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ON SOME GENERALIZATIONS OF CLOSED SUBMODULES

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Abstract. Characterizations of closed subgroups in abelian groups have been generalized to modules in essentially different ways; they are in general inequivalent. Here we consider the relations between these generalizations over commutative rings, and we characterize the commutative rings over which they coincide. These are exactly the commutative noetherian distributive rings. We also give a characterization of c-injective modules over commutative noetherian distributive rings. For a noetherian distributive ring R, we prove that, (1) direct product of simple R-modules is c-injective; (2) an R-module D is c-injective if and only if it is isomorphic to a direct summand of a direct product of simple R-modules and injective R-modules.

1. Introduction

Throughout the paper, we shall assume that all rings are associative with identity and all modules are unitary left modules. Let R be any ring. A submodule K of an R-module M is called closed (in M) provided K has no proper essential extension in M. Moreover, if L is any submodule of M, then there exists, by Zorn's Lemma, a submodule K of M maximal with respect to the property that L is an essential submodule of K, and in this case K is a closed submodule of M. A module M is called an extending module if every closed submodule is a direct summand, and in this case every submodule of M is essential in a direct summand of M. For the properties of closed submodules and extending modules see [5].

A submodule K of an R-module M is called pure provided for every (finitely presented) right R-module U, the induced homomorphism $U \otimes_R K \to U \otimes_R M$ of abelian groups is a monomorphism. When R is a Dedekind domain (more generally a Prüfer domain), a submodule K of an R-module M is pure if and only if $K \cap aM = aK$ for all $a \in R$. Inspired by this characterization of pure submodules over Dedekind domains, Honda [9] introduced neat subgroups in order to characterize the closed subgroup in abelian groups. Namely, a

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subgroup A of an abelian group B is called neat in B if $Ap = A \cap Bp$ for every prime numbers p. It is easy to see that a closed exact sequence of abelian groups can also be defined in terms of either of the following homological properties (p denotes primes): $0 \to A \xrightarrow{\iota} B \to C \to 0$ is a closed exact sequence of abelian groups, i.e., $\iota(A)$ is closed in B if and only if

(a) $\iota(A)p = \iota(A) \cap Bp$ for all p;

(b) the sequence $0 \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, A) \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, B) \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, C) \to 0$ is exact for all p;

(c) the sequence $0 \to \mathbb{Z}/p\mathbb{Z} \otimes A \to \mathbb{Z}/p\mathbb{Z} \otimes B \to \mathbb{Z}/p\mathbb{Z} \otimes C \to 0$ is exact for all p;

(d) the sequence $0 \to \mathrm{Hom}(C, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(B, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z}/p\mathbb{Z}) \to 0$ is exact for all p.

The definition of closed subgroup can be extended to arbitrary commutative rings R either via (a), (b), (c) or (d).

Neatness over arbitrary associative rings in the sense of (a) has been considered by Mermut et al. in [13], a submodule A of an R-module B is called P-pure if $PA = A \cap PB$ for every maximal right ideal P of R. Generalization in the sense of (b) was discussed by Renault [16], a submodule A of an R-module B is called neat if $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A) \to 0$ is surjective for each simple R-module S. In the sense of (c) has been considered by I. Crivei [2], a submodule A of an R-module B is called s-pure if the map $S \otimes A \to S \otimes B$ is monic for each simple right R-module S. In the sense of (d) was discussed by Fuchs [7], a submodule A of an R-module B is called coneat if $\mathrm{Hom}(B, S) \to \mathrm{Hom}(B/A, S) \to 0$ is surjective for each simple R-module S.

Let A be a submodule of a left R-module B. For a right ideal I of R, $A \cap IB = IA$ if and only if the map $R/I \otimes A \to R/I \otimes B$ is monic, [20, Lemma 6.1]. This result can be used to show that P-pure submodules and s-pure submodules coincide. P-pure submodules coincide with coneat submodules over commutative rings by [7, Proposition 3.1]. Closed submodules are neat (see [22, Proposition 5]). Neat submodules of each R-module are closed if and only if R is left C-ring, i.e., for every proper essential left ideal I of R, the module R/I has a simple submodule (see [8, Theorem 5]). As one may see from [7, Examples 3.2–3.3], neat submodules and coneat submodules are not only inequivalent, but even incomparable. The closed submodules and the coneat submodules are also incomparable (see, Examples 2.3 and 2.6). Summing up, on the contrary to the case in abelian groups, the concepts of closed, neat and coneat do not coincide. Motivated by this fact, we consider the following question:

Question A. For which commutative rings the concepts of closed, neat and coneat coincide?

Fuchs has recently considered the problem of comparing neatness and coneatness, and he proved that for an integral domain R the two concepts coincide if and only if every maximal ideal of R is finitely generated (projective), that is, invertible [7, Theorem 5.2]. Over arbitrary commutative rings, he also mentioned that: (1) if coneat submodules in each R-module are neat, then every simple R-module is finitely presented and neat submodules in each R-module are coneat; (2) if neat submodules in each R-module are coneat and every simple R-module is finitely presented, then coneat submodules in each R-module are neat [7, p. 138]. But if R is a ring whose simple R-modules are finitely presented, then the concepts of coneat and neat are still inequivalent, (see [7, Example 3.2]). Crivei proved in [4] that if R is a commutative ring whose maximal ideals are principal, then neat and coneat submodules of every module coincide.

A submodule K of M is called small in M if $M \neq K + T$ for every proper submodule T of M. Given submodules $K \subseteq L \subseteq M$, the inclusion $K \subseteq L$ is called cosmall in M if $L/K \ll M/K$. A submodule $L \subseteq M$ is called coclosed in M if L has *no* proper submodule K for which the inclusion $K \subseteq L$ is cosmall in M. Recently, Zöschinger showed in [25] that, over commutative noetherian rings, coclosed submodules in every R-module are closed if and only if R is distributive. In general coclosed submodules are coneat, [1]. Coneat submodules are coclosed if and only if R is left K-ring (i.e., every small left R-module is coatomic), [1, Theorem 2.21].

Regarding Question A, we prove that for a commutative ring R the following conditions are equivalent: (1) R is noetherian distributive ring; (2) the concepts of closed, neat and coneat are equivalent; (3) coneat submodules in each R-module are closed; (4) neat submodules in each R-module are coclosed, and R is noetherian ring (Theorem 2.11, Corollary 2.12).

Let M be any R-module. In [18] an R-module X is called M-c-injective provided, for every closed submodule K of M, every homomorphism $\varphi : K \to X$ can be extended to a homomorphism $\theta : M \to X$. Moreover, X is called c-injective provided X is M-c-injective for every R-module M. Note that if M is an extending module, then every R-module is M-c-injective. It is proved in [19, Theorem 6] that if R is a Dedekind domain, then a direct product of simple R-modules is c-injective. In [13], it is shown that if R is a Dedekind domain, then an R-module X is c-injective if and only if X is isomorphic to a direct summand of a direct product of simple R-modules and injective R-modules. Inspired by this result, we consider the following question:

Question B. Is it true that an R-module X is c-injective if and only if X is isomorphic to a direct summand of a direct product of simple R-modules and injective R-modules on a commutative noetherian hereditary rings?

We prove that if R is a commutative noetherian distributive ring, then an R-module X is c-injective if and only if X is isomorphic to a direct summand of a direct product of simple R-modules and injective R-modules. This result gives an affirmative answer to Question B.

For a left (right) R-module M, its character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. We use the notation $E(M)$ for the injective hull of M, $\mathrm{Soc}(M)$

for the socle of M and $Z(M)$ for the singular submodule of M. By $N \subseteq M$, we mean that N is a submodule of M.

2. Closed, neat and coneat submodules coincide

In the sequel we will use the proposition below.

Proposition 2.1 ([2, Proposition 3.1])**.** *Let R be a commutative ring. Then a submodule A of an R-module B is s-pure in B if and only if it is coneat in B.*

Remark 2.2. If A is a pure submodule of B, then $AI = A \cap BI$ for every left ideal I of R (see, [10, Corollary 4.92]). Therefore, pure submodules in every R-module are coneat by Proposition 2.1 over commutative rings.

An R-module A is said to be m-injective (absolutely pure) if it is neat (pure) in every module that contains it as a submodule, equivalently it is a neat (pure) submodule of an injective R-module. It is time to substantiate our claim that *closed* is in general inequivalent to *coneat*.

Example 2.3. This example exhibits a pure submodule that is not closed. Let R be a commutative ring that is not noetherian. Then there exists an absolutely pure R-module M that is not injective by [12, Theorem 3]. This means that the sequence $0 \to M \hookrightarrow E(M) \to E(M)/M \to 0$ is pure and does not split. On the other hand, since M is essential in $E(M)$, M is not closed in $E(M)$.

Example 2.3 is also an example for coneat submodule that is not closed by Remark 2.2.

Remark 2.4. A commutative domain R is called almost perfect if R/I is a perfect ring for each nonzero ideal I of R. It is clear that almost perfect domains are C-rings. In [17], the authors proved that, if R is an almost perfect domain, then an R-module M is injective if and only if $\mathrm{Ext}(S, M) = 0$ (i.e., M is m-injective) for each simple module S. Actually, one of the characterizations of right C-rings is the following: R is a right C-ring if and only if every m-injective right R-module is injective (see, [21, Lemma 4]). Note that right perfect rings are examples for left C-rings.

Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. For a right R-module M, $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact if and only if $0 \to \mathrm{Hom}(C^+, M) \to \mathrm{Hom}(B^+, M) \to \mathrm{Hom}(A^+, M) \to 0$ is exact (see [20, Theorem 8.1]).

Proposition 2.5. *Let R be a right C-ring. Then the s-pure submodules in each projective R-module coincide with the pure submodules.*

Proof. Let F be a projective R-module and A an s-pure submodule of F. Then the exact sequence $0 \to (F/A)^+ \to F^+ \to A^+ \to 0$ is neat exact. By [6, Theorem 3.2.10], F^+ is injective, and so $(F/A)^+$ is m-injective. Since R is right C-ring, $(F/A)^+$ is also injective. Then F/A is flat by [6, Theorem 3.2.10]. Hence A is a pure submodule of F by [10, Theorem 4.85]. □

An R-module M is called extending if every closed submodule of M is a direct summand. Extending modules are a generalization of injective modules. It is well known that every projective R-module is extending if and only if R is a left co-H ring, [15]. If R is commutative ring, then R is quasi-Frobenious if and only if R is a co-H ring (see [15, Theorem 4.4]).

Example 2.6. We give an example for a closed submodule that fails to be coneat. Let R be a commutative perfect ring which is not quasi-Frobenius. Then there is a projective R-module F which is not extending. Assume that K is the closed submodule of F which is not direct summand in F. We claim that K is not coneat in F. If K is coneat in F, then it is pure in F since R is C-ring by Proposition 2.5. So that F/K is a flat R-module, and it is projective since R is a perfect ring. Then K is direct summand of F, a contradiction.

Let $R = F[x_1, x_2, \ldots]$, where F is a field and the x_i are commuting indeterminants satisfying the relations

$$x_i^3 = 0 \text{ for all } i,\ x_i x_j = 0 \text{ for all } i \neq j,\ x_i^2 = x_j^2 \text{ for all } i \text{ and } j.$$

The ring R is a commutative, semiprimary, local, but not quasi-Frobenius (see [14, p. 77]). Thus, R is an example of a commutative perfect ring which is not quasi-Frobenius.

A submodule A of an R-module B is called complement of K in B if $K \cap A = 0$ and A is maximal with respect to this property. It is known that closed submodules and complement submodules in a module coincide (see [5, §1]).

Lemma 2.7. *Let R be a ring. The following are equivalent*:

(1) *R is left noetherian.*
(2) *Pure submodules in any left R-module are closed.*
(3) *Pure submodules in any injective left R-module are closed.*

Proof. (1) ⇒ (2) Let B be an R-module and A a pure submodule of B. Let K be a complement of A in B and let A' be a complement of K in B containing A. Then A is an essential submodule of A'. Suppose that $A \neq A'$ and let $a \in A' \setminus A$. Then A is essential in $Ra + A$, and $(Ra + A)/A$ is finitely presented since R is a left noetherian ring. As A is a pure submodule of A', it is pure in $Ra + A$. But $(Ra + A)/A \cong Ra/(Ra \cap A)$ is finitely presented, so the sequence $0 \to A \hookrightarrow Ra + A \to (Ra + A)/A \to 0$ splits. But this is a contradiction with the essentiality of A in $Ra + A$. Hence $A = A'$, and so A is closed in B.

(2) ⇒ (1) Note that closed submodules of an injective R-module are injective. Let M be an absolutely pure left R-module. Then M is pure in $E(M)$ and is closed in $E(M)$ by the assumption. Therefore M is injective, and so R is left noetherian by [12, Theorem 3].

(1) ⇔ (3) By [12, Theorem 3]. □

Proposition 2.8. *A finite direct product of left C-rings is also a left C-ring.*

Proof. Assume that R is a finite direct product of the left C-rings $R_1, R_2, \ldots$, R_n. We will show that $\mathrm{Soc}(R/I) \neq 0$ for each essential left ideal I of R. By assumption, $I = I_1 \times I_2 \times \ldots \times I_n$, where $I_i \leq R_i$ for $i = 1, 2, \ldots, n$. Since I is essential ideal of R, I_i is essential ideal of R_i for $i = 1, 2, \ldots, n$. Then $\mathrm{Soc}(R_i/I_i) \neq 0$ for $i = 1, 2, \ldots, n$. $\mathrm{Soc}(R/I) \cong \prod_i^n \mathrm{Soc}(R_i/I_i) \neq 0$, as desired. □

Proposition 2.9. *A left-right noetherian hereditary ring is left-right C-ring.*

Proof. By [11, Proposition 5.4.5], the left (right) R-module R/I has finite length for every essential (proper) left (right) ideal I of R. Since R is left (right) noetherian, R is left (right) C-ring by [16, Corollary to Theorem 1.2)]. □

A ring R is called right distributive (or arithmetic) if for any right ideals I, J and K of R, $(I + J) \cap K = (I \cap K) + (J \cap K)$. All strongly regular rings, all valuation rings in division rings, all commutative hereditary rings and all commutative Dedekind rings are distributive, see [23]. It has been shown that R is a noetherian right distributive ring if and only if R is a finite direct product of artinian right uniserial rings and invariant hereditary noetherian domains; R is a distributive right noetherian ring if and only if R is a distributive left noetherian ring if and only if R is a finite direct product of uniserial artinian rings and invariant hereditary noetherian domains (see [23, Theorem 9.18]).

We have the following by Proposition 2.9 and Proposition 2.8.

Corollary 2.10. *Noetherian right distributive rings and distributive right* (*left*) *noetherian rings are C-rings.*

We can now state the main result of this section.

Theorem 2.11. *Let R be a commutative ring. The following are equivalent.*

(1) *Neat submodules in any R-module are coclosed, and R is noetherian.*
(2) *Coneat submodules in any R-module are closed.*
(3) *R is noetherian distributive.*

Proof. (1) ⇒ (2) We claim that R is a C-ring. Let M be an m-injective R-module. $\mathrm{Soc}(E(M)/M) = 0$ by [3, Theorem 3]. Consider the exact sequence $0 \to K \hookrightarrow F \to E(M)/M \to 0$ where K is a submodule of a free R-module F. K is neat submodule of F, since $\mathrm{Hom}(S, E(M)/M) = 0$ for any simple R-module S. Then K is also coclosed in F by (1). But coclosed submodules in every projective R-module coincide with pure submodules over noetherian ring by [24, Satz 3.4]. Hence $E(M)/M$ is a flat R-module by [10, Corollary 4.86]. Then M is pure in $E(M)$ by [10, Theorem 4.85], and so M is absolutely pure. By [12, Theorem 3], M is injective. Then R is C-ring since every m-injective R-module is injective by [21, Lemma 4].

Since every coclosed submodules in R-module are coneat, neat submodules in R-module are coneat by (1). Then neat and coneat submodules coincide by

[7, p. 138], and hence coneat submodules in R-module are closed since R is C-ring by [8, Theorem 5].

(2) $\Rightarrow$ (3) By Remark 2.2, pure submodules in every R-module are coneat. Then pure submodules in any R-module are closed, and by Lemma 2.7, R is noetherian.

Since coclosed submodules are coneat, they are also closed by (2). Then R is distributive by [25, Lemma 3.1].

(3) $\Rightarrow$ (1) By Corollary 2.10, R is a C-ring. Hence, neat submodules are closed, and they are coclosed by [25, Lemma 3.1]. □

By Theorem 2.11, we have the following:

Corollary 2.12. *Let R be a commutative ring. Closed, neat and coneat submodules in each R-module coincide if and only if R is a noetherian distributive ring.*

Note that commutative distributive domains are semihereditary domains. It has been shown that if R is a Dedekind domain, then closed and coneat submodules in an R-module coincide (see [13, Corollary 4.6]). The following corollary shows that the converse is also true.

Corollary 2.13. *Let R be a commutative domain. Closed and coneat submodules coincide if and only if R is a Dedekind domain.*

3. On *c*-injective modules

Recall that an R-module M is said to be c-injective if it has the injective property with respect to all closed exact sequences. c-injective modules have been discussed in [13], [18], [19]. Throughout this section, R is a commutative ring.

We begin with the following:

Definition 3.1. An R-module D is called coneat-injective if it is injective with respect to the coneat monomorphisms.

Clearly, injective modules are coneat-injective, and simple modules are coneat-injective by Proposition 2.1.

The following lemma can be proved by using similar arguments as for injective modules.

Lemma 3.2. *Let $\{D_i\}_{i\in I}$ be a class of R-modules. Then $\prod_{i\in I} D_i$ is a coneat injective R-module if and only if D_i is coneat-injective for all $i \in I$.*

In particular, direct product of simple R-modules is coneat-injective since simple R-modules coneat-injective.

Lemma 3.3. *For any R-module M, there is an extension D of M such that D is coneat-injective and M is a coneat submodule of D.*

Proof. Let $\{S_j\}_{j\in J}$ be a set of representative simple R-modules. Consider the R-module

$$D = (\prod_{m\in M} E(Rm)) \otimes (\prod_{j\in J}(\prod_{\phi_j} S_j)),$$

where $E(Rm)$ is the injective hull of Rm for each $m \in M$, while ϕ_j runs over the non-zero elements of $\mathrm{Hom}_R(M, S_j)$. By Lemma 3.2, D is a coneat-injective module. The map $\phi : M \to D$ is defined by mapping $m \in M$ to $m \in Rm$ and acting on S_j as ϕ_j. It is obvious that ϕ is injective. By Proposition 2.1, M is coneat submodule of D, since, by construction, all the simple R-modules have the injective property with respect to ϕ. □

Theorem 3.4. *Let R be a ring and E an R-module. The following are equivalent.*

(1) *E is a coneat-injective module.*
(2) *For any coneat monomorphism $f : A \to B$, every homomorphism from A to E can be extended to a homomorphism from B to E.*
(3) *E is a direct summand of every R-module L such that E is a coneat submodule of L.*
(4) *E is isomorphic to a direct summand of a direct product of simple modules and injective hull of some cyclic modules.*

Proof. (1) ⇒ (2) ⇒ (3) are clear.

(3) ⇒ (4) In view of the proof of Lemma 3.3, E is a coneat submodule of D. Then, by (3), E is isomorphic to a direct summand of a direct product of simple modules and injective hull of some cyclic modules.

(4) ⇒ (1) By Lemma 3.2. □

The following is an immediate consequence of Theorem 3.4 and Corollary 2.12.

Corollary 3.5. *Let R be a noetherian distributive ring. Then an R-module D is coneat-injective if and only if it is c-injective. In particular, every direct product of simple R-modules is c-injective.*

Recall that Dedekind domains are exactly noetherian distributive domains. Thus, we have the following as a consequence of Corollary 3.5:

Corollary 3.6 ([13, Lemma 4.7])**.** *Let R be a Dedekind domain. Then an R-module D is c-injective if and only if it is isomorphic to a direct summand of a direct product of simple R-modules and injective R-modules.*

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