Bull. Korean Math. Soc. **52** (2015), No. 5, pp. 1433–1443
http://dx.doi.org/10.4134/BKMS.2015.52.5.1433

CIRCLE-FOLIATED MINIMAL SURFACES IN 4-DIMENSIONAL SPACE FORMS

Sung-Ho Park

Abstract. Catenoid and Riemann's minimal surface are foliated by circles, that is, they are union of circles. In $\mathbb{R}^3$, there is no other nonplanar example of circle-foliated minimal surfaces. In $\mathbb{R}^4$, the graph G_c of $w = c/z$ for real constant c and $\zeta \in \mathbb{C}\backslash\{0\}$ is also foliated by circles. In this paper, we show that every circle-foliated minimal surface in $\mathbb{R}^n$ is either a catenoid or Riemann's minimal surface in some 3-dimensional Affine subspace or a graph surface G_c in some 4-dimensional Affine subspace. We use the property that G_c is circle-foliated to construct circle-foliated minimal surfaces in S^4 and H^4.

1. Introduction

A surface $M \in \mathbb{R}^n$ is said to be circle-foliated if there is a one-parameter family of planes whose intersection with M are circles. The catenoid and Riemann's minimal surface are examples of circle-foliated minimal surfaces in $\mathbb{R}^3$. Enneper proved that the planes containing the circles of a circle-foliated minimal surface in $\mathbb{R}^3$ should be parallel [2] and [7]. Then it is easy to see that the plane, catenoid and Riemann's minimal surface are the only circle-foliated minimal surfaces in $\mathbb{R}^3$ [7].

One may consider $\mathbb{R}^4$ as $\mathbb{C}^2$ with complex coordinates (z, w). For a real constant $c \neq 0$, the graph $G_c = \{(w, z) \in \mathbb{C}^2 \,|\, wz = c\}$ is circle-foliated. In fact, the image g_r of the circle $\{|z| = r\}$ on the z-plane is $\{(z, c/z) \,|\, |z| = r\}$. Considering $\mathbb{C}^2$ as $\mathbb{R}^4$, g_r lies on the plane through $(0, 0, 0, 0)$, $(1, 0, -r^2, 0)$ and $(0, 1, 0, r^2)$ (cf. Remark 2). Since $|(z, c/z)|^2 = r^2 + c^2/r^2$, g_r is a circle. Therefore G_c is circle-foliated. Since every complex submanifold of a Kaehler manifold is minimal [6], G_c is minimal. Moreover, G_c is complete, doubly-connected and has finite total curvature -4π with two planar ends, which are asymptotic to the planes $\{z = 0\}$ and $\{w = 0\}$ (cf. Remark 2). Hoffman and Osserman classified complete simply-connected and doubly-connected minimal surfaces in $\mathbb{R}^n$ with total curvature -4π including G_c [4]. They showed that

Received February 10, 2014; Revised May 24, 2014.
2010 *Mathematics Subject Classification.* 53A10, 53C12.
Key words and phrases. circle-foliated surface, minimal surface in $\mathbb{S}^4$ and $\mathbb{H}^4$.
This work was supported by Hankuk University of Foreign Studies Research Fund.

such minimal surfaces are foliated by ellipses or circles, and called them as the *skew catenoids*.

In this paper, we show that every circle-foliated minimal surface in $\mathbb{R}^n$ is either a catenoid or a Riemann's minimal surface in a 3-dimensional Affine subspace or the graph surface G_c in a 4-dimensional Affine subspace. Therefore there is no counterpart of the Riemann's minimal surface in $\mathbb{R}^n$, for $n \geq 4$. We then use this property of G_c to construct circle-foliated minimal surfaces in $\mathbb{S}^4$ and $\mathbb{H}^4$.

2. Circle-foliated minimal surfaces in $\mathbb{R}^n$

Let Σ be a circle-foliated surface in $\mathbb{R}^n$. Let $\{P_t\}$ be the one-parameter family of planes on which the circles of the foliation is on. Let $\tilde{P}_t$ be the plane parallel to P_t and passes through the origin of $\mathbb{R}^n$. There is a one-parameter family of orthonormal basis of $\mathbb{R}^n$ satisfying Frenet type equations [3].

Theorem A. *Let $\{\tilde{P}_t\}$ be a smooth one-parameter family of planes in $\mathbb{R}^n$. There is a one-parameter family of orthonormal basis $e_1(t), e_2(t), \ldots, e_n(t)$ of $\mathbb{R}^n$ such that $e_1(t)$ and $e_2(t)$ span $\tilde{P}_t$, and the following equations hold*

$$\begin{array}{lll} & e_i' = \alpha_i^j e_j + \kappa^i e_{2+i} & (i, l = 1, 2) \\ (1) & e_{2+i}' = -\kappa^i e_i + \tau_i^l e_{2+l} + \gamma_i^\lambda e_{4+\lambda} & (\lambda = 1, \ldots, n-4) \\ & e_{4+\xi}' = -\gamma_l^\xi e_{2+l} + \beta_\xi^\lambda e_{4+\lambda} & (\alpha_i^j = -\alpha_j^i,\ \tau_i^l = -\tau_l^i,\ \beta_i^\lambda = -\beta_\lambda^i), \end{array}$$

where $\left(\kappa^1\right)^2 \geq \left(\kappa^2\right)^2$, and $' = \frac{d}{dt}$.

Using the above orthonormal basis of $\mathbb{R}^n$, we can parameterize a circle-foliated surface by

$$X(t, \theta) = c(t) + r(t)(\cos\theta e_1 + \sin\theta e_2), \tag{2}$$

where $c(t)$ and $r(t)$ are the center and the radius of the circle on P_t.

Theorem 1. *Circle-foliated minimal surface in $\mathbb{R}^n$ is either* i) *a catenoid or a Riemann's minimal surface in* 3*-dimensional Affine subspace or* ii) *a graph surface G_c* (*defined in* §1) *in some* 4*-dimensional Affine subspace.*

To prove the above theorem, we have to show that every circle-foliated minimal surface in $\mathbb{R}^n$, $n \geq 5$, actually lies in a (at most) 4-dimensional Affine subspace. First, let us assume that a circle-foliated surface X lies in $\mathbb{R}^5$. (The case of $n \geq 6$ is analogous to the case of $\mathbb{R}^5$.) For the simplicity of notations, we write (1) as

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}' = \begin{pmatrix} 0 & -\beta & -\kappa & 0 & 0 \\ \beta & 0 & 0 & \tau & 0 \\ \kappa & 0 & 0 & -\eta & -\nu \\ 0 & -\tau & \eta & 0 & -\mu \\ 0 & 0 & \nu & \mu & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}. \tag{3}$$

Let $c'(t)=\sum_{i=1}^{5}\alpha_i e_i$, where α_i's are smooth functions. Then we have

$$\begin{aligned} X_t =& (\alpha_1 + r'\cos\theta + r\beta\sin\theta)e_1 + (\alpha_2 + r'\sin\theta - r\beta\cos\theta)e_2 \\ &+ (\alpha_3 - r\kappa\cos\theta)e_3 + (\alpha_4 + r\tau\sin\theta)e_4 + \alpha_5 e_5, \\ X_\theta =& -r\sin\theta e_1 + r\cos\theta e_2. \end{aligned}$$

Let N be a normal vector of X given by

$$N = \cos\theta e_1 + \sin\theta e_2 + \gamma e_3 + \delta e_4 + \rho e_5. \tag{4}$$

Then γ, δ and ρ satisfy

$$\begin{aligned} X_t\cdot N = \alpha_1\cos\theta + \alpha_2\sin\theta + r' + \gamma(\alpha_3 - r\kappa\cos\theta) \\ + \delta(\alpha_4 + r\tau\sin\theta) + \rho\alpha_5 = 0. \end{aligned} \tag{5}$$

Let E, F, G be the coefficients of the first fundamental form of X. Then

$$\begin{aligned} E =& |X_t|^2 \\ =& \sum_{i=1}^{5}\alpha_i^2 + r'^2 + r^2\beta^2 + 2r'\alpha_1\cos\theta + 2r'\alpha_2\sin\theta \\ &+ 2r\alpha_1\beta\sin\theta - 2r\alpha_2\beta\cos\theta - 2r\alpha_3\kappa\cos\theta \\ &+ r^2(\kappa^2-\tau^2)(\cos\theta)^2 + 2r\alpha_4\tau\sin\theta + r^2\tau^2, \\ F =& X_t\cdot X_\theta = -r\alpha_1\sin\theta + r\alpha_2\cos\theta - r^2\beta, \\ G =& |X_\theta|^2 = r^2. \end{aligned}$$

Lemma 1. *The surface $X(t,\theta)$ defined by* (2) *with $\tau\neq 0$ is minimal only if* i) *$\alpha_i=0$ for all $i=1,\dots,5$,* ii) *$\mu=\nu=0$ and* iii) *$\kappa^2=\tau^2$, $\beta\kappa=\tau\eta$ and $\beta\tau=\kappa\tau$. Hence $X(t,\theta)$ lies in a 4-dimensional Affine subspace.*

Proof. Let $l=X_{tt}\cdot N$, $m=X_{t\theta}\cdot N$ and $n=X_{\theta\theta}\cdot N$, where N is given by (4). Since $X(t,\theta)$ is minimal, we must have

$$\mathcal{H} := lG + nE - 2mF = 0.$$

Direct computation shows that

$$\mathcal{H} = \begin{array}{l} r^2\left\{\begin{array}{l} \alpha_1'\cos\theta + r'' + \alpha_2\beta\cos\theta - \alpha_1\beta\sin\theta + \alpha_2'\sin\theta - r\beta^2 \\ +\alpha_3\kappa\cos\theta - r\kappa^2(\cos\theta)^2 - \alpha_4\tau\sin\theta - r\tau^2(\sin\theta)^2 \\ +\gamma\left(\begin{array}{l}\alpha_3' - 2r'\kappa\cos\theta - \alpha_1\kappa - r\kappa'\cos\theta \\ -r\beta\kappa\sin\theta + \alpha_4\eta + r\tau\eta\sin\theta + \alpha_5\nu\end{array}\right) \\ +\delta\left(\begin{array}{l}\alpha_4' + 2r'\tau\sin\theta + \alpha_2\tau + r\tau'\sin\theta \\ -r\beta\tau\cos\theta - \alpha_3\eta + r\kappa\eta\cos\theta + \alpha_5\mu\end{array}\right) \\ +\rho\left(\ \alpha_5' + r\kappa\nu\cos\theta - \alpha_3\nu - \alpha_4\mu - r\tau\mu\sin\theta\ \right) \end{array}\right\} \\ -r\left\{\begin{array}{l} \sum_{i=1}^{5}\alpha_i^2 + r'^2 + r^2\beta^2 + 2r'\alpha_1\cos\theta + 2r'\alpha_2\sin\theta \\ +2r\alpha_1\beta\sin\theta - 2r\alpha_2\beta\cos\theta - 2r\alpha_3\kappa\cos\theta \\ +r^2(\kappa^2-\tau^2)(\cos\theta)^2 + 2r\alpha_4\tau\sin\theta + r^2\tau^2 \end{array}\right\} \\ -2(r\beta + r\gamma\kappa\sin\theta + r\delta\tau\cos\theta)(-r\alpha_1\sin\theta + r\alpha_2\cos\theta - r^2\beta). \end{array} \tag{6}$$

Since γ, δ and ρ satisfy (5), we first let $\gamma = -(\alpha_1 \cos\theta + \alpha_2 \sin\theta + r')/(\alpha_3 - r\kappa\cos\theta)$, $\delta = \rho = 0$ and $\mathcal{S} := \mathcal{H}(\alpha_3 - r\kappa\cos\theta)$. Then the coefficients of $\cos(3\theta)$ and $\sin(3\theta)$ of $\mathcal{S}$ are $r^2\kappa\left(r^2(\kappa^2 - \tau^2) + \alpha_1^2 - \alpha_2^2\right)/2$ and $r^2\kappa\alpha_1\alpha_2$ respectively. Since these should be equal to zero and $\kappa^2 \geq \tau^2$, we necessarily have $\alpha_1 = 0$.

Let $\delta = -(\alpha_2 \sin\theta + r')/(\alpha_4 + r\tau\sin\theta)$, $\gamma = \rho = 0$ and $\mathcal{T} := \mathcal{H}(\alpha_4 + r\tau\cos\theta)$. The coefficients of $\cos(2\theta)$ of $\mathcal{S}$ and $\sin(2\theta)$ of $\mathcal{T}$ are $r^3\left(-5\alpha_3\kappa^2 + 2\alpha_3\tau^2 + \alpha_2\tau\eta\right)$ and $\left(3r^3\alpha_3\kappa\tau - r^3\alpha_2\kappa\eta\right)/2$ respectively, which are equal to 0. Hence we have either $\kappa^2 - \tau^2 = 0$ or $\alpha_3 = 0$. On the other hand, the coefficient of $\cos(3\theta)$ of $\mathcal{T}$ is $-r^4\tau\left(\kappa^2 - \tau^2\right)$, which implies that $\kappa^2 = \tau^2$. Then we have $\alpha_2 = 0$, and since we assumed that $\tau^2 > 0$, we also have $\alpha_3 = 0$. Substituting these into $\mathcal{S}$, we have $\alpha_4 = 0$ from the coefficient of $\sin(2\theta)$.

Suppose that $\alpha_5 \neq 0$, and let $\gamma = \delta = 0$ and $\rho = -r'/\alpha_5$. Then $\mathcal{H}$ becomes

$$r^2[r'' - 2r\kappa^2(\cos\theta)^2 - 2r\tau^2(\sin\theta)^2 - \frac{r'}{\alpha_5}(\alpha_5' + r\kappa\nu\cos\theta - r\tau\nu\sin\theta)] \\ -r(\alpha_5^2 + {r'}^2).$$

Therefore we have $\kappa = \tau = 0$, which contradicts the assumption $\kappa, \tau \neq 0$. From (5), (6) and $\alpha_5 = 0$, it follows that $\mu = \nu = 0$. This completes the proofs of i) and ii).

From the coefficients of $\sin\theta$ of $\mathcal{S}$ and $\cos\theta$ of $\mathcal{T}$, we have

$$\beta\kappa = \tau\eta \tag{7}$$

and

$$\beta\tau = \kappa\eta. \tag{8}$$

Since $\mu = \nu = 0$ and $\alpha_i = 0$ for all $i = 1, \ldots, 5$, the surface $X(t,\theta)$ lies in a 4-dimensional Affine subspace. □

Remark 1. When $n \geq 6$ and $X(t,\theta)$ is minimal, it is easy to see in the above proof that $\alpha_k = 0$ for $k \geq 5$ and $\gamma_i^\lambda = 0$ and $\beta_\xi^\lambda = 0$. Hence $X(t,\theta)$ should lie in a 4-dimensional Affine subspace.

Lemma 2. *If the surface $X(t,\theta)$ defined by* (2) *is minimal with $\tau \equiv 0$, then the planes $\tilde{P}$ lie in some* 3*-dimensional Affine subspace.*

Proof. When $\tau \equiv 0$, we consider two cases; $\alpha_4 \equiv 0$ or $\alpha_4 \not\equiv 0$. First of all, we have $\alpha_1 = 0$ as in the proof of the above lemma. If $\tau \equiv 0$ and $\alpha_4 \not\equiv 0$, then we let $\gamma = \rho = 0$ and $\delta = -(\alpha_2 \sin\theta + r')/\alpha_4$. The coefficient of $\cos(2\theta)$ of $\mathcal{H}$ is $-2r^3\kappa^3$. Since this must be 0, we have $\kappa \equiv 0$. Then η, ν and μ can be chosen to be zero, and $\tilde{p}$ are parallel planes in a 3-dimensional Affine subspace.

If $\alpha_4 \equiv 0$, then $\mathcal{H}$ is independent of the choice of δ. Hence we have $\eta \equiv 0$ and $\alpha_5\mu \equiv 0$. If $\alpha_5 \equiv 0$, then we should have $\nu \equiv 0$. From (3), e_4 and e_5 are independent of e_1, e_2 and e_3, and $\tilde{p}$ lie in a 3-dimensional Affine subspace. If $\alpha_5 \neq 0$, let $\gamma = \delta = 0$ and $\rho = -(\alpha_2 \sin\theta + r')/\alpha_5$. Then the coefficient of $\cos(2\theta)$ of $\alpha_5\mathcal{H}$ is $-2r^3\kappa$, which should be 0. Therefore e_1 and e_2 are independent of e_3, e_4 and e_5, and $\tilde{p}$ lie in a 3-dimensional Affine subspace. □

Lemma 3. *The circle-foliated minimal surfaces in* $\mathbb{R}^4$ *of Lemma 1 is the graph* G_c *for some real* c.

Proof. From (7), (8) and $\kappa^2 = \tau^2$, we see that $\beta^2 = \kappa^2 = \tau^2 = \eta^2$. Suppose that $\beta = \kappa$ and $\tau = \eta$ (the case $\beta = -\kappa$ and $\tau = -\eta$ can be dealt with in the same way). It follows that $(e_2+e_3)' = 0$ and $(e_1 \pm e_4)' = 0$ (depending on the sign of κ/τ). We may suppose that $e_1 + e_4 = (0, 0, 0, \sqrt{2})$ and $e_2 + e_3 = (0, 0, \sqrt{2}, 0)$. Then we have

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{2}}(\cos\psi(t), \sin\psi(t), 0, 1), \\ e_2 &= \frac{1}{\sqrt{2}}(\cos\phi(t), \sin\phi(t), 1, 0), \\ e_3 &= \frac{1}{\sqrt{2}}(-\cos\phi(t), -\sin\phi(t), 1, 0), \\ e_4 &= \frac{1}{\sqrt{2}}(-\cos\psi(t), -\sin\psi(t), 0, 1). \end{aligned}$$

From $e_1' = -\beta e_2 + \beta e_3$, we see that $2\beta = \pm\psi'$. If $\psi' = 2\beta$, then we have $\psi = \pi/2 + \phi$. Moreover $e_3' = -\beta e_1 + \tau e_4$ implies that $\beta = \eta$. Similarly, when $\psi' = -2\beta$, we have $\kappa = \eta$. Therefore we may assume that $\beta = \kappa = \tau = \eta = 1$. Then direct computation shows that

$$\mathcal{H} = r\left(rr'' - 3{r'}^2 - 2r^2\right)$$

for the normals of $X(t, \theta)$ corresponding to the cases i) $\gamma = -r'/r\cos\theta$, $\delta = 0$ and ii) $\gamma = 0$, $\delta = r'/r\sin\theta$. Hence r satisfies

$$rr'' - 3{r'}^2 - 2r^2 = 0. \tag{9}$$

The solution of (9) is $r = C_1(\cos(2t+C_2))^{-1/2}$, where C_1 and C_2 are constants.

We may let $C_1 = c$ and $C_2 = 0$ and $-\pi/4 < t < \pi/4$. Let A be the 4×4 orthogonal matrix given by

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\tilde{X}(t, \theta) = A \circ X(t, \theta)$ represents the graph

$$G_c = \left\{\left(\zeta, \frac{c}{\zeta}\right) \mid \zeta \in \mathbb{C} - \{0\}\right\}.$$

□

Remark 2. i) The parametrization of G_c is given by

$$\tilde{X}(t, \theta) = \left(r\cos\theta, r\sin\theta, \frac{c}{r}\cos\theta, -\frac{c}{r}\sin\theta\right).$$

Clearly, G_c has two ends that are asymptotic to the planes $\{(w, z) \mid w = 0\}$ and $\{(w, z) \mid z = 0\}$.

ii) Let $g_r = \left(r\cos\theta, r\sin\theta, \frac{c}{r}\cos\theta, -\frac{c}{r}\sin\theta\right)$ be the circle on G_c for fixed r. The geodesic curvature of g_r is

$$\kappa_g = \frac{r|c^2 - r^4|}{(c^2 + r^4)^{3/2}}.$$

Hence we have

$$\int_{g_r} \kappa_g \, ds = 2\pi \frac{|c^2 - r^4|}{c^2 + r^4}.$$

Since G_c is doubly-connected, Gauss-Bonnet theorem implies that

$$\int_{G_c} K \, dA = -\lim_{r\to 0}\int_{g_r} \kappa_g \, ds - \lim_{r\to\infty}\int_{g_r} \kappa_g \, ds = -4\pi.$$

3. Circle-foliated minimal surfaces in $\mathbb{S}^4$ and $\mathbb{H}^4$

To construct a circle-foliated minimal surface in $\mathbb{S}^4$, we consider $\mathbb{R}^4$ with the conformal metric $ds_s^2 = ds_0^2 / \left((1 + \langle x, x\rangle)/2\right)^2$, where ds_0^2 is the Euclidean metric of $\mathbb{R}^4$ and $\langle , \rangle$ is the Euclidean inner product. Let H_s and H_0 be the mean curvatures of a surface M in $\mathbb{R}^4$ with respect to the metrics ds_s^2 and ds_0^2 respectively with respect to fixed Euclidean normal N satisfying (4). We have

$$H_s = \frac{1 + \langle x, x\rangle}{2|N|} H_0 + \left\langle x, \frac{N}{|N|}\right\rangle.$$

Similarly, to construct a circle-foliated minimal surfaces in $\mathbb{H}^4$, we equip the conformal metric $ds_h^2 = ds_0^2 / \left((1 - \langle x, x\rangle)/2\right)^2$ on the unit ball $B(O, 1)$ of $\mathbb{R}^4$. Then the mean curvature H_h of a surface M in $B(O, 1)$ with respect to ds_h^2 satisfies

$$H_h = \frac{1 - \langle x, x\rangle}{2|N|} H_0 - \left\langle x, \frac{N}{|N|}\right\rangle.$$

Examples of circle-foliated minimal surfaces in $\mathbb{S}^4$ and $\mathbb{H}^4$. Let e_1, e_2 be defined as in the proof of Theorem 3:

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{2}}(-\sin 2t, \cos 2t, 0, 1), \\ e_2 &= \frac{1}{\sqrt{2}}(\cos 2t, \sin 2t, 1, 0). \end{aligned} \tag{10}$$

The mean curvature of the circle-foliated surface

$$X(t, \theta) = r(t)\left(\cos\theta e_1 + \sin\theta e_2\right) \tag{11}$$

satisfies

$$H_0\,|N| = \frac{r\left(rr'' - 3r'^2 - 2r^2\right)}{2r^2\left(r^2 + r'^2\right)}.$$

Hence as a surface in $\mathbb{S}^4$, $X(t,\theta)$ has mean curvature

$$H_s\,|N| = \frac{1+r^2}{2}\cdot\frac{r\left(rr''-3r'^2-2r^2\right)}{2r^2\left(r^2+r'^2\right)}+r$$

for all normal direction. Therefore the circles centered at the origin on the planes spanned by $e_1(t)$, $e_2(t)$ with radius function $r(t)$ satisfying

$$\frac{1+r^2}{2}\cdot\frac{rr''-3r'^2-2r^2}{2r^2\left(r^2+r'^2\right)}+1=0 \tag{12}$$

define a circle-foliated minimal surface in $\mathbb{S}^4$.

Lemma 4. *Solution of* (12) *with the initial conditions*

$$r(0)=a^2>0 \text{ and } r'(0)=0$$

is periodic.

Proof. Note that if $r(t)$ is a solution of (12), then $r(-t)$ is also a solution of (12). Hence each solution r of (12) is an even function. Moreover if $r'(t_1)=0$, then $r(t_1+t)=r(t_1-t)$. Therefore it suffices to show that $r'(t_1)=0$ for some $t_1>0$.

Suppose that r' does not vanish except for $t=0$, therefore, $r'(t)>0$ for all $t>0$. From (12), we have

$$r''=\frac{\left(3-r^2\right)r'^2+2r^2\left(1-r^2\right)}{r\left(1+r^2\right)}. \tag{13}$$

We may assume that $a<1$. Then we have $r''(0)>0$ and $r'(t)>0$ for t close to 0. If r is not bounded, then $r''\to-\infty$ as $t\to\infty$ by (13). Then $r'\to-\infty$, which contradicts $r'(t)>0$ for all $t>0$.

If r is bounded, then $r''\to 0$ and $r'\to 0$ as $t\to\infty$. From (13) and the fact that $r''\to 0$ as $t\to\infty$, it follows that $r\nearrow 1$ as $t\to\infty$. From (13), we have $r''(t)>0$ for all $t>0$. On the other hand, since r is bounded and increasing, we should have $r''(t)<0$ for sufficiently large t. Hence we conclude that $r'(t_1)=0$ for some t_1 and r is periodic. □

To estimate the period of (13), we use the integrating factor to obtain a first integral

$$\frac{\left(1+r^2\right)^4}{r^6}\left(r'\right)^2+\left(\frac{1}{r^4}+\frac{4}{r^2}+4r^2+r^4\right)=C,$$

or

$$\left(r+\frac{1}{r}\right)^4\left(\frac{r'}{r}\right)^2+\left(r+\frac{1}{r}\right)^4=C. \tag{14}$$

We suppose that $r'(0) = 0$ and the minimum value $r_{\min}$ is attained at $t = 0$, that is, $r_{\min} = r(0)$. Then

$$C = \left(r_{\min} + \frac{1}{r_{\min}} \right)^4 .$$

Let $r_{\max}$ be the maximum value of r and $r_{\max} = r(t_{\max})$ so that the period of r is $2\, t_{\max}$. Then have

$$r_{\min} = \frac{1}{r_{\max}}.$$

Since r is strictly increasing on $(0, t_{\max})$, we consider the inverse function $t = t(r)$ of $r(t)$. From (14), we have

$$t'^2 = \frac{1}{r^2} \frac{\left(r^2+1\right)^4}{Cr^4 - \left(r^2+1\right)^4}.$$

Then

$$t_{\max} = \int_{r_{\min}}^{r_{\max}} \frac{1}{r} \frac{\left(r^2+1\right)^2}{\sqrt{Cr^4 - \left(r^2+1\right)^4}}\, dr.$$

Since $1/r$ also satisfies (12), we have

$$t_{\max} = 2\int_{r_{\min}}^{1} \frac{1}{r} \frac{\left(r^2+1\right)^2}{\sqrt{Cr^4 - \left(r^2+1\right)^4}}\, dr.$$

Substituting $R = r^2$, we get

$$t_{\max} = 2\int_{r_{\min}^2}^{1} \frac{1}{2R} \frac{\left(R+1\right)^2}{\sqrt{CR^2 - \left(R+1\right)^4}}\, dR. \tag{15}$$

For convenience, we let $c^2 = C$ with $c \geq 4$, $p = \sqrt{(c+4)/c}$ and $k^2 = (c+4)/(c-4)$.

Substituting $R = (\rho - p)/(\rho + p)$, we have

$$\begin{aligned}\frac{dR}{\sqrt{c^2R^2 - (R+1)^4}} &= \frac{2p\, d\rho}{\sqrt{(c+4)(\rho^2-1)\left((c-4)\rho^2 - (c+4)\right)}} \\ &= \frac{2}{\sqrt{c(c-4)}} \frac{d\rho}{\sqrt{(\rho^2-1)(\rho^2-k^2)}}.\end{aligned}$$

Then (15) becomes

$$t_{\max} = \frac{8}{\sqrt{c(c-4)}} \int_{\rho_0}^{\infty} \frac{\rho^2\, d\rho}{(\rho^2-p^2)\sqrt{(\rho^2-1)(\rho^2-k^2)}},$$

where $\rho_0 = p(1+r_{\min}^2)/(1-r_{\min}^2) = k$.

Substituting $\tau = k/\rho$, we get

$$(16)\qquad \begin{aligned} t_{\max} =& \frac{8}{\sqrt{c(c+4)}} \int_0^1 \frac{d\tau}{\left(1 - \left(\frac{p}{k}\right)^2 \tau^2\right) \sqrt{\left(1-\tau^2\right)\left(1 - \frac{\tau^2}{k^2}\right)}} \\ =& \frac{8}{\sqrt{c(c+4)}} \Pi\left(\frac{c-4}{c} \mid \frac{c-4}{c+4}\right), \end{aligned}$$

where $\Pi\left((c-4)/c \mid (c-4)/(c+4)\right)$ is the complete elliptic integral of the third kind. For the following lemma, we introduce the elliptic integral of the 1st kind $F(\phi, \alpha)$ and the elliptic integral of the 2nd kind $E(\phi, \alpha)$:

$$\begin{aligned} F(\phi, \alpha) &= \int_0^\phi \frac{d\theta}{\sqrt{1 - \sin^2\alpha \sin^2\theta}}, \\ E(\phi, \alpha) &= \int_0^\phi \sqrt{1 - \sin^2\alpha \sin^2\theta}\, d\theta. \end{aligned}$$

Moreover, $K(\alpha) = F(\pi/2, \alpha)$ and $E(\alpha) = E(\pi/2, \alpha)$ are the complete elliptic integrals of the first and second kinds respectively. Letting $k = \sin\alpha$, we also let $E(k) = E(\pi/2\,; k)$, $K(k) = F(\pi/2\,; k)$.

Note that $c^2 = \left(r_{\min} + 1/r_{\min}\right)^2$. If $c \to 4$ or $r_{\min} \to 1$, then $t_{\max} \to \pi/\sqrt{2}$. In this case, we have $r \equiv 1$ and the resulting minimal surface is a torus.

Lemma 5. *As a function of* $c \geq 4$, $t_{\max}$ *is decreasing and satisfies*

$$\lim_{c\to\infty} t_{\max} = \frac{\pi}{2}.$$

Hence the period of the solution of (12) *is between* π *and* $\sqrt{2}\pi$.

Proof. Straightforward computation shows that

$$\frac{d}{dc}\left(\frac{8}{\sqrt{c(c+4)}} \Pi\left(\frac{c-4}{c} \mid \frac{c-4}{c+4}\right)\right) = \frac{E\left(\frac{c-4}{c+4}\right) - K\left(\frac{c-4}{c+4}\right)}{2(c-4)\sqrt{c(c+4)}}.$$

Since

$$E\left(\frac{c-4}{c+4}\right) - K\left(\frac{c-4}{c+4}\right) < 0,$$

$t_{\max}$ is a decreasing function of c.

Let $\alpha = \sin^{-1}\sqrt{(c-4)/(c+4)}$ with $0 < \alpha < \pi/2$ and let $\nu = (c-4)/c$. According to [1], the integral $\Pi\left((c-4)/c\,; \pi/2, (c-4)/(c+4)\right) = \Pi\left(\nu; \pi/2, \alpha\right)$ belongs to the circular case with $\sin^2\alpha < \nu < 1$, and

$$\Pi\left(\nu; \pi/2, \alpha\right) = K(\alpha) + \frac{\pi}{2}\delta_2\left(1 - \Lambda_0(\phi, \alpha)\right),$$

where Λ_0 is the Heuman's Lambda function satisfying

$$\Lambda_0(\phi, \alpha) = \frac{2}{\pi}\left[K(\alpha)\left(E(\phi, \pi/2 - \alpha) - F(\phi, \pi/2 - \alpha)\right) + E(\alpha) F(\phi, \pi/2 - \alpha)\right],$$

and

$$\delta_2 = \sqrt{\nu/(1-\nu)(\nu - \sin^2\alpha)} = \sqrt{c(c+4)}/4,$$
$$\phi = \sin^{-1}\sqrt{(1-\nu)/\cos^2\alpha} = \sin^{-1}\sqrt{(c+4)/2c}.$$

Therefore $\phi \to \pi/4$ and $\alpha \to \pi/2$ as $c \to \infty$.

Clearly,

$$\lim_{c\to\infty}(E(\phi,\pi/2-\alpha) - F(\phi,\pi/2-\alpha)) = -\lim_{c\to\infty}\int_0^{\phi}\frac{\cos^2\alpha\sin^2\theta}{\sqrt{1-\cos^2\alpha\sin^2\theta}}d\theta.$$

We note that $\cos\alpha = \sqrt{8/(c+4)}$ and that $\lim_{\alpha\nearrow\pi/2}\cos\alpha K(\alpha) = 0$ (cf. Lemma 8 of [5]). Then

$$\lim_{c\to\infty} t_{\max} = \lim_{c\to\infty}\frac{8}{\sqrt{c(c+4)}}\Pi(\nu;\pi/2,\alpha) = \frac{\pi}{2}.$$
□

Theorem 2. *The circle-foliated surface given by* (11) *with* e_1*,* e_2 *satisfying* (10) *and* r *satisfying* (12) *defines a one-parameter family of circle-foliated minimal surfaces in* $\mathbb{S}^4$*. Moreover, the radius function* r *is periodic with the period between* π *and* $\sqrt{2}\pi$*. Hence there are infinitely many circle-foliated immersed minimal tori in* $\mathbb{S}^4$*.*

In $\mathbb{H}^4$, we let e_1, e_2 and $X(t,\theta)$ be as in (10) and (11). Then the mean curvature of $X(t,\theta)$ with respect to ds_h^2 satisfies

$$H_h|N| = \frac{1-r^2}{2}\cdot\frac{r\left(rr'' - 3r'^2 - 2r^2\right)}{2r^2\left(r^2+r'^2\right)} - r$$

with N satisfying (4). Hence if $X(t,\theta)$ is minimal, then r satisfies

$$\frac{1-r^2}{2}\cdot\frac{rr'' - 3r'^2 - 2r^2}{2r^2\left(r^2+r'^2\right)} - 1 = 0. \tag{17}$$

We note that r'' blows up as $r \to 1$, when $X(t,\theta)$ approaches the ideal boundary of $\mathbb{H}^4$. For each initial condition $r(0) = b^2 < 1$, $r'(0) = 0$ of (17), $X(t,\theta)$ gives a complete circle-foliated minimal surface in $\mathbb{H}^4$.

Theorem 3. *The parametrization* (11) *with* e_1*,* e_2 *satisfying* (10) *and* r *satisfying* (17) *gives a one-parameter family of circle-foliated minimal surfaces in* $\mathbb{H}^4$*.*

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series **55**, Government Printing Office, 1964.

[2] A. Enneper, *Die cyklischen flächen*, Z. Math Phys. **14** (1869), 393–421.

[3] H. Frank and O. Giering, *Verallgemeinerte Regelflächen*, Math. Z. **150** (1976), no. 3, 261–271.

[4] D. Hoffman and R. Osserman, *The geometry of the generalized Gauss map*, Mem. Amer. Math. Soc. **28** (1980), no. 236, 105 pp.
[5] R. Hynd, S. Park, and J. McCuan, *Symmetric surfaces of constant mean curvature in* $\mathbb{S}^3$, Pacific J. Math. **241** (2009), no. 1, 63–115.
[6] H. Lawson, *Complete minimal surfaces in* $\mathbb{S}^3$, Ann. Math. **92** (1970), no. 3, 335–374.
[7] J. Nitsche, *Lectures on Minimal Surfaces*, Cambridge Univ. Press, Cambridge, 1989.

Major in Mathematics
Graduate School of Education
Hankuk University of Foreign Studies
Seoul 130-791, Korea
E-mail address: sunghopark@hufs.ac.kr