

## LEAST-SQUARES SPECTRAL COLLOCATION PARALLEL METHODS FOR PARABOLIC PROBLEMS

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**Abstract.** In this paper, we study the first-order system least-squares (FOSLS) spectral method for parabolic partial differential equations. There were lots of least-squares approaches to solve elliptic partial differential equations using finite element approximation. Also, some approaches using spectral methods have been studied in recent. In order to solve the parabolic partial differential equations in parallel, we consider a parallel numerical method based on a hybrid method of the frequency-domain method and first-order system least-squares method. First, we transform the parabolic problem in the space-time domain to the elliptic problems in the space-frequency domain. Second, we solve each elliptic problem in parallel for some frequencies using the first-order system least-squares method. And then we take the discrete inverse Fourier transforms in order to obtain the approximate solution in the space-time domain. We will introduce such a hybrid method and then present a numerical experiment.

### 1. Introduction

Let  $\Omega$  be an open convex polygon in  $\mathbb{R}^2$ . We consider the following parabolic problem:

$$(1.1) \quad \begin{cases} c p_t - \nabla \cdot (a \nabla p) = f & \text{in } \Omega \times (0, \infty), \\ p = 0 & \text{on } \Gamma_D \times (0, \infty), \\ \mathbf{n} \cdot (a \nabla p) = 0 & \text{on } \Gamma_N \times (0, \infty), \\ p(\cdot, 0) = 0 & \text{in } \Omega, \end{cases}$$

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where  $c \in L^2(\Omega)$  and  $a \in W^{1,\infty}(\Omega)$  are bounded positive functions of only  $x$ , and  $f(\cdot, t) \in L^2(\Omega)$  for  $t > 0$  but  $f(\cdot, t) \equiv 0$  for  $t > T$ ;  $\partial\Omega = \Gamma_D \cup \Gamma_N$  denotes the partition of the boundary of  $\Omega$ ; and  $\mathbf{n}$  is the outward unit vector normal to the boundary. For simplicity, assume that both  $\Gamma_D$  and  $\Gamma_N$  are nonempty, with the obvious generalization to quotient spaces when one of them is empty in the subsequent sections. We further assume that the functions  $c$ ,  $a$  and  $f$  are all real-valued functions, and  $c$  and  $a$  are bounded positive functions.

The Fourier transform  $\hat{p}(\cdot, \omega)$  of a function  $p(\cdot, t)$  in time and the Fourier inversion are given by

$$\hat{p}(\cdot, \omega) = \int_{-\infty}^{\infty} p(\cdot, t) \exp(-i\omega t) dt$$

and

$$p(\cdot, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(\cdot, \omega) \exp(i\omega t) d\omega.$$

In order to take the Fourier transformation for the space-time problem (1.1) we extend  $f$  and  $p$  by zeros for  $t < 0$ . Then, the Fourier transform  $\hat{p}$  satisfies the following set of elliptic problems depending on  $\omega$ : for all  $\omega \in \mathbb{R}$

$$(1.2) \quad \begin{cases} i\omega c \hat{p} - \nabla \cdot (a \nabla \hat{p}) &= \hat{f} & \text{in } \Omega, \\ \hat{p} &= 0 & \text{on } \Gamma_D, \\ \mathbf{n} \cdot (a \nabla \hat{p}) &= 0 & \text{on } \Gamma_N. \end{cases}$$

Since  $p(x, t)$  is a real function, its Fourier transform satisfies the conjugate relation:

$$\hat{p}(x, -\omega) = \overline{\hat{p}(x, \omega)} \quad \text{for all } \omega \in \mathbb{R}$$

and the Fourier inversion is given by

$$p(x, t) = \frac{1}{\pi} \operatorname{Re} \left( \int_0^{\infty} \hat{p}(x, \omega) \exp(i\omega t) d\omega \right).$$

The approximate solution for the problem (1.1) was obtained by time stepping methods such as backward Euler and Crank-Nicolson methods traditionally. In recent a natural parallel algorithm which does not require any significant communication costs was introduced by transforming the parabolic problem (1.1) in the space-time domain into the independent elliptic problems (1.2) in the space-frequency domain [11, 12, 15, 20, 21]. See [22, 23, 27, 28] for the Laplace transformation.

Recently there has been lots of interest in the use of first-order system least-squares method (FOSLS) for numerical approximations of elliptic

partial differential equations, Stokes equations and Navier-Stokes equations. For a use of finite element method, the least-squares approach was widely studied in [1, 2, 4, 5, 6, 13, 14, 19, 24] and a least-squares method using pseudo spectral approximation were studied in [16, 17, 18, 25]. The least-squares methods have several benefits such that the resulting algebraic system is always symmetric positive definite and the methods can avoid LBB compatibility condition. For more details we refer to [1] and references therein.

In this paper, we consider a parallel numerical method based on a hybrid method of the frequency-domain method and first-order system least-squares method to solve the parabolic partial differential equation (1.1) in parallel. We apply the first-order system pseudo spectral least-squares method to the frequency-domain formulation (1.2) using the similar approaches given in [17]. And then we take the discrete inverse Fourier transformation using adaptive Gaussian quadrature rule in frequency variable to obtain the approximate solution in the space-time domain.

The paper consists of as follows. In section 2, we deliver the first-order system least-squares and show the existence of solution. In sections 3, we develop the least-squares pseudo spectral collocation method including the norm equivalence and spectral convergence. In section 4, we provide the formulation for the inverse Fourier transformation and provide numerical experiment in section 5.

## 2. First-Order System Least-Squares (FOSLS)

In this section, we will apply the first-order system least-squares approach to the problem (1.2). We suppress the hat mark on the notation of all space-frequency function  $\hat{p}$  for simplicity.

Introducing an independent vector variable

$$\mathbf{u} = a^{\frac{1}{2}} \nabla p$$

and using the following identities

$$\nabla \times (a^{-\frac{1}{2}} \mathbf{u}) = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\tau} \cdot (a^{-\frac{1}{2}} \mathbf{u}) = 0 \quad \text{on } \Gamma_D,$$

we obtain an equivalent extended system to problem (1.2): for all  $\omega \in \mathbb{R}$

$$(2.1) \quad \left\{ \begin{array}{ll} -\nabla \cdot (a^{\frac{1}{2}} \mathbf{u}) + i\omega c p &= f \quad \text{in } \Omega, \\ \mathbf{u} - a^{\frac{1}{2}} \nabla p &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla \times (a^{-\frac{1}{2}} \mathbf{u}) &= 0 \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma_D, \\ \mathbf{n} \cdot (a^{\frac{1}{2}} \mathbf{u}) &= 0 \quad \text{on } \Gamma_N, \\ \boldsymbol{\tau} \cdot (a^{-\frac{1}{2}} \mathbf{u}) &= 0 \quad \text{on } \Gamma_D. \end{array} \right.$$

We use standard notations and definitions for the real-valued Sobolev spaces  $H^s(\Omega)$ , associated with the norms  $\|\cdot\|_s$ ,  $s \geq 0$ . But,  $H^0(\Omega)$  coincides with  $L^2(\Omega)$ , in which the associated inner product and norm are denoted by  $(\cdot, \cdot)$  and  $\|\cdot\|$ , respectively.

Denote by  $v_r$  and  $v_i$  the real-part and imaginary-part of a complex-valued vector or scalar function  $v$ , respectively. Then, the  $L^2$ -inner product and  $L^2$ -norm for complex-valued functions  $u = u_r + iu_i$  and  $v = v_r + iv_i$  are given by

$$(u, v)_c = \int_{\Omega} u \bar{v} dx = (u, \bar{v}) \quad \text{and} \quad \|v\|_c = (v, \bar{v})^{\frac{1}{2}} = (\|v_r\|^2 + \|v_i\|^2)^{\frac{1}{2}}.$$

From now on, let us identify the complex-valued function  $v = v_r + iv_i$  by  $v := [v_r, v_i]$  with real-valued functions  $v_r$  and  $v_i$ . Also, denote by  $H_c^s(\Omega) = H^s(\Omega) \times H^s(\Omega)$  and  $L_c^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ .

Let  $V$  be a subspace of  $H_c^1(\Omega)^2$  given by

$$V = \{q = [q_r, q_i] \in H_c^1(\Omega) : q = 0 \text{ on } \Gamma_D\}$$

equipped with the norm

$$\|q\|_V = (\|q_r\|_1^2 + \|q_i\|_1^2)^{\frac{1}{2}}.$$

Denote the curl operator and its formal adjoint by

$$\nabla \times = (-\partial_2, \partial_1) \quad \text{and} \quad \nabla^\perp = \begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix}.$$

Let

$$H(\operatorname{div} a^{\frac{1}{2}}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}) \in L^2(\Omega)\}$$

and

$$H(\operatorname{curl} a^{-\frac{1}{2}}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla \times (a^{-\frac{1}{2}} \mathbf{v}) \in L^2(\Omega)\},$$

which are Hilbert spaces under norms

$$\|\mathbf{v}\|_{H(\operatorname{div} a^{\frac{1}{2}}; \Omega)} = \left( \|\mathbf{v}\|^2 + \|\nabla \cdot (a^{\frac{1}{2}} \mathbf{v})\|^2 \right)^{\frac{1}{2}}$$

and

$$\|\mathbf{v}\|_{H(\operatorname{curl} a^{-\frac{1}{2}}; \Omega)} = \left( \|\mathbf{v}\|^2 + \|\nabla \times (a^{-\frac{1}{2}} \mathbf{v})\|^2 \right)^{\frac{1}{2}},$$

respectively. Define the subspaces

$$H_0(\operatorname{div} a^{\frac{1}{2}}; \Omega) = \{\mathbf{v} \in H(\operatorname{div} a^{\frac{1}{2}}; \Omega) : \mathbf{n} \cdot (a^{\frac{1}{2}} \mathbf{v}) = 0 \text{ on } \Gamma_D\}$$

$$H_0(\operatorname{curl} a^{-\frac{1}{2}}; \Omega) = \{\mathbf{v} \in H(\operatorname{curl} a^{-\frac{1}{2}}; \Omega) : \boldsymbol{\tau} \cdot (a^{-\frac{1}{2}} \mathbf{v}) = 0 \text{ on } \Gamma_N\}$$

where  $\boldsymbol{\tau}$  represents the unit vector tangent to the boundary oriented counterclockwise. Let

$\mathcal{U} = \{\mathbf{v} = [\mathbf{v}_r, \mathbf{v}_i] \in L_c^2(\Omega)^2 : \mathbf{v}_r, \mathbf{v}_i \in H_0(\operatorname{div} a^{\frac{1}{2}}; \Omega) \cap H_0(\operatorname{curl} a^{-\frac{1}{2}}; \Omega)\}$ ,  
equipped with the norm

$$\|\mathbf{v}\|_{\mathcal{U}}^2 = \|\mathbf{v}\|_c^2 + \|\nabla \cdot (a^{\frac{1}{2}} \mathbf{v})\|_c^2 + \|\nabla \times (a^{-\frac{1}{2}} \mathbf{v})\|_c^2.$$

Now, define the first-order least-squares functional as summing the  $L^2$ -norms of residual equations

(2.2)

$$G_\omega(\mathbf{v}, q; f) = \|f + \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}) - i\omega c q\|_c^2 + \|\mathbf{v} - a^{\frac{1}{2}} \nabla q\|_c^2 + \|\nabla \times (a^{-\frac{1}{2}} \mathbf{v})\|_c^2.$$

Using the similar techniques for the first-order least-squares approaches given in [5] and [6], one may prove the following lemma.

**Lemma 2.1.** *For each frequency  $\omega$ , there exists a constant  $C$ , only dependent on  $a$ ,  $c$ ,  $\Omega$ , such that*

(2.3)

$$\frac{1}{C\omega} \left( \|\mathbf{v}\|_{\mathcal{U}}^2 + \|q\|_V^2 \right) \leq G_\omega(\mathbf{v}, q; 0) \leq C\omega \left( \|\mathbf{v}\|_{\mathcal{U}}^2 + \|q\|_V^2 \right) \quad \forall (\mathbf{v}, q) \in \mathcal{U} \times V.$$

*Proof.* The upper bound is an immediate consequence of the triangle inequality and the bounds of coefficients  $a$  and  $c$ . Now, let us show the lower bound. It follows from the definition and the Cauchy-Schwarz inequality that

$$\begin{aligned} \|a^{\frac{1}{2}} \nabla q\|_c^2 &= \left( a^{\frac{1}{2}} \nabla q - \mathbf{v}, a^{\frac{1}{2}} \nabla q \right)_c + \left( a^{\frac{1}{2}} \mathbf{v}, \nabla q \right)_c \\ &= \left( a^{\frac{1}{2}} \nabla q - \mathbf{v}, a^{\frac{1}{2}} \nabla q \right)_c - \left( \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}), q \right)_c \\ &= \left( a^{\frac{1}{2}} \nabla q - \mathbf{v}, a^{\frac{1}{2}} \nabla q \right)_c - \left( \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}) - i\omega c q, q \right)_c - (i\omega c q, q)_c \\ &\leq \|a^{\frac{1}{2}} \nabla q - \mathbf{v}\|_c \|a^{\frac{1}{2}} \nabla q\|_c + \|\nabla \cdot (a^{\frac{1}{2}} \mathbf{v}) - i\omega c q\|_c \|q\|_c + C\omega \|q\|_c^2. \end{aligned}$$

Using the Poincaré-Friedrich's inequality yields that

$$\|a^{\frac{1}{2}} \nabla q\|_c \leq C\omega \left( \|a^{\frac{1}{2}} \nabla q - \mathbf{v}\|_c + \|\nabla \cdot (a^{\frac{1}{2}} \mathbf{v}) - i\omega c q\|_c + \|q\|_c \right).$$

It also follows from the triangle inequality that

$$\|\mathbf{v}\|_{\mathcal{U}} \leq C\omega \left( \|a^{\frac{1}{2}}\nabla q - \mathbf{v}\|_c + \|\nabla \cdot (a^{\frac{1}{2}}\mathbf{v}) - i\omega c q\|_c + \|\nabla \times (a^{-\frac{1}{2}}\mathbf{v})\|_c + \|q\|_c \right).$$

Combining the last two estimates together with the Poincaré-Friedrich's inequality, we have

$$\|\mathbf{v}\|_{\mathcal{U}}^2 + \|q\|_V^2 \leq C \left( G_\omega(\mathbf{v}, q; 0) + \|q\|_c^2 \right).$$

Now, using the standard compactness argument, one may easily show the conclusion (2.1).  $\square$

The functional  $G_\omega(\mathbf{v}, q; f)$  can be also written by, with  $\mathbf{v} = [\mathbf{v}_r, \mathbf{v}_i]$  and  $q = [q_r, q_i]$

$$(2.4) \quad \begin{aligned} G_\omega(\mathbf{v}, q; f) = & \|f_r + \nabla \cdot (a^{\frac{1}{2}}\mathbf{v}_r) + \omega c q_i\|^2 + \|f_i + \nabla \cdot (a^{\frac{1}{2}}\mathbf{v}_i) - \omega c q_r\|^2 \\ & + \|\mathbf{v}_r - a^{\frac{1}{2}}\nabla q_r\|^2 + \|\mathbf{v}_i - a^{\frac{1}{2}}\nabla q_i\|^2 + \|\nabla \times (a^{-\frac{1}{2}}\mathbf{v}_r)\|^2 + \|\nabla \times (a^{-\frac{1}{2}}\mathbf{v}_i)\|^2. \end{aligned}$$

Then the corresponding minimization problem for (1.2) is to minimize the quadratic functional  $G_\omega(\mathbf{v}, q; f)$  over  $\mathcal{U} \times V$ : find  $(\mathbf{u}, p) := ([\mathbf{u}_r, \mathbf{u}_i], [p_r, p_i]) \in \mathcal{U} \times V$  such that

$$(2.5) \quad G_\omega(\mathbf{u}, p; f) = \inf_{(\mathbf{v}, q) \in \mathcal{U} \times V} G_\omega(\mathbf{v}, q; f).$$

The corresponding variational problem is as follows:

$$(2.6) \quad \begin{aligned} & \text{Find } (\mathbf{u}, p) := ([\mathbf{u}_r, \mathbf{u}_i], [p_r, p_i]) \in \mathcal{U} \times V \text{ such that} \\ & \mathcal{A}_\omega(\mathbf{u}, p; \mathbf{v}, q) = \mathcal{F}_\omega(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in \mathcal{U} \times V, \end{aligned}$$

where the bilinear form  $\mathcal{A}_\omega(\cdot; \cdot)$  is given by

$$(2.7) \quad \begin{aligned} & \mathcal{A}_\omega \left( [\mathbf{u}_r, \mathbf{u}_i], [p_r, p_i]; [\mathbf{v}_r, \mathbf{v}_i], [q_r, q_i] \right) \\ &= \left( \nabla \cdot (a^{\frac{1}{2}}\mathbf{u}_r) + \omega c p_i, \nabla \cdot (a^{\frac{1}{2}}\mathbf{v}_r) + \omega c q_i \right) \\ &+ \left( \nabla \cdot (a^{\frac{1}{2}}\mathbf{u}_i) - \omega c p_r, \nabla \cdot (a^{\frac{1}{2}}\mathbf{v}_i) - \omega c q_r \right) \\ &+ \left( \mathbf{u}_r - a^{\frac{1}{2}}\nabla p_r, \mathbf{v}_r - a^{\frac{1}{2}}\nabla q_r \right) + \left( \mathbf{u}_i - a^{\frac{1}{2}}\nabla p_i, \mathbf{v}_i - a^{\frac{1}{2}}\nabla q_i \right) \\ &+ \left( \nabla \times (a^{-\frac{1}{2}}\mathbf{u}_r), \nabla \times (a^{-\frac{1}{2}}\mathbf{v}_r) \right) + \left( \nabla \times (a^{-\frac{1}{2}}\mathbf{u}_i), \nabla \times (a^{-\frac{1}{2}}\mathbf{v}_i) \right) \end{aligned}$$

and the linear form  $\mathcal{F}_\omega(\cdot)$  is given by

$$(2.8) \quad \begin{aligned} \mathcal{F}_\omega \left( [\mathbf{v}_r, \mathbf{v}_i], [q_r, q_i] \right) = & - \left( f_r, \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}_r) + \omega c q_i \right) \\ & - \left( f_i, \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}_i) - \omega c q_r \right). \end{aligned}$$

Now, we establish the well-posedness of the problem (2.6) in the following theorem.

**Theorem 2.2.** *For each frequency  $\omega$ , there exists a unique solution to the problem (2.6).*

*Proof.* Note that

$$\mathcal{A}_\omega(\mathbf{v}, q; \mathbf{v}, q) = \mathcal{A}_\omega([\mathbf{v}_r, \mathbf{v}_i], [q_r, q_i]; [\mathbf{v}_r, \mathbf{v}_i], [q_r, q_i]) = G_\omega(\mathbf{v}, q; 0).$$

Then, the coerciveness and continuity of the bilinear form  $\mathcal{A}_\omega(\cdot; \cdot)$  are immediate consequences of the Lemma (2.1). Also it is apparent that  $\mathcal{F}_\omega(\cdot, \cdot)$  is a continuous linear form. Hence, the problem (2.6) has a unique solution from the Lax-Milgram lemma.  $\square$

### 3. Spectral least-squares approximation

In this section, we assume that the functions  $a$  and  $c$  are constants and  $\Omega = (-1, 1)^2$ . we further assume that the right hand side  $f$  in (2.1) is a continuous function. Let  $\mathcal{P}_N$  be the space of all polynomials of degree less than or equal to  $N$ . Let  $\{\xi_i\}_{i=0}^N$  be the Legendre-Gauss-Lobatto (LGL) points on  $[-1, 1]$  such that  $-1 =: \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N := 1$ , in which  $\{\xi_i\}_{i=0}^N$  are the zeros of  $(1-t^2)L'_N(t)$  where  $L_N$  is the  $N^{th}$  Legendre polynomial and the corresponding quadrature weights  $\{w_i\}_{i=0}^N$  are given by

$$(3.1) \quad w_0 = w_N = \frac{2}{N(N+1)}, \quad w_j = \frac{2}{N(N+1)} \frac{1}{[L_N(\xi_j)]^2}, \quad 1 \leq j \leq N-1.$$

The Gaussian quadrature rule yields the exactness of numerical integration such that

$$(3.2) \quad \int_{-1}^1 p(t) dt = \sum_{i=0}^N w_i p(\xi_i), \quad \forall p \in \mathcal{P}_{2N-1}.$$

The two-dimensional LGL points  $\{\mathbf{x}_{ij}\}$  and the corresponding weights  $\{\mathbf{w}_{ij}\}$  are given by

$$\mathbf{x}_{ij} = (\xi_i, \xi_j), \quad \mathbf{w}_{ij} = w_i w_j, \quad i, j = 0, 1, \dots, N.$$

Let  $\mathcal{Q}_N$  be the space of all polynomials of degree less than or equal to  $N$  with respect to each single variable  $x$  and  $y$ . For any continuous functions  $u$  and  $v$  over  $\bar{\Omega}$ , the associated discrete scalar product and norm are given by

$$(3.3) \quad \langle u, v \rangle_N = \sum_{i,j=0}^N \mathbf{w}_{ij} u(\mathbf{x}_{ij}) v(\mathbf{x}_{ij}) \quad \text{and} \quad \|v\|_N = \langle v, v \rangle_N^{1/2}.$$

Then, we have from (4.1) that

$$(3.4) \quad \langle u, v \rangle_N = (u, v) \quad \text{for } uv \in \mathcal{Q}_{2N-1}$$

and it is well-known that

$$(3.5) \quad \|v\| \leq \|v\|_N \leq \gamma^* \|v\|, \quad \forall v \in \mathcal{Q}_N,$$

where  $\gamma^* = (2 + \frac{1}{N})$ . For any continuous function  $v$ , we denote by  $I_N v \in \mathcal{Q}_N$  the interpolant of  $v$  at the LGL-points  $\{\mathbf{x}_{ij}\}$ . Then

$$(3.6) \quad I_N v(\mathbf{x}) = \sum_{i,j=0}^N v(\mathbf{x}_{ij}) \psi_{ij}(\mathbf{x}), \quad \forall \mathbf{x} \in \bar{\Omega}$$

where  $\psi_{ij}$  are the Lagrange interpolation polynomials of degree  $N$  for  $i, j = 0, 1, 2, \dots, N$ .

The interpolation error estimate is given by

$$(3.7) \quad \|v - I_N v\|_k \leq C N^{k-s} \|v\|_s, \quad k = 0, 1,$$

provided  $v \in H^s(\Omega)$  for  $s \geq 2$  (see [3, 8, 26]) and using (3.4)–(3.7) yields that for all  $u \in H^s(\Omega)$ ,  $s \geq 2$ , and  $v_N \in \mathcal{Q}_N$

$$(3.8) \quad |(u, v_N) - \langle u, v_N \rangle_N| \leq C N^{-s} \|u\|_s \|v_N\|.$$

Let  $\mathcal{Q}_N^D = \mathcal{Q}_N \cap H_D^1(\Omega)$ . We now recall the orthogonal projection  $P_{1,N}^D : H_D^1(\Omega) \rightarrow \mathcal{Q}_N^D$  through

$$(3.9) \quad (\nabla P_{1,N}^D u, \nabla \theta_N) = (\nabla u, \nabla \theta_N), \quad \forall \theta_N \in \mathcal{Q}_N^D.$$

Then, for all  $u \in H^s(\Omega) \cap H_D^1(\Omega)$ , with  $s \geq 1$ , (see [26])

$$(3.10) \quad \|u - P_{1,N}^D u\|_k \leq C N^{k-s} \|u\|_s, \quad k = 0, 1.$$

Let us recall the inverse inequality (see [26])

$$(3.11) \quad \|v_N\|_1 \leq C N^2 \|v_N\|, \quad \forall v_N \in \mathcal{Q}_N.$$

Let  $\mathcal{U}^N$  and  $V^N$  be a finite dimensional subspaces of  $\mathcal{U}$  and  $V$ , respectively:

$$\mathcal{U}^N = \mathcal{U} \cap [\mathcal{Q}_N \times \mathcal{Q}_N]^2 \quad \text{and} \quad V^N = V \cap [\mathcal{Q}_N \times \mathcal{Q}_N].$$

Define the discrete first-order least-squares functional as summing the discrete Legendre spectral norms of the residual equations: with  $\mathbf{v} = [\mathbf{v}_r, \mathbf{v}_i]$  and  $q = [q_r, q_i]$

(3.12)

$$\begin{aligned} G_{\omega,N}(\mathbf{v}, q; f) = & \|f_r + \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}_r) + \omega c q_i\|_N^2 + \|f_i + \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}_i) - \omega c q_r\|_N^2 \\ & + \|\mathbf{v}_r - a^{\frac{1}{2}} \nabla q_r\|_N^2 + \|\mathbf{v}_i - a^{\frac{1}{2}} \nabla q_i\|_N^2 \\ & + \|\nabla \times (a^{-\frac{1}{2}} \mathbf{v}_r)\|_N^2 + \|\nabla \times (a^{-\frac{1}{2}} \mathbf{v}_i)\|_N^2. \end{aligned}$$

Then the corresponding discretized minimization problem for (1.2) is to minimize the quadratic functional  $G_{\omega,N}(\mathbf{v}^N, q^N; f)$  over  $\mathcal{U}^N \times V^N$ : find  $(\mathbf{u}^N, p^N) := ([\mathbf{u}_r^N, \mathbf{u}_i^N], [p_r^N, p_i^N]) \in \mathcal{U}^N \times V^N$  such that

$$(3.13) \quad G_{\omega,N}(\mathbf{u}^N, p^N; f) = \inf_{(\mathbf{v}^N, q^N) \in \mathcal{U}^N \times V^N} G_{\omega,N}(\mathbf{v}^N, q^N; f).$$

And, the corresponding variational problem is as follows:

(3.14)

Find  $(\mathbf{u}^N, p^N) := ([\mathbf{u}_r^N, \mathbf{u}_i^N], [p_r^N, p_i^N]) \in \mathcal{U}^N \times V^N$  such that

$$\mathcal{A}_{\omega,N}(\mathbf{u}^N, p^N; \mathbf{v}^N, q^N) = \mathcal{F}_{\omega,N}(\mathbf{v}^N, q^N) \quad \forall (\mathbf{v}^N, q^N) \in \mathcal{U}^N \times V^N,$$

where the bilinear form  $\mathcal{A}_{\omega,N}(\cdot; \cdot)$  is given by

(3.15)

$$\begin{aligned} \mathcal{A}_{\omega,N} & \left( [\mathbf{u}_r, \mathbf{u}_i], [p_r, p_i]; [\mathbf{v}_r, \mathbf{v}_i], [q_r, q_i] \right) \\ = & \left\langle \nabla \cdot (a^{\frac{1}{2}} \mathbf{u}_r) + \omega c p_i, \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}_r) + \omega c q_i \right\rangle_N \\ & + \left\langle \nabla \cdot (a^{\frac{1}{2}} \mathbf{u}_i) - \omega c p_r, \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}_i) - \omega c q_r \right\rangle_N \\ & + \left\langle \mathbf{u}_r - a^{\frac{1}{2}} \nabla p_r, \mathbf{v}_r - a^{\frac{1}{2}} \nabla q_r \right\rangle_N + \left\langle \mathbf{u}_i - a^{\frac{1}{2}} \nabla p_i, \mathbf{v}_i - a^{\frac{1}{2}} \nabla q_i \right\rangle_N \\ & + \left\langle \nabla \times (a^{-\frac{1}{2}} \mathbf{u}_r), \nabla \times (a^{-\frac{1}{2}} \mathbf{v}_r) \right\rangle_N + \left\langle \nabla \times (a^{-\frac{1}{2}} \mathbf{u}_i), \nabla \times (a^{-\frac{1}{2}} \mathbf{v}_i) \right\rangle_N \end{aligned}$$

and the linear form  $\mathcal{F}_{\omega,N}(\cdot)$  is given by

$$(3.16) \quad \begin{aligned} \mathcal{F}_{\omega,N} & \left( [\mathbf{v}_r, \mathbf{v}_i], [q_r, q_i] \right) = - \left\langle f_r, \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}_r) + \omega c q_i \right\rangle_N \\ & - \left\langle f_i, \nabla \cdot (a^{\frac{1}{2}} \mathbf{v}_i) - \omega c q_r \right\rangle_N. \end{aligned}$$

Now, we establish the well-posedness of the problem (3.14) using the similar arguments of [17] in the following theorem.

**Theorem 3.1.** *For each frequency  $\omega$ , there exists a constant  $C$ , only dependent on  $a$ ,  $c$ ,  $\Omega$ , such that*

$$(3.17) \quad \frac{1}{C\omega} \left( \|\mathbf{v}^N\|_{\mathcal{U}}^2 + \|q^N\|_V^2 \right) \leq G_{\omega,N}(\mathbf{v}^N, q^N; 0) \leq C\omega \left( \|\mathbf{v}^N\|_{\mathcal{U}}^2 + \|q^N\|_V^2 \right)$$

for all  $(\mathbf{v}^N, q^N) \in \mathcal{U}^N \times V^N$ .

*Proof.* Since the fact that  $a$ ,  $c$  and  $\omega$  are constants, we can easily show that the real parts and imaginary parts of the functions

$$\nabla \cdot (a^{\frac{1}{2}} \mathbf{u}^N) + \omega c p^N, \quad \mathbf{u}^N - a^{\frac{1}{2}} \nabla p^N \quad \text{and} \quad \nabla \times (a^{-\frac{1}{2}} \mathbf{u}^N)$$

are polynomial functions of degree less than or equal to  $N$  if  $(\mathbf{v}^N, q^N) \in \mathcal{U}^N \times V^N$ . Hence, we have the following equivalence from (3.5):

$$\frac{1}{C} G_{\omega}(\mathbf{v}^N, q^N; 0) \leq G_{\omega,N}(\mathbf{v}^N, q^N; 0) \leq C G_{\omega}(\mathbf{v}^N, q^N; 0)$$

for all  $(\mathbf{v}^N, q^N) \in \mathcal{U}^N \times V^N$ . Now, the bound (3.1) is an immediate consequence of Lemma 2.1 together with the last inequality.  $\square$

Using the standard techniques for spectral convergence (see [3], [17], [26]), one may easily show that our Legendre pseudo-spectral least-squares approach has the following spectral convergence:

**Theorem 3.2.** *Assume that the solution  $(\mathbf{u}, p)$  of the problem (2.6) is in  $H_c^s(\Omega)^3$  for some  $s \geq 1$  and  $f \in H_c^\ell(\Omega)$  for some integer  $\ell \geq 2$ . Let  $(\mathbf{u}^N, p^N) \in \mathcal{U}^N \times V^N$  be the discrete solution of the problem (3.14). Then there exists a constant  $C$  such that*

$$(3.18) \quad \|\mathbf{u} - \mathbf{u}^N\|_{\mathcal{U}} + \|p - p^N\|_V \leq C [N^{1-s} (\|\mathbf{u}\|_{H_c^s(\Omega)} + \|p\|_{H_c^s(\Omega)}) + N^{-\ell} \|f\|_{H_c^\ell(\Omega)}].$$

#### 4. Inverse Fourier Transformation of Approximate Solution

Denote by  $(\hat{\mathbf{u}}^N(\omega), \hat{p}^N(\omega)) := (\hat{\mathbf{u}}^N(\mathbf{x}, \omega), \hat{p}^N(\mathbf{x}, \omega))$  the solution of the problem (3.14) for each frequency  $\omega > 0$ . In this section, with a fixed sufficiently large  $\omega^* > 0$  such that  $(\hat{\mathbf{u}}^N(\omega), \hat{p}^N(\omega))$  is negligible for  $\omega > \omega^*$ , we will approximate the time-domain solution  $p(x, t)$  of the problem (1.1) and its flux  $\mathbf{u} = a^{\frac{1}{2}} \nabla p$  using the adaptive Gaussian quadrature rule.

Let  $\{a_j\}_{j=0}^m$  be a partition of interval  $[0, \omega^*]$  satisfying  $0 = a_0 < a_1 < \dots < a_m = \omega^*$ . Let  $h_j = a_j - a_{j-1}$  for  $j = 1, \dots, m$  and let

$h = \max\{h_j : i = 1, 2, \dots, m\}$ . Then, the LGL-points of each interval  $[a_{j-1}, a_j]$  are given by

$$\xi_i^j = \frac{a_{j-1} + a_j}{2} + \frac{(a_j - a_{j-1})}{2} \xi_i \quad \forall i = 0, 1, \dots, n,$$

where  $\xi_i$  are the LGL-points of the interval  $[-1, 1]$ . The corresponding quadrature weights  $w_i^j$  are given by

$$w_i^j = w_i \quad \forall j = 1, 2, \dots, m, i = 0, 1, \dots, n,$$

where  $w_i$  are quadrature weights corresponding to  $\xi_i$ .

Then, the adaptive Gaussian quadrature rule yields the exactness of numerical integration such as

$$(4.1) \quad \int_0^{\omega^*} p(\omega) d\omega = \sum_{j=1}^m \sum_{i=0}^n w_i^j p(\xi_i^j),$$

for all piecewise polynomial  $p(\omega)$  of degree less than or equal to  $2n - 1$ .

Recall that the Fourier inversion of  $\hat{q}(x, \omega)$  is given by

$$q(x, t) = \frac{1}{\pi} \operatorname{Re} \left( \int_0^\infty \hat{q}(x, \omega) \exp(i\omega t) d\omega \right).$$

Since  $\hat{q}(x, \omega) = \hat{q}_r(x, \omega) + i\hat{q}_i(x, \omega)$ , we have the following Fourier inversion of  $\hat{q}(x, \omega)$ :

$$q(x, t) = \frac{1}{\pi} \int_0^\infty \left( \hat{q}_r(x, \omega) \cos \omega t - \hat{q}_i(x, \omega) \sin \omega t \right) d\omega.$$

Denote by

$$q_{\omega^*}(x, t) = \frac{1}{\pi} \operatorname{Re} \left( \int_0^{\omega^*} \hat{q}(x, \omega) \exp(i\omega t) d\omega \right)$$

and

$$q_{\omega^*,g}(x, t) = \frac{1}{\pi} \operatorname{Re} \left( \sum_{j=1}^m \sum_{i=0}^n \hat{q}(x, \xi_i^j) \exp(it \xi_i^j) w_i^j \right).$$

Now, the approximate solutions of the solution  $p(x, t)$  and its flux  $\mathbf{u} = a^{\frac{1}{2}} \nabla p$  of the time-domain problem (1.1) are given by

$$p_{\omega^*,g}^N(x, t) = \frac{1}{\pi} \sum_{j=1}^m \sum_{i=0}^n \left( \hat{p}_r^N(x, \xi_i^j) \cos(t \xi_i^j) - \hat{p}_i^N(x, \xi_i^j) \sin(t \xi_i^j) \right) w_i^j$$

and, for  $k = 1, 2$ ,

$$u_{\omega^*,g}^{N,k}(x,t) = \frac{1}{\pi} \sum_{j=1}^m \sum_{i=0}^n \left( \hat{u}_r^{N,k}(x, \xi_i^j) \cos(t\xi_i^j) - \hat{u}_i^{N,k}(x, \xi_i^j) \sin(t\xi_i^j) \right) w_i^j$$

where  $\mathbf{u}_{\omega^*,g}^N = (u_{\omega^*,g}^{N,1}, u_{\omega^*,g}^{N,2})$  and  $\hat{\mathbf{u}}^N = (\hat{u}^{N,1}, \hat{u}^{N,2})$ .

Denote by, for  $t > 0$

$$E(p, t) = \|p(x, t) - p_{\omega^*,g}^N(x, t)\| \quad \text{and} \quad E(\mathbf{u}, t) = \|\mathbf{u}(x, t) - \mathbf{u}_{\omega^*,g}^N(x, t)\|.$$

Then, we have from the triangle inequality that

$$E(p, t) \leq \|p(x, t) - p_{\omega^*}(x, t)\| + \|p_{\omega^*}(x, t) - p_{\omega^*,g}(x, t)\| + \|p_{\omega^*,g}(x, t) - p_{\omega^*,g}^N(x, t)\|.$$

Then, using the similar argument of [20] with appropriate assumptions, we can show that, for  $t > 0$

$$E(p, t) \longrightarrow 0 \quad \text{and} \quad E(\mathbf{u}, t) \longrightarrow 0 \quad \text{as} \quad N, \omega^* \longrightarrow \infty$$

provided with appropriate  $m$  and  $n$ .

## 5. Computational experiments

As the numerical example of exact solution we take  $p(x, y, t) = g(x, y)h(t)$  and its Fourier transformation is then given by  $\hat{p}(x, y, \omega) = g(x, y)\hat{h}(\omega)$  where

$$g(x, y) = \sin(3\pi(x+1)/4) \sin(3\pi(y+1)/4),$$

$$h(t) = \frac{t}{t^2 + 1} \quad \text{and} \quad \hat{h}(\omega) = -ie^{-\omega},$$

here  $h(t)$  and  $\hat{h}(\omega)$  are a well-known Fourier transform pair. Figure 1 and 2 show their variation of function values along with time  $t$  and frequency  $\omega$ , respectively.

We apply above  $\hat{p}(x, y, \omega)$  to

$$\left\{ \begin{array}{ll} -\nabla \cdot (a^{\frac{1}{2}} \mathbf{u}) + i\omega c p &= f \quad \text{in} \quad \Omega, \\ \mathbf{u} - a^{\frac{1}{2}} \nabla p &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \nabla \times (a^{-\frac{1}{2}} \mathbf{u}) &= 0 \quad \text{in} \quad \Omega, \\ p &= 0 \quad \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot (a^{\frac{1}{2}} \mathbf{u}) &= 0 \quad \text{on} \quad \Gamma_N, \\ \boldsymbol{\tau} \cdot (a^{-\frac{1}{2}} \mathbf{u}) &= 0 \quad \text{on} \quad \Gamma_D. \end{array} \right.$$

and we take  $a = 1$  and  $c = 1$ . For computing, Use of spectral collocation approximation we compute the algebraic problems using PCGM with high performance parallel machine; we use a linux-cluster named APPC

which is consist of 8 nodes. And we program a code using Fortran 90 and MPI.

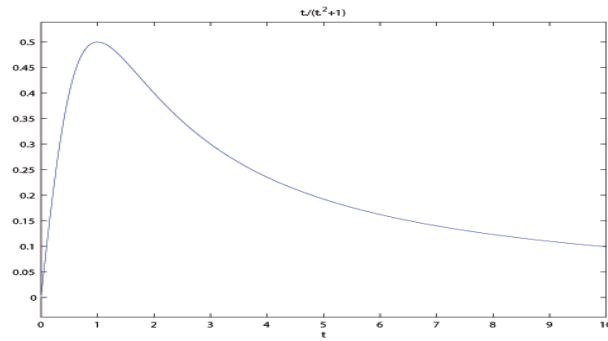


FIGURE 1.  $h(t) = \frac{t}{t^2+1}$

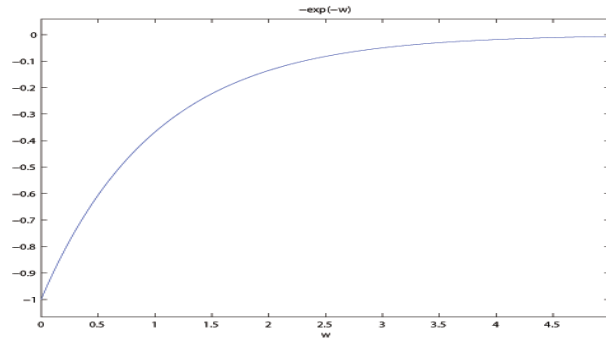


FIGURE 2.  $\hat{h}(\omega) = -ie^{-\omega}$

Fixing  $\omega^* = 40$ , we denote  $N_{\omega^*}$  by the number of LGL-points for a gaussian quadrature of the inverse Fourier transform. Fixing  $N_{\omega^*} = 40$ , we present some numerical results listed in Table 1 and 2. Recall that  $N$  denotes the degree of polynomials for spectral least-squares approximation.

$N$	time	$\ E(p, t)\ $	cputime	$N$	time	$\ E(p, t)\ $	cputime
5	0.25	6.0968D-002	0.10	17	0.25	1.5402D-013	696.02
	0.50	5.0840D-002			0.50	1.2466D-013	
	0.75	3.8152D-002			0.75	9.0899D-014	
	1.00	2.8694D-002			1.00	6.6873D-014	
9	0.25	3.9379D-005	5.33	21	0.25	1.3931D-013	3336.01
	0.50	3.1689D-005			0.50	1.3834D-013	
	0.75	2.2744D-005			0.75	1.2811D-013	
	1.00	1.6294D-005			1.00	1.1416D-013	
13	0.25	4.5284D-009	27.93	25	0.25	8.6443D-014	6808.57
	0.50	3.6534D-009			0.50	9.5164D-014	
	0.75	2.6440D-009			0.75	9.6722D-014	
	1.00	1.9231D-009			1.00	9.4099D-014	

Table 1. Norm of error fixing  $N_{\omega^*} = 40$ .Table 2. Norm of error fixing  $N_{\omega^*} = 40$ .

In Table 1 and 2 we list the norm of error between  $p(x, y, t)$  and  $p_{\omega^*, g}^N(x, y, t)$  for several numbers of degree of polynomials for spectral least-squares approximation at some fixed times of  $t$ .

On the other hand, fixing  $N = 17$ , we present some other numerical results listed in Table 3 and 4. In Table 3 and 4 we list the norm of error between  $p(x, y, t)$  and  $p_{\omega^*, g}^N(x, y, t)$  for several numbers of degree of polynomials for the approximation of Fourier inversion at some fixed times of  $t$ .

In those numerical result, referring to Table 2 and 3 we find out an interesting fact that choosing  $N_{\omega^*} = 24$  for Fourier inversion, the approximate solution  $p_{\omega^*, g}^N(x, y, t)$  appears to have a similar scales of norms of error but rather much smaller computational times to be taken compared to the choice of  $N_{\omega^*} = 40$ ; of course taking  $N_{\omega^*} = 24$  it gets much shorter times of computation than it takes for  $N_{\omega^*} = 40$ , and this result reflects a well known typical situation happening when we use gaussian quadrature rule in integration.

$N_{\omega^*}$	time	$\ E(p, t)\ $	cputime	$N_{\omega^*}$	time	$\ E(p, t)\ $	cputime
8	0.25	0.1061	172.54	32	0.25	1.5382D-013	624.51
	0.50	0.2129			0.50	1.2444D-013	
	0.75	0.3720			0.75	9.0836D-014	
	1.00	0.7995			1.00	6.7119D-014	
16	0.25	1.4911D-008	291.70	40	0.25	1.5402D-013	763.57
	0.50	1.4258D-007			0.50	1.2466D-013	
	0.75	9.8006D-007			0.75	9.0899D-014	
	1.00	4.5734D-005			1.00	6.6873D-014	
24	0.25	1.5404D-013	494.95				
	0.50	1.2472D-013					
	0.75	9.0983D-014					
	1.00	5.4001D-013					

Table 3. Norm of error fixing  
 $N = 17$ .Table 4. Norm of error fixing  
 $N = 17$ .

## 6. Conclusion

In this paper, we have applied the first-order least-squares spectral collocation methods to the time-dependent problems.

First, we transformed the time-domain problem to the frequency-domain problems which are independent of frequencies  $\omega$ . For each frequency  $\omega$ , we applied the least-squares spectral collocation method to the frequency-domain problems. From the least-squares approaches, we derive the positive definite symmetric algebraic system for each frequency so that we can further compute the algebraic problems using very fast iteration methods like PCGM with high performance parallel machine. Use of spectral collocation approximate allows us to have high accurate convergence. Also, it can be very easily implemented with relatively small system.

After we solve the frequency-domain problem, we transform the approximate solution to the time-domain approximate solution inversely using the adaptive Gaussian quadrature rule. Using the property that the Fourier transform  $\hat{q}(\omega)$  of a function  $q(t)$  goes to zero as  $\omega \rightarrow \infty$ , we can apply the adaptive strategy for inverse Fourier transform and save the cost to find the inversion.

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