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ON A NEW CLASS OF SERIES IDENTITIES

Nidhi Shekhawat, Junesang Choi*, Arjun K. Rathie and Om Prakash

Abstract. We aim at giving explicit expressions of

$$\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho+i)_n m!n!},$$

where $i = 0, \pm1, \ldots, \pm9$ and $\{\Delta_n\}$ is a bounded sequence of complex numbers. The main result is derived with the help of the generalized Kummer's summation theorem for the series ${}_2F_1$ obtained earlier by Choi. Further some special cases of the main result considered here are shown to include the results obtained earlier by Kim and Rathie and the identity due to Bailey.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by (see, *e.g.*, [2, 6, 7])

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\; z\right] &= {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.1}$$

where $(a)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number a by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)\ldots(a+n-1), & n \in \mathbb{N} := \{1, 2, 3, \ldots\}. \end{cases} \tag{1.2}$$

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*Corresponding author

Using the fundamental relation $\Gamma(a+1) = a\Gamma(a)$, $(a)_n$ can be written in the form

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.3}$$

where Γ is the well-known gamma function (see, *e.g.*, [7, Section 1.1]).

It is well known that whenever a generalized hypergeometric function reduces to quotient of the products of the gamma function, the results become very important, in particular, from the application point of view. In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems for the series ${}_2F_1$, ${}_3F_2$ and others play an important role. Here we recall the following known identity involving product of generalized hypergeometric series:

$$ {}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix} \; x\right] \times {}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix} \; -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad}; \\ \rho, \, \frac{1}{2}\rho, \, \frac{1}{2}\rho + \frac{1}{2}; \end{matrix} \; -\frac{x^2}{4}\right], \tag{1.4}$$

which was established by Bailey [1] who used the following classical Kummer's summation theorem [2]:

$$ {}_2F_1\left[\begin{matrix} a, \, b; \\ 1+a-b; \end{matrix} \; -1\right] = \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1+a-b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(1+\frac{1}{2}a-b)}. \tag{1.5}$$

In 2009, Kim and Rathie [4] generalized the result (1.4) in the form

$$ {}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix} \; x\right] \times_0 F_1\left[\begin{matrix} \text{---} \; ; \\ \rho+i; \end{matrix} \; -x\right] \quad (i = 0, \, \pm 1, \, \ldots, \, \pm 5)$$

by using the following generalization of the classical Kummer summation theorem due to Lavoie, *et al.* [5]. In 2010, Choi [3] obtained the generalization of Kummer's summation theorem in the form:

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a, \, b; \\ 1+a-b+i; \end{matrix} \; -1\right] &= \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+|i|))} \\ &\times \left\{ \frac{\mathcal{A}_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)} \right. \\ &\left. + \frac{\mathcal{B}_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)} \right\}, \end{aligned} \tag{1.6}$$

where $i = 0, \, \pm 1, \, \ldots, \, \pm 9$. Here $[x]$ is the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients $\mathcal{A}_i$ and $\mathcal{B}_i$ are given in the tables.

In this paper, we aim at giving explicit expressions of the following double summation:

$$\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho+i)_n m!n!} \quad (i = 0, \pm 1, \ldots, \pm 9)$$

by employing the known result (1.6). Furthermore the results obtained earlier by Kim and Rathie [4] are shown to belong to special cases of our main result. It is also noted that the easily-derived results here are simple, interesting, and (potentially) useful.

2. Main Results

We state a new class of series identities to be established as in the following theorem.

Theorem 2.1. *Let $\{\Delta_n\}$ be a bounded sequences of complex numbers. Then the following general result holds true:*

(2.1)

$$\begin{aligned}
\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho+i)_n m!n!} &= \frac{\Gamma(\frac{1}{2})\Gamma(\rho)\Gamma(\rho+i)}{\Gamma(\rho+\frac{1}{2}(i+|i|))} \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-1)^m x^{2m}}{(\frac{1}{2})_m m!(\rho+\frac{1}{2}(i+|i|))_{2m}} \\
&\times \left\{ \frac{\mathcal{A}'_i}{\Gamma(\rho+\frac{1}{2}i)\Gamma(\frac{1}{2}+\frac{1}{2}i-[\frac{1+i}{2}])} \frac{(\frac{1}{2}-\frac{1}{2}i+[\frac{1+i}{2}])_m}{(\rho+\frac{1}{2}i)_m} \right. \\
&\left. + \frac{\mathcal{B}'_i}{\Gamma(\rho-\frac{1}{2}+\frac{1}{2}i)\Gamma(\frac{1}{2}i-[\frac{i}{2}])} \frac{(1-\frac{1}{2}i+[\frac{i}{2}])_m}{(\rho+\frac{1}{2}i-\frac{1}{2})_m} \right\} \\
&+ \frac{2\Gamma(\frac{1}{2})\Gamma(\rho)\Gamma(\rho+i)}{\Gamma(1+\rho+\frac{1}{2}(i+|i|))} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-1)^m x^{2m+1}}{(\frac{3}{2})_m m!(\rho+1+\frac{1}{2}(i+|i|))_{2m}} \\
&\times \left\{ \frac{\mathcal{A}''_i}{\Gamma(\rho+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}i-[\frac{1+i}{2}])} \frac{(1-\frac{1}{2}i+[\frac{1+i}{2}])_m}{(\rho+\frac{1}{2}i+\frac{1}{2})_m} \right. \\
&\left. + \frac{\mathcal{B}''_i}{\Gamma(\rho+\frac{1}{2}i)\Gamma(\frac{1}{2}i-\frac{1}{2}-[\frac{i}{2}])} \frac{(\frac{3}{2}-\frac{1}{2}i+[\frac{i}{2}])_m}{(\rho+\frac{1}{2}i)_m} \right\}
\end{aligned}$$

for $i = 0, \pm 1, \ldots, \pm 9$. The coefficients $\mathcal{A}'_i$ and $\mathcal{B}'_i$ can be obtained from the table of $\mathcal{A}_i$ and $\mathcal{B}_i$ by changing a by $-2m$ and b by $1-\rho-2m$, respectively, and the coefficients $\mathcal{A}''_i$ and $\mathcal{B}''_i$ can be obtained from the table of $\mathcal{A}_i$ and $\mathcal{B}_i$ by changing a by $-2m-1$ and b by $-\rho-2m$, respectively.

Proof. For simplicity, let S denote the left-hand side of (2.1). Then, by using the following manipulation of double series (see, *e.g.*, [6, pp. 56-57]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k),$$

we have

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{\Delta_m (-1)^n x^m}{(\rho)_{m-n}(\rho+i)_n (m-n)!n!}.$$

Here using the following well-known identities (see, *e.g.*, [7, p. 5]):

$$(a)_{m-n} = \frac{(-1)^n (a)_m}{(1-a-m)_n} \quad (0 \leq n \leq m;\ n,\, m \in \mathbb{N}_0)\,,$$

and, for $a = 1$,

$$(m-n)! = \frac{(-1)^n\, m!}{(-m)_n} \quad (0 \leq n \leq m;\ n,\, m \in \mathbb{N}_0)\,,$$

after a little simplification, we get

$$S = \sum_{m=0}^{\infty} \frac{\Delta_m x^m}{(\rho)_m m!} \sum_{n=0}^{m} \frac{(-m)_n (1-\rho-m)_n (-1)^n}{(\rho+i)_n n!}.$$

Expressing the inner sum of the resulting double series in ${}_2F_1$, we find

$$S = \sum_{m=0}^{\infty} \frac{\Delta_m x^m}{(\rho)_m m!}\, {}_2F_1 \left[\begin{matrix} -m,\ 1-\rho-m\,; \\ \rho+i\,; \end{matrix} \ -1 \right].$$

Separating the series into even and odd powers of x yields

$$\begin{aligned} S = & \sum_{m=0}^{\infty} \frac{\Delta_{2m} x^{2m}}{(\rho)_{2m} (2m)!}\, {}_2F_1 \left[\begin{matrix} -2m,\ 1-\rho-2m\,; \\ \rho+i\,; \end{matrix} \ -1 \right] \\ & + \sum_{m=0}^{\infty} \frac{\Delta_{2m+1} x^{2m+1}}{(\rho)_{2m+1} (2m+1)!}\, {}_2F_1 \left[\begin{matrix} -2m-1,\ -\rho-2m\,; \\ \rho+i\,; \end{matrix} \ -1 \right]. \end{aligned}$$

Now using the generalized Kummer's summation formula (1.6) in each ${}_2F_1$, we simplify to get

$$S = \sum_{m=0}^{\infty} \frac{\Delta_{2m} x^{2m}}{(\rho)_{2m}(\frac{1}{2})_m m!} \frac{\Gamma(\frac{1}{2})\Gamma(\rho+i)\Gamma(\rho+2m)}{\Gamma(\rho+2m+\frac{1}{2}(i+|i|)}$$
$$\times \left\{ \frac{\mathcal{A}_i'}{\Gamma(-m+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(\rho+m+\frac{1}{2}i)} + \frac{\mathcal{B}_i'}{\Gamma(-m+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\rho+m-\frac{1}{2}+\frac{1}{2}i)} \right\}$$
$$+ \sum_{m=0}^{\infty} \frac{\Delta_{2m+1} x^{2m+1}}{(\rho)_{2m+1}(\frac{3}{2})_m m!} \frac{2\Gamma(\frac{1}{2})\Gamma(\rho+i)\Gamma(1+\rho+2m)}{\Gamma(1+\rho+2m+\frac{1}{2}(i+|i|))}$$
$$\times \left\{ \frac{\mathcal{A}_i''}{\Gamma(-m+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(\rho+\frac{1}{2}+m+\frac{1}{2}i)} + \frac{\mathcal{B}_i''}{\Gamma(-m-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\rho+m+\frac{1}{2}i)} \right\}.$$

Finally, using the following known easily-derivable identities:

$$(a)_{2m} = 2^{2m} \left(\frac{1}{2}a\right)_m \left(\frac{1}{2}a+\frac{1}{2}\right)_m,$$
$$(a)_{2m+1} = a 2^{2m} \left(\frac{1}{2}a+\frac{1}{2}\right)_m \left(\frac{1}{2}a+1\right)_m,$$
$$(2m)! = 2^{2m}\, m! \left(\frac{1}{2}\right)_m,$$
$$(2m+1)! = 2^{2m}\, m! \left(\frac{3}{2}\right)_m$$

and

$$\Gamma(a-m) = \frac{(-1)^m \Gamma(a)}{(1-a)_m},$$

after a simplification, we arrive at the right-hand side of the main result (2.1), This completes the proof of (2.1). □

3. Special Cases

We consider some interesting special cases of our main result (2.1). In this regard, we begin by setting $i = 0$, ± 1, ± 2, ± 3 and ± 4 in (2.1) to give the following nine results:

$$\text{(3.1)} \qquad \sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho)_n m! n!} = \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho)_m(\rho)_{2m} m!};$$

(3.2)
$$\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho+1)_n m!n!} = \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho)_m(\rho+1)_{2m} m!} + \frac{x}{\rho(\rho+1)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-x^2)^m}{(\rho+1)_m(\rho+2)_{2m} m!};$$

(3.3)
$$\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho-1)_n m!n!} = \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho-1)_m(\rho)_{2m} m!} - \frac{x}{\rho(\rho-1)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-x^2)^m}{(\rho)_m(\rho+1)_{2m} m!};$$

(3.4)
$$\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho+2)_n m!n!} = \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho+1)_m(\rho+2)_{2m} m!} + \frac{2x}{\rho(\rho+2)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-x^2)^m}{(\rho+1)_m(\rho+3)_{2m} m!};$$

(3.5)
$$\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho-2)_n m!n!} = \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho-1)_m(\rho)_{2m} m!} - \frac{2x}{\rho(\rho-2)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-x^2)^m}{(\rho-1)_m(\rho+1)_{2m} m!};$$

(3.6)
$$\begin{aligned}\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho+3)_n m!n!} &= \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho+1)_m(\rho+3)_{2m} m!} + \frac{3x}{\rho(\rho+3)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-x^2)^m}{(\rho+2)_m(\rho+4)_{2m} m!} \\ &+ \frac{2x^2}{\rho(\rho+1)(\rho+3)(\rho+4)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+2}(-x^2)^m}{(\rho+2)_m(\rho+5)_{2m} m!} \\ &- \frac{2x^3}{\rho(\rho+1)(\rho+2)(\rho+3)(\rho+4)(\rho+5)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+3}(-x^2)^m}{(\rho+3)_m(\rho+6)_{2m} m!};\end{aligned}$$

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho-3)_n m!n!} \\
&= \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho-2)_m(\rho)_{2m}m!} - \frac{3x}{\rho(\rho-3)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-x^2)^m}{(\rho-1)_m(\rho+1)_{2m}m!} \\
&+ \frac{2x^2}{\rho(\rho+1)(\rho-2)(\rho-3)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+2}(-x^2)^m}{(\rho-1)_m(\rho+2)_{2m}m!} \\
&+ \frac{2x^3}{\rho(\rho+1)(\rho+2)(\rho-1)(\rho-2)(\rho-3)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+3}(-x^2)^m}{(\rho)_m(\rho+3)_{2m}m!};
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho+4)_n m!n!} \\
&= \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho+2)_m(\rho+4)_{2m}m!} + \frac{4x}{\rho(\rho+4)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-x^2)^m}{(\rho+3)_m(\rho+5)_{2m}m!} \\
&+ \frac{2(2\rho+5)x^2}{\rho(\rho+1)(\rho+2)(\rho+4)(\rho+5)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+2}(-x^2)^m}{(\rho+3)_m(\rho+6)_{2m}m!} \\
&- \frac{4x^3}{\rho(\rho+2)(\rho+3)(\rho+4)(\rho+5)(\rho+6)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+3}(-x^2)^m}{(\rho+4)_m(\rho+7)_{2m}m!} \\
&- \frac{4x^4}{\rho(\rho+1)(\rho+2)(\rho+3)(\rho+4)(\rho+5)(\rho+6)(\rho+7)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+4}(-x^2)^m}{(\rho+4)_m(\rho+8)_{2m}m!};
\end{aligned} \tag{3.8}$$

(3.9)

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{\Delta_{m+n}(-1)^n x^{m+n}}{(\rho)_m(\rho-4)_n m!n!} \\
&= \sum_{m=0}^{\infty} \frac{\Delta_{2m}(-x^2)^m}{(\rho-2)_m(\rho)_{2m}m!} - \frac{4x}{\rho(\rho-4)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+1}(-x^2)^m}{(\rho-1)_m(\rho+1)_{2m}m!} \\
&\quad + \frac{2(2\rho-3)x^2}{\rho(\rho+1)(\rho-2)(\rho-3)(\rho-4)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+2}(-x^2)^m}{(\rho-1)_m(\rho+2)_{2m}m!} \\
&\quad + \frac{4x^3}{\rho(\rho+1)(\rho+2)(\rho-1)(\rho-2)(\rho-4)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+3}(-x^2)^m}{(\rho)_m(\rho+3)_{2m}m!} \\
&\quad - \frac{4x^4}{\rho(\rho+1)(\rho+2)(\rho+3)(\rho-1)(\rho-2)(\rho-3)(\rho-4)} \sum_{m=0}^{\infty} \frac{\Delta_{2m+4}(-x^2)^m}{(\rho)_m(\rho+4)_{2m}m!}.
\end{aligned}$$

Further setting $\Delta_m = 1$ $(m \in \mathbb{N}_0)$ in the identities (3.1) to (3.9) yields the following interesting formulas for the product of the generalized hypergeometric series:

$$(3.10) \quad {}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\; -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad}\; ; \\ \rho,\ \frac{1}{2}\rho,\ \frac{1}{2}\rho+\frac{1}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right];$$

(3.11)

$$\begin{aligned}
{}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \overline{\quad}\; ; \\ \rho+1\,; \end{matrix}\; -x\right] &= {}_0F_3\left[\begin{matrix} \overline{\qquad}\; ; \\ \rho,\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2}\rho+1\,; \end{matrix}\; -\frac{x^2}{4}\right] \\
&\quad + \frac{x}{\rho(\rho+1)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad}\; ; \\ \rho+1,\ \frac{1}{2}\rho+1,\ \frac{1}{2}\rho+\frac{3}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right];
\end{aligned}$$

(3.12)

$$\begin{aligned}
{}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \overline{\quad}\; ; \\ \rho-1\,; \end{matrix}\; -x\right] &= {}_0F_3\left[\begin{matrix} \overline{\qquad}\; ; \\ \rho-1,\ \frac{1}{2}\rho,\ \frac{1}{2}\rho+\frac{1}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right] \\
&\quad - \frac{x}{\rho(\rho-1)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad}\; ; \\ \rho,\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2}\rho+1\,; \end{matrix}\; -\frac{x^2}{4}\right];
\end{aligned}$$

(3.13)

$$ {}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \text{—} ; \\ \rho+2; \end{matrix}\; -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad}\; ; \\ \rho+1,\; \frac{1}{2}\rho+1,\; \frac{1}{2}\rho+\frac{3}{2}; \end{matrix}\; -\frac{x^2}{4}\right] $$

$$ + \frac{2x}{\rho(\rho+2)}\, {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad}\; ; \\ \rho+1,\; \frac{1}{2}\rho+2,\; \frac{1}{2}\rho+\frac{3}{2}; \end{matrix}\; -\frac{x^2}{4}\right]; $$

(3.14)

$$ {}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \text{—} ; \\ \rho-2; \end{matrix}\; -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad}\; ; \\ \rho-1,\; \frac{1}{2}\rho,\; \frac{1}{2}\rho+\frac{1}{2}; \end{matrix}\; -\frac{x^2}{4}\right] $$

$$ - \frac{2x}{\rho(\rho-2)}\, {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad}\; ; \\ \rho-1,\; \frac{1}{2}\rho+\frac{1}{2},\; \frac{1}{2}\rho+1; \end{matrix}\; -\frac{x^2}{4}\right]; $$

(3.15)

$$ {}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \text{—} ; \\ \rho+3; \end{matrix}\; -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad}\; ; \\ \rho+1,\; \frac{1}{2}\rho+2,\; \frac{1}{2}\rho+\frac{3}{2}; \end{matrix}\; -\frac{x^2}{4}\right] $$

$$ + \frac{3x}{\rho(\rho+3)}\, {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad}\; ; \\ \rho+2,\; \frac{1}{2}\rho+2,\; \frac{1}{2}\rho+\frac{5}{2}; \end{matrix}\; -\frac{x^2}{4}\right] $$

$$ + \frac{2x^2}{\rho(\rho+1)(\rho+3)(\rho+4)}\, {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad}\; ; \\ \rho+2,\; \frac{1}{2}\rho+3,\; \frac{1}{2}\rho+\frac{5}{2}; \end{matrix}\; -\frac{x^2}{4}\right] $$

$$ - \frac{2x^3}{\rho(\rho+1)(\rho+2)(\rho+3)(\rho+4)(\rho+5)}\, {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad}\; ; \\ \rho+3,\; \frac{1}{2}\rho+3,\; \frac{1}{2}\rho+\frac{7}{2}; \end{matrix}\; -\frac{x^2}{4}\right]; $$

(3.16)

$$
{}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\ x\right] \times {}_0F_1\left[\begin{matrix} \text{—} ; \\ \rho-3; \end{matrix}\ -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} ; \\ \rho-2,\ \frac{1}{2}\rho,\ \frac{1}{2}\rho+\frac{1}{2}; \end{matrix}\ -\frac{x^2}{4}\right]
$$

$$
-\frac{3x}{\rho(\rho-3)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} ; \\ \rho-1,\ \frac{1}{2}\rho+1,\ \frac{1}{2}\rho+\frac{1}{2}; \end{matrix}\ -\frac{x^2}{4}\right]
$$

$$
+\frac{2x^2}{\rho(\rho+1)(\rho-2)(\rho-3)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} ; \\ \rho-1,\ \frac{1}{2}\rho+1,\ \frac{1}{2}\rho+\frac{3}{2}; \end{matrix}\ -\frac{x^2}{4}\right]
$$

$$
+\frac{2x^3}{\rho(\rho+1)(\rho+2)(\rho-1)(\rho-2)(\rho-3)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} ; \\ \rho,\ \frac{1}{2}\rho+2,\ \frac{1}{2}\rho+\frac{3}{2}; \end{matrix}\ -\frac{x^2}{4}\right];
$$

(3.17)

$$
{}_0F_1\left[\begin{matrix} -; \\ \rho; \end{matrix}\ x\right] \times {}_0F_1\left[\begin{matrix} \text{—} ; \\ \rho+4; \end{matrix}\ -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} ; \\ \rho+2,\ \frac{1}{2}\rho+2,\ \frac{1}{2}\rho+\frac{5}{2}; \end{matrix}\ -\frac{x^2}{4}\right]
$$

$$
+\frac{4x}{\rho(\rho+4)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} ; \\ \rho+3,\ \frac{1}{2}\rho+\frac{5}{2},\ \frac{1}{2}\rho+3; \end{matrix}\ -\frac{x^2}{4}\right]
$$

$$
+\frac{2(2\rho+5)x^2}{\rho(\rho+1)(\rho+2)(\rho+4)(\rho+5)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} ; \\ \rho+3,\ \frac{1}{2}\rho+3,\ \frac{1}{2}\rho+\frac{7}{2}; \end{matrix}\ -\frac{x^2}{4}\right]
$$

$$
-\frac{4x^4}{\rho(\rho+1)(\rho+2)(\rho+3)(\rho+4)(\rho+5)(\rho+6)(\rho+7)}
$$

$$
\times\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} ; \\ \rho+4,\ \frac{1}{2}\rho+4,\ \frac{1}{2}\rho+\frac{9}{2}; \end{matrix}\ -\frac{x^2}{4}\right];
$$

(3.18)

$$
\begin{aligned}
&{}_0F_1\left[\begin{matrix}-\,;\\ \rho\,;\end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix}\text{---}\;;\\ \rho-4\,;\end{matrix}\; -x\right] = {}_0F_3\left[\begin{matrix}\overline{\qquad\qquad}\;;\\ \rho-2,\ \frac{1}{2}\rho,\ \frac{1}{2}\rho+\frac{1}{2}\,;\end{matrix}\; -\frac{x^2}{4}\right]\\
&-\frac{4x}{\rho(\rho-4)}\,{}_0F_3\left[\begin{matrix}\overline{\qquad\qquad}\;;\\ \rho-1,\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2}\rho+1\,;\end{matrix}\; -\frac{x^2}{4}\right]\\
&+\frac{2(2\rho-3)x^2}{\rho(\rho+1)(\rho-2)(\rho-3)(\rho-4)}\,{}_0F_3\left[\begin{matrix}\overline{\qquad\qquad}\;;\\ \rho-1,\ \frac{1}{2}\rho+1,\ \frac{1}{2}\rho+\frac{3}{2}\,;\end{matrix}\; -\frac{x^2}{4}\right]\\
&+\frac{4x^3}{\rho(\rho+1)(\rho+2)(\rho-1)(\rho-2)(\rho-4)}\,{}_0F_3\left[\begin{matrix}\overline{\qquad\qquad}\;;\\ \rho,\ \frac{1}{2}\rho+\frac{3}{2},\ \frac{1}{2}\rho+2\,;\end{matrix}\; -\frac{x^2}{4}\right]\\
&-\frac{4x^4}{\rho(\rho+1)(\rho+2)(\rho+3)(\rho-1)(\rho-2)(\rho-3)(\rho-4)}\\
&\quad\times{}_0F_3\left[\begin{matrix}\overline{\qquad\qquad}\;;\\ \rho,\ \frac{1}{2}\rho+2,\ \frac{1}{2}\rho+\frac{5}{2}\,;\end{matrix}\; -\frac{x^2}{4}\right].
\end{aligned}
$$

It is noted that the results (3.10) to (3.18) are also obtained by Kim and Rathie [4] who used a different method. The result (3.10) is the well known identity due to Bailey [1]. It is also remarked that our main result (2.1) may yield many other interesting and (potentially) useful identities as its special cases.

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Nidhi Shekhawat
Research Scholar, Department of Mathematics & Statistics, Bansthali University,
Rajasthan-304 022, India.
E-mail: shekhawat.nidhi1@gmail.com

Junesang Choi
Department of Mathematics, Dongguk University,
Gyeongju 780-714, Republic of Korea.
E-mail: junesang@mail.dongguk.ac.kr

Arjun K. Rathie
Department of Mathematics, School of Mathematical and Physical Sciences, Central University of Kerala,
Kasaragod-671 328, Kerala, India.
E-mail: akrathie@gmail.com

Om Prakash
Department of Mathematics, Indian Institute of Technology Patna,
Patna-800 013, Bihar, India.
E-mail: om@iitp.ac.in

TABLE 1. Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
9	$-16a^4+72a^3b-108a^2b^2+60ab^3+23b^4-328a^3+972a^2b-792ab^2+150b^3-2240a^2+3612ab-999b^2-5696a+3162b-3984$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4+248a^3-516a^2b+240ab^2-10b^3+1160a^2-1028ab+35b^2+1576a-50b-24$
8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4+128a^3-312a^2b+176ab^2-10b^3+624a^2-672ab+35b^2+896a-50b+24$	$8b^3-40ab^2+48a^2b-16a^3-192a^2+312ab-88b^2-640a+352b-512$
7	$7b^3-28ab^2+28a^2b-8a^3-100a^2+196ab-70b^2-352a+245b-302$	$8a^3-20a^2b+12ab^2-b^3+68a^2-76ab+6b^2+128a-11b+6$
6	$4a^3-12a^2b+9ab^2-b^3+36a^2-51ab+6b^2+74a-11b+6$	$16ab-8a^2-6b^2-48a+34b-52$
5	$10ab-4a^2-5b^2-26a+25b-32$	$4a^2-6ab+b^2+14a-3b+2$
4	$2a^2-4ab+b^2+8a-3b+2$	$4(b-a-2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0

TABLE 2. Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
-9	$16a^4-72a^3b+108a^2b^2-60ab^3+9b^4-320a^3+972a^2b-828ab^2+174b^3+2240a^2-3936ab+1323b^2-6400a+4614b+6144$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4-256a^3+564a^2b-300ab^2+26b^3+1376a^2-1568ab+251b^2-2816a+1066b+1680$
-8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4-128a^3+328a^2b-208ab^2+22b^3+688a^2-928ab+179b^2-1408a+638b+840$	$16a^3-48a^2b+40ab^2-8b^3-192a^2+328ab-104b^2+704a-480b-768$
-7	$8a^3-28a^2b+28ab^2-7b^3-96a^2+196ab-77b^2+352a-294b-384$	$8a^3-20a^2b+12ab^2-b^3-72a^2+92ab-15b^2+184a-74b-120$
-6	$4a^3-12a^2b+9ab^2-b^3-36a^2+57ab-12b^2+92a-47b-60$	$8a^2-16ab+6b^2-48a+38b+64$
-5	$4a^2-10ab+5b^2-24a+25b+32$	$4a^2-6ab+b^2-16a+7b+12$
-4	$2a^2-4ab+b^2-8a+5b+6$	$4(a-b-2)$
-3	$2a-3b-4$	$2a-b-2$
-2	$a-b-1$	2
-1	1	1