_n, b_n), \quad a_n, b_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The following results are straightforward from simple integration.

Theorem 2. Let $\{\varphi_n\}$, $\varphi \in M_{c,d}^{\text{loc}}$, and $\psi, \psi_1 \in D(\mathbb{R}^+)$, $\alpha^* \in \mathbb{C}$ and $\varphi_n \rightarrow \varphi$ in $M_{c,d}^{\text{loc}}$ as $n \rightarrow \infty$. Then we have

- (i) $\varphi \times_{\alpha}^{\beta} (\psi + \psi_1) = \varphi \times_{\alpha}^{\beta} \psi + \varphi \times_{\alpha}^{\beta} \psi_1$;
- (ii) $(\alpha^* \varphi) \times_{\alpha}^{\beta} \psi = \alpha^* (\varphi \times_{\alpha}^{\beta} \psi)$;
- (iii) $\varphi_n \times_{\alpha}^{\beta} \psi \rightarrow \varphi \times_{\alpha}^{\beta} \psi$ in $M_{c,d}^{\text{loc}}$ as $n \rightarrow \infty$.

Theorem 3. Let $\varphi \in M_{c,d}^{\text{loc}}$ and $\psi \in D(\mathbb{R}^+)$. Then we have $\varphi \times_{\alpha}^{\beta} \psi \in M_{c,d}^{\text{loc}}$.

Proof. In view of (4), we find

$$\left| \varrho_{c,d}^{\tau,k} \left(\int_0^{\infty} \varphi(yx) \Delta_{\alpha}^{\beta} \psi(x) dx \right) \right| \leq \int_K \left| \varrho_{c,d}^{\tau,k}(\varphi(yx)) \right| |\Delta_{\alpha}^{\beta} \psi(x)| dx,$$

where K is a compact subset of \mathbb{R}^+ containing the support of ψ . Hence, taking supremum over the compact subset K of \mathbb{R}^+ gives

$$\xi_{c,d,k}(\varphi \times_{\alpha}^{\beta} \psi) \leq \xi_{c,d,k}(\varphi) \int_K |\Delta_{\alpha}^{\beta} \psi(x)| dx.$$

That is,

$$\xi_{c,d,k}(\varphi \times_{\alpha}^{\beta} \psi) \leq N \xi_{c,d,k}(\varphi),$$

where

$$N := \int_K |\Delta_{\alpha}^{\beta} \psi(x)| dx.$$

This completes the proof. □

Theorem 4. Let $\varphi \in M_{c,d}^{\text{loc}}$ and $\{\delta_n\} \in \Delta$. Then we have $\varphi \times_{\alpha}^{\beta} \delta_n \rightarrow \varphi$ in $M_{c,d}^{\text{loc}}$ as $n \rightarrow \infty$.

Proof. By using (3) and Prop. 1 of Δ we write

$$\begin{aligned} (8) \quad & \left| \varrho_{c,d}^{\tau,k} \left(\tau \left(\varphi \times_{\alpha}^{\beta} \delta_n - \varphi \right) (\tau) \right) \right| \\ &= \left| \varrho_{c,d}^{\tau,k} \tau \left(\int_0^{\infty} \varphi(\tau x) \Delta_{\alpha}^{\beta} \delta_n(x) dx \right) \right| - \left| \varrho_{c,d}^{\tau,k} \tau \left(\varphi(\tau) \int_0^{\infty} \delta_n(x) dx \right) \right| \\ &\leq \int_{a_n}^{b_n} \left| \varrho_{c,d}^{\tau,k} \left(\tau \left(\varphi(\tau x) - \Delta_{\alpha}^{-\beta} \varphi(\tau) \right) \right) \right| |\Delta_{\alpha}^{\beta} \delta_n(x)| dx. \end{aligned}$$

In view of Props. 2 and 3 of Δ , (8) gives

$$\begin{aligned} (9) \quad & \left| \varrho_{c,d}^{\tau,k} \left(\tau \left(\varphi \times_{\alpha}^{\beta} \delta_n - \varphi \right) (\tau) \right) \right| \\ &\leq M \int_{a_n}^{b_n} \left| \varrho_{c,d}^{\tau,k} \left(\tau \left(\varphi(\tau x) - \Delta_{\alpha}^{-\beta} \varphi(\tau) \right) \right) \right| |x^{\alpha+\beta}| dx \\ &\leq M \int_{a_n}^{b_n} (|x^{\alpha+\beta}| \xi_{c,d,k}(\varphi) - \xi_{c,d,k}(\varphi)) dx. \end{aligned}$$

Integrating the last expression in (9), we obtain

$$\left| \varrho_{c,d}^{\tau,k} \left(\tau \left(\varphi \times_{\alpha}^{\beta} \delta_n - \varphi \right) (\tau) \right) \right|$$

$$\begin{aligned} &\leq M\xi_{c,d,k}(\varphi) \int_{a_n}^{b_n} (|x^{\alpha+\beta}| - 1) dx \\ &= M\xi_{c,d,k}(\varphi) \left\{ \frac{(b_n)^{\alpha+\beta+1} - (a_n)^{\alpha+\beta+1}}{\alpha + \beta + 1} - (b_n - a_n) \right\}. \end{aligned}$$

Finally, by using Prop. 3 of Δ sequences, as $n \rightarrow \infty$, since $\alpha, \beta \in \mathbb{R}^+$, we obtain

$$\left| \varrho_{c,d}^{\tau,k}(\tau(\varphi \times_{\alpha}^{\beta} \delta_n - \varphi)(\tau)) \right| \rightarrow 0.$$

This completes the proof. □

It is noted that $\beta(M_{c,d}^{\text{loc}}, \times_{\alpha}^{\beta})$ is recognized as a space of Boehmians.

Theorem 5. *Let $\varphi \in M_{c,d}^{\text{loc}}$, and $\psi, \psi_1 \in D(\mathbb{R}^+)$. Then we have*

$$\varphi \times_{\alpha}^{\beta}(\psi \Upsilon \psi_1) = (\varphi \times_{\alpha}^{\beta} \psi) \times_{\alpha}^{\beta} \psi_1.$$

Proof. By using (4) and (5), we apply Fubini's theorem to change the order of integrals to get

$$\begin{aligned} (\varphi \times_{\alpha}^{\beta}(\psi \Upsilon \psi_1))(y) &= \int_0^{\infty} \varphi(yx) \Delta_{\alpha}^{\beta}(\psi \Upsilon \psi_1)(x) dx \\ &= \int_0^{\infty} \varphi(yx) x^{\alpha+\beta} \left(\int_0^{\infty} \psi(x\tau^{-1}) \Delta_{-1} \psi_1(\tau) d\tau \right) dx \\ &= \int_0^{\infty} \Delta_{-1} \psi_1(\tau) \left(\int_0^{\infty} x^{\alpha+\beta} \varphi(yx) \psi(x\tau^{-1}) dx \right) d\tau. \end{aligned}$$

Changing variables $x\tau^{-1} = z$ gives

$$\begin{aligned} (\varphi \times_{\alpha}^{\beta}(\psi \Upsilon \psi_1))(y) &= \int_0^{\infty} \Delta_{\alpha}^{\beta} \psi_1(\tau) \left(\int_0^{\infty} \varphi(y\tau z) \Delta_{\alpha}^{\beta} \psi(z) dz \right) \\ &= \int_0^{\infty} \Delta_{\alpha}^{\beta} \psi_1(\tau) (\varphi \times_{\alpha,\beta} \psi)(y\tau) d\tau \\ &= ((\varphi \times_{\alpha}^{\beta} \psi) \times_{\alpha}^{\beta} \psi_1)(y). \end{aligned}$$

This completes the proof. □

Construction of the space $\beta(M_{c,d}^{\text{loc}}, \Upsilon)$ may follow from that of the space $\beta(M_{c,d}^{\text{loc}}, \times_{\alpha}^{\beta})$ in a similar way.

The sum of two Boehmians and multiplication by a scalar in $\beta(M_{c,d}^{\text{loc}}, \times_{\alpha}^{\beta})$ can be defined as follows:

$$\left[\frac{\varphi_n}{\delta_n} \right] + \left[\frac{\psi_n}{\varepsilon_n} \right] = \left[\frac{\varphi_n \times_{\alpha}^{\beta} \varepsilon_n + \psi_n \times_{\alpha}^{\beta} \delta_n}{\delta_n \Upsilon \varepsilon_n} \right]$$

and

$$\alpha \left[\frac{\varphi_n}{\delta_n} \right] = \left[\alpha \frac{\varphi_n}{\delta_n} \right] = \left[\frac{\alpha \varphi_n}{\delta_n} \right] \quad (\alpha \in \mathbb{C}).$$

The operation \times_α^β and the differentiation are defined by

$$\left[\frac{\varphi_n}{\delta_n} \right] \times_\alpha^\beta \left[\frac{\psi_n}{\varepsilon_n} \right] = \left[\frac{\varphi_n \times_\alpha^\beta \psi_n}{\delta_n \Upsilon \varepsilon_n} \right]$$

and

$$D^\alpha \left[\frac{\varphi_n}{\delta_n} \right] = \left[\frac{D^\alpha \varphi_n}{\delta_n} \right].$$

Let $\left[\frac{\varphi_n}{\delta_n} \right] \in \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$ and $\omega \in M_{c,d}^{\text{loc}}$. Then the operation \times_α^β can be extended to $\beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right) \times M_{c,d}^{\text{loc}}$ by

$$\left[\frac{\varphi_n}{\delta_n} \right] \times_\alpha^\beta \omega = \left[\frac{\varphi_n \times_\alpha^\beta \omega}{\delta_n} \right].$$

A sequence of Boehmians $\{\beta_n\}$ in $\beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$ is said to be δ -convergent to a Bohmian β in $\beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence $\{\delta_n\}$ such that

$$\beta_n \times_\alpha^\beta \delta_n, \beta \times_\alpha^\beta \delta_n \in M_{c,d}^{\text{loc}} \quad \text{for all } k, n \in \mathbb{N}$$

and

$$\beta_n \times_\alpha^\beta \delta_n \rightarrow \beta \times_\alpha^\beta \delta_n \text{ in } M_{c,d}^{\text{loc}}, \text{ as } n \rightarrow \infty \text{ for every } k \in \mathbb{N}.$$

Here is an equivalent statement for δ -convergence:

$\beta_n \xrightarrow{\delta} \beta$ in $\beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$ as $n \rightarrow \infty$, if and only if there is $\varphi_{n,k}, \varphi_k \in M_{c,d}^{\text{loc}}$ and $\delta_k \in \Delta$ such that $\beta_n = \left[\frac{\varphi_{n,k}}{\delta_k} \right]$, $\beta = \left[\frac{\varphi_k}{\delta_k} \right]$ and, for each $k \in \mathbb{N}$, $\varphi_{n,k} \rightarrow \varphi_k$ in $M_{c,d}^{\text{loc}}$ as $n \rightarrow \infty$.

A sequence of Boehmians $\{\beta_n\}$ in $\beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$ is said to be Δ -convergent to a Bohmian β in $\beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a sequence $\{\delta_n\} \in \Delta$ such that $(\beta_n - \beta) \times_\alpha^\beta \delta_n \in M_{c,d}^{\text{loc}}$ ($n \in \mathbb{N}$), and $(\beta_n - \beta) \times_\alpha^\beta \delta_n \rightarrow 0$ in $M_{c,d}^{\text{loc}}$ as $n \rightarrow \infty$.

It is noted that, similarly as above, operations of addition and multiplication for Υ , and convergence on $\beta \left(M_{c,d}^{\text{loc}}, \Upsilon \right)$ can be defined.

3. \mathcal{L}_α^β of Boehmians

Here we begin by recalling the following required lemma (see [24, Lemma 4.1]).

Lemma. Let $y > 0$ be fixed and $c < 1 + \alpha$, and $d > 0$. Then we have

$$t^{-\alpha-\beta} H_{0,2}^{1,0} \left[y\tau \left| \begin{matrix} - \\ (\alpha, 1), (\beta, 1) \end{matrix} \right. \right] \in M_{c,d}.$$

The following theorem is easily established.

Theorem 6. Let $y > 0$ be fixed, $c < 1 + \alpha$, $d > 0$ and let $\varphi \in M_{c,d}^{\text{loc}}$. Then we have

$$\mathcal{L}_\alpha^\beta \varphi \in M_{c,d}^{\text{loc}}.$$

Proof. Let K be a compact subset of I . Then we find from Theorem 5 and Lemma that

$$\begin{aligned} & \xi_{c,d,k} \left(\mathcal{L}_\alpha^\beta \varphi (y) \right) \\ &= \sup_{y \in I} \left| \varrho_{c,d}^{\tau,y} \left(y \mathcal{L}_\alpha^\beta \varphi (y) \right) \right| \\ &= \int_K |\varphi(\tau)| \left(\left| \varrho_{c,d}^{\tau,y} \left(y \left(y^{-\alpha-\beta} H_{0,2}^{1,0} \left[y\tau \left| \begin{matrix} - \\ (\alpha, 1), (\beta, 1) \end{matrix} \right. \right] \right) \right) \right| \right) d\tau \\ &\leq \xi_{c,d,k} \left(y^{-\alpha-\beta} H_{0,2}^{1,0} \left[y\tau \left| \begin{matrix} - \\ (\alpha, 1), (\beta, 1) \end{matrix} \right. \right] \right) \int_K |\varphi(\tau)| d\tau < \infty, \end{aligned}$$

since φ is locally integrable over \mathbb{R}^+ .

This completes the proof. □

By the aid of Theorem 6, we define the generalized Hankel-Clifford transform of Fox kernel type of $\left[\frac{\varphi_n}{\delta_n} \right] \in \beta \left(M_{c,d}^{\text{loc}}, \gamma \right)$ as

$$\widehat{\mathcal{L}}_\alpha^\beta \left[\frac{\varphi_n}{\delta_n} \right] = \left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n} \right]$$

in the space $\beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$.

Theorem 7. The operator $\widehat{\mathcal{L}}_\alpha^\beta$ is well-defined and linear from $\beta \left(M_{c,d}^{\text{loc}}, \gamma \right)$ into $\beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$.

Proof. Let $\left[\frac{\varphi_n}{\delta_n} \right] = \left[\frac{\psi_n}{\varepsilon_n} \right] \in \beta \left(M_{c,d}^{\text{loc}}, \gamma \right)$. Then we have

$$\varphi_n \Upsilon \varepsilon_m = \psi_m \Upsilon \delta_n = \psi_n \Upsilon \delta_m.$$

Applying \mathcal{L}_α^β on both sides of the above equation and using Theorem 1 imply

$$\mathcal{L}_\alpha^\beta \varphi_n \times_\alpha^\beta \varepsilon_m = \mathcal{L}_\alpha^\beta \psi_n \times_\alpha^\beta \delta_m \quad (n, m \in \mathbb{N}).$$

That is,

$$\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n} \right] = \left[\frac{\mathcal{L}_\alpha^\beta \psi_n}{\varepsilon_n} \right].$$

We show that the $\widehat{\mathcal{L}}_\alpha^\beta : \beta \left(M_{c,d}^{\text{loc}}, \gamma \right) \rightarrow \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$ is linear. Indeed, let $\left[\frac{\varphi_n}{\delta_n} \right], \left[\frac{\psi_n}{\varepsilon_n} \right] \in \beta \left(M_{c,d}^{\text{loc}}, \gamma \right)$. Then by Theorem 1 we can write

$$\begin{aligned} \widehat{\mathcal{L}}_\alpha^\beta \left(\left[\frac{\varphi_n}{\delta_n} \right] + \left[\frac{\psi_n}{\varepsilon_n} \right] \right) &= \widehat{\mathcal{L}}_\alpha^\beta \left(\left[\frac{\varphi_n \gamma \varepsilon_n + \psi_n \gamma \delta_n}{\delta_n \gamma \varepsilon_n} \right] \right) \\ &= \left[\frac{\mathcal{L}_\alpha^\beta(\varphi_n \gamma \varepsilon_n) + \mathcal{L}_\alpha^\beta(\psi_n \gamma \delta_n)}{\delta_n \gamma \varepsilon_n} \right] \\ &= \left[\frac{\mathcal{L}_\alpha^\beta \varphi_n \times_\alpha^\beta \varepsilon_n + \mathcal{L}_\alpha^\beta \psi_n \times_\alpha^\beta \delta_n}{r_n \gamma \varepsilon_n} \right] \\ &= \left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n} \right] + \left[\frac{\mathcal{L}_\alpha^\beta \psi_n}{\varepsilon_n} \right]. \end{aligned}$$

Hence

$$\widehat{\mathcal{L}}_\alpha^\beta \left(\left[\frac{\varphi_n}{\delta_n} \right] + \left[\frac{\psi_n}{\varepsilon_n} \right] \right) = \widehat{\mathcal{L}}_\alpha^\beta \left[\frac{\varphi_n}{\delta_n} \right] + \widehat{\mathcal{L}}_\alpha^\beta \left[\frac{\psi_n}{\varepsilon_n} \right].$$

Also, if $\alpha \in \mathbb{C}$, then we have

$$\alpha \widehat{\mathcal{L}}_\alpha^\beta \left[\frac{\varphi_n}{\delta_n} \right] = \alpha \left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n} \right] = \left[\frac{\mathcal{L}_\alpha^\beta (\alpha \varphi_n)}{\delta_n} \right].$$

Hence

$$\alpha \widehat{\mathcal{L}}_\alpha^\beta \left[\frac{\varphi_n}{\delta_n} \right] = \widehat{\mathcal{L}}_\alpha^\beta \left(\alpha \left[\frac{\varphi_n}{\delta_n} \right] \right).$$

This completes the proof. \square

Theorem 8. *The mapping $\widehat{\mathcal{L}}_\alpha^\beta : \beta \left(M_{c,d}^{\text{loc}}, \gamma \right) \rightarrow \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$ is an isomorphism.*

Proof. Let $\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n} \right] = \left[\frac{\mathcal{L}_\alpha^\beta \psi_n}{\varepsilon_n} \right] \in \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$. Then, by using Theorem 1, we get $\mathcal{L}_\alpha^\beta \varphi_n \times_\alpha^\beta \varepsilon_n = \mathcal{L}_\alpha^\beta \psi_n \times_\alpha^\beta \delta_n$. Once again, Theorem 1 implies

$$\mathcal{L}_\alpha^\beta (\varphi_n \times_\alpha^\beta \varepsilon_n) = \mathcal{L}_\alpha^\beta (\psi_n \times_\alpha^\beta \delta_n).$$

We thus have $\varphi_n \times_\alpha^\beta \varepsilon_n = \psi_n \times_\alpha^\beta \delta_n$. Therefore,

$$\left[\frac{\varphi_n}{\delta_n} \right] = \left[\frac{\psi_n}{\varepsilon_n} \right] \in \beta \left(M_{c,d}^{\text{loc}}, \gamma \right).$$

This proves that the mapping is injective. Next the surjection of $\widehat{\mathcal{L}}_\alpha^\beta$ is obvious, since, for every $\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n} \right] \in \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta \right)$, there is $\left[\frac{\varphi_n}{\delta_n} \right] \in \beta \left(M_{c,d}^{\text{loc}}, \gamma \right)$ such that

$$\widehat{\mathcal{L}}_\alpha^\beta \left[\frac{\varphi_n}{\delta_n} \right] = \left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n} \right].$$

Finally, together with Theorem 7, the proof is complete. \square

Definition 4. Let $\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n}\right] \in \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta\right)$. Then we define the inverse transform of $\widehat{\mathcal{L}_\alpha^\beta}$ as

$$\left(\widehat{\mathcal{L}_\alpha^\beta}\right)^{-1} \left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n}\right] = \left[\frac{(\mathcal{L}_\alpha^\beta)^{-1} (\mathcal{L}_\alpha^\beta \varphi_n)}{\delta_n}\right] = \left[\frac{\varphi_n}{\delta_n}\right]$$

for each $\{\delta_n\} \in \Delta$.

Theorem 9. Let $\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n}\right] \in \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta\right)$ for some $\left[\frac{\varphi_n}{\delta_n}\right] \in \beta \left(M_{c,d}^{\text{loc}}, \gamma\right)$ and $\phi, \psi \in D(I)$. Then we have

- (i) $\left(\widehat{\mathcal{L}_\alpha^\beta}\right)^{-1} \left(\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n}\right] \times_\alpha^\beta \phi\right) = \left[\frac{\varphi_n}{\delta_n}\right] \times_\alpha^\beta \phi;$
- (ii) $\widehat{\mathcal{L}_\alpha^\beta} \left(\left[\frac{\varphi_n}{\delta_n}\right] \gamma \psi\right) = \left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n}\right] \times_\alpha^\beta \psi.$

Proof. Let $\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n}\right] \in \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta\right)$ be given. Then, by Theorem 1, we write

$$\begin{aligned} \left(\widehat{\mathcal{L}_\alpha^\beta}\right)^{-1} \left(\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n}\right] \times_\alpha^\beta \phi\right) &= \left(\widehat{\mathcal{L}_\alpha^\beta}\right)^{-1} \left(\left[\frac{\mathcal{L}_\alpha^\beta \varphi_n \times_\alpha^\beta \phi}{\delta_n}\right]\right) \\ &= \left[\frac{(\mathcal{L}_\alpha^\beta)^{-1} (\mathcal{L}_\alpha^\beta \varphi_n \times_\alpha^\beta \phi)}{\delta_n}\right]. \end{aligned}$$

To prove the second identity, we use Theorem 1 and Definition 4 to obtain

$$\widehat{\mathcal{L}_\alpha^\beta} \left(\left[\frac{\varphi_n}{\delta_n}\right] \gamma \psi\right) = \widehat{\mathcal{L}_\alpha^\beta} \left(\left[\frac{\varphi_n \gamma \psi}{\delta_n}\right]\right) = \left[\frac{\mathcal{L}_\alpha^\beta \varphi_n}{\delta_n}\right] \times_\alpha^\beta \psi.$$

This complete the proof. □

Theorem 10. $\widehat{\mathcal{L}_\alpha^\beta} : \beta \left(M_{c,d}^{\text{loc}}, \gamma\right) \rightarrow \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta\right)$ and $\left(\widehat{\mathcal{L}_\alpha^\beta}\right)^{-1} : \beta \left(M_{c,d}^{\text{loc}}, \times_\alpha^\beta\right) \rightarrow \beta \left(M_{c,d}^{\text{loc}}, \gamma\right)$ are continuous with respect to δ and Δ -convergence.

Proof. A similar proof of this theorem is available in the references of the second-named author cited here. We omit the proof. □

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