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A Singular Perturbation-Like Approach to EDFA Gain Control Based on Observer Techniques

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Abstract – In this paper, we propose a singular perturbation-like approach to EDFA gain controller design and analysis. Considering a three-level model of EDFA, a gain controller containing a state observer and a channel add/drop estimator is designed based on a singular perturbation - like concept. The proposed design methodology is shown to be effective and advantageous not only in theoretically verifying the asymptotic stability of systems with multi-time scales such as EDFA but also in designing an asymptotic estimator for channel add/drops which does not satisfy the matching condition.

Keywords: EDFA, Singular perturbation, State observer, Gain control, Channel add/drop estimator

1. Introduction

EDFA(Erbium Doped Fiber Amplifier) is widely used for the amplification of channel signals in a WDM optical network. In an EDFA, it is important to maintain the gain of each channel when channel add/drops or active rearrangements of the network occur. The change of the number of channel signals called a channel add/drop causes a change of the amplifier gain of each channel signal due to the cross gain saturation effect [1].

There have been suggested several methods to handle this issue. One of them uses an EDFA output as feedback signal in an optical feedback control loop [2]. It, however, has the drawback that the frequency of channel add/drops should be less than that of the relaxation oscillation frequency of the EDFA, which is several hundred hertz. On the other hand, the gain fluctuation due to channel add / drops can be effectively compensated for by controlling the pump laser output electrically according to the EDFA output signal level [3]. In the previous papers [4-7], we proposed a novel technique which minimizes the gain-transient time effectively under the assumption that the rate of Erbium ions at level 3 converges relatively fast to the desired equilibrium compared with the one at level 2. A simplified two-level EDFA model was considered to design a gain controller and a disturbance observer (DOB) technique [8, 9, 10], and a proportional / integral (PI) controller was applied to the control of the EDFA gain in WDM add/drop networks. However, in order to compensate for the gain fluctuation due to channel add/drops as fast as in the order of micro-seconds, a full three-level model should be considered and a nominal gain controller should be designed considering the state of the population of Erbium ions at level 3. In a simplified two-level model, the matching condition is satisfied and channel add / drops can be easily controlled by a disturbance observer. However, the matching condition is not satisfied by the three-level model, so a new EDFA gain controller design methodology based on the three level model is necessary. In [5], a PID gain control algorithm considering the three-level EDFA model was applied to a nominal control. Since a channel add / drop compensator was still designed using a DOB based on a simplified two-level model, theoretical analysis of asymptotic stability could not be provided rigorously.

In this paper, a theoretical design and analysis of EDFA gain control system is carried out based on a mathematical three level EDFA model [11] using a singular perturbation technique [12]. In order to compensate for channel add / drop effects, a channel add/drop estimator is designed based on an internal model of EDFA, and an EDFA gain controller is proposed combining a state observer with the channel add/drop estimator. With successive applications of time scale separation to the designed EDFA control system, a singular perturbation technique gives a theoretical performance analysis of the proposed EDFA gain control algorithm even in the case that the matching condition is not satisfied. Through simulations, the practicality of the proposed control algorithm is also confirmed.

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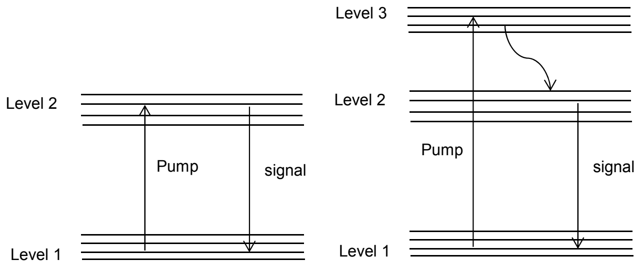
2. Design of EDFA Gain Control System

2.1 Three-level EDFA Model

In order to design an EDFA gain controller, the following three-level model is considered [11]. The energy level of



(a) two-level model (b) three-level model

Fig. 1. Models of EDFA

EDFA is shown in Fig. 1 and the equations for the three-level process are given as

$$\frac{dN_3}{dt} = -\Gamma_{32} N_3 - (N_3 \sigma_p^e - N_1 \sigma_p^a)\varphi_p \tag{1}$$

$$\frac{dN_2}{dt} = -\Gamma_{21} N_2 + (N_1 \sigma_s^a - N_2 \sigma_s^e)\varphi_s + \Gamma_{32} N_3 \tag{2}$$

$$\frac{dN_1}{dt} = \Gamma_{21} N_1 - (N_1 \sigma_s^a - N_2 \sigma_s^e)\varphi_s + (N_3 \sigma_p^e - N_1 \sigma_p^a)\varphi_p \tag{3}$$

Where Γ_{21}, Γ_{32} are positive constants; ϕ_s, ϕ_p are photon flux densities per second of the signal and the pump; $\sigma^e{}_s$, $\sigma^a{}_s$, $\sigma^e{}_p$, $\sigma^a{}_p$ are absorption and emission cross section of the signal and the pump ($\sigma^T = \sigma^e + \sigma^a$); and N_1, N_2, and N_3 are the number of erbium-ions at each energy level ($N = N_1 + N_2 + N_3 = 1$). The power P_s of the signal and the power P_p of the pump obey the following equations:

$$\frac{dP_p}{dt} = \rho \Gamma_p \left(\sigma_p^T N_3 + \sigma_p^a N_2 - \sigma_p^a \right) P_p \tag{4}$$

$$\frac{dP_s}{dt} = \rho \Gamma_s \left(\sigma_s^T N_2 + \sigma_s^a N_3 - \sigma_s^a \right) P_s \tag{5}$$

where ρ is the Erbium density, and Γ_s and Γ_p are respectively the geometric correction factor for the overlap between the power and the erbium-ions.

Define a reservoir $r_i(t)$, $i = 2,3$ that represents the number of excited Erbium-ions at each level and the EDFA gain of the k-th channel as follows:

$$r_i(t) \equiv \rho A \int_0^L N_i(z,t)dz, i = 2,3 \tag{6}$$

$$G_k(t) = \ln \frac{P_k^{out}}{P_k^{in}} \tag{7}$$

where L is the length of the Erbium-doped fiber, A is the cross-section area of erbium-doped fiber core, and P_k^{in} and P_k^{out} are respectively the k-th channel input power and output power. Without loss of generality, each channel input power is assumed to be an average power that is a positive constant until channel signals drop. Then, by integrating (1)-(3) along the whole length of EDF, we can obtain the following three-level EDFA model equations from definitions of reservoir $r_i(t)$, $i = 2, 3$ and $G_k(t) = ln(P_k{}^{out}/P_k{}^{in})$:

$$\frac{dr_3}{dt} = -\Gamma_{32} r_3 + \left(1 - e^{G_P(t)}\right) P_p^{in}(t) \tag{8}$$

$$\frac{dr_2}{dt} = -\Gamma_{21} r_2 + \Gamma_{32} r_3 + \sum_{k=1}^{N} \left(1 - e^{G_k(t)}\right) P_k^{in} \tag{9}$$

$$G_k(t) = B_k r_2 - A_k \tag{10}$$

where N is the number of channels, $G_p(t)$ is the gain of input pump channel and

$$\varphi_p = \Gamma_p \frac{P_p}{A}, \varphi_s = \Gamma_s \frac{P_p}{A}, A_k = \rho \Gamma_k \sigma_k^a L, B_k = \frac{\Gamma_k \sigma_k^T}{A} \tag{11}$$

Suppose that the k-th channel gain $G_k(t)$ should be maintained to be a desired constant channel gain G_k^C. Then, the state variable r_2 in the EDFA model Eqs. (8) and (9) must satisfy

$$r_2 = r_2^* = \frac{1}{B_k}(G_k^C + A_k) \tag{12}$$

at the steady state or equilibrium. Define an error variable as

$$e_2 = r_2 - r_2^*. \tag{13}$$

Then, the error dynamics are written as

$$\frac{dr_3}{dt} = -\Gamma_{32} r_3 + \left(1 - e^{G_P(t)}\right) P_p^{in}(t) \tag{14}$$

$$\frac{de_2}{dt} = -\Gamma_{21} e_2 + \Gamma_{32} r_3 - \Gamma_{21} r_2^* + \sum_{k=1}^{N} \left(1 - e^{G_k(t)}\right) P_k^{in}. \tag{15}$$

Our goal is to design a stabilizing controller for the system described by (14) and (15). The term $\sum_{k=1}^{N} \left(1 - e^{G_k(t)}\right) P_k^{in}$ consists of channel signals and varies according to channel add/drops which is not predictable in advance and is considered as a disturbance. So a disturbance observer technique can be adopted to reject the influence of channel add/drops on the channel gain variation. However, the control input P_p^{in} does not appear in the same equation with this term and thus it is nontrivial to compensate for this channel add/drops. The system (14) and (15) does not satisfy the so-called matching condition. In order to overcome this difficulty, we employ a singular perturbation method. If the dynamics (14) can be made much faster than the dynamics (15) by a control, a singular perturbation can be applied and we can reduce the dynamics such that the reduced dynamics satisfy the matching condition. We can then design a stabilizing controller for this reduced dynamic system using error state feedback and a disturbance estimator.

2.2 EDFA gain controller

In order to design a stabilizing controller for the system in (14) and (15), let us make the following assumption:

(A1) The gain $G_p(t)$ of the input pump channel is measurable.

Then, a stabilizing controller for the error dynamics (14) and (15) is designed as follows.

$$P_p^{in}(t) = -\frac{1}{\left(1-e^{G_P(t)}\right)} \left[k_1 e_2(t) + k_2 \hat{r}_3(t) + k_3 \hat{c}(t) - \frac{\Gamma_{21}}{\Gamma_{32}}\left(\Gamma_{32}+k_2\right) r_2^* \right] \tag{16}$$

where $\hat{r}_3(t)$ and $\hat{c}(t)$ are respectively estimation variables of the state $r_3(t)$ and the disturbance $c(t) = \sum_{k=1}^{N}\left(1-e^{G_k(t)}\right)P_k^{in}$ and $k_i, i=1,2,3$ are positive constants. In (16), the term $k_2\hat{r}_3(t)$ is to make the dynamics in (14) much faster than the one in (15). Since $r_3(t)$ is not measurable, we use its estimated value $\hat{r}_3(t)$ instead. The term $k_3\hat{c}(t)$ is to reject the term $c(t)$. So we need to design a state estimator for $r_3(t)$ and a disturbance observer for $c(t)$.

2.3 Design of a channel add/drop estimator

As mentioned in the previous section, we need to estimate the term $\sum_{k=1}^{N}\left(1-e^{G_k(t)}\right)P_k^{in}$ in (15) including channel add/drops. It usually costs a lot to measure all the channel powers P_k^{in} and all the channel gains G_k of the EDFA in optical networks. So it is inevitable to estimate the term $\sum_{k=1}^{N}\left(1-e^{G_k(t)}\right)P_k^{in}$. In order to estimate this, we consider the following internal nominal model of the EDFA:

$$\frac{d\tilde{r}_3}{dt} = -\Gamma_{32}\tilde{r}_3 + \left(1-e^{G_P(t)}\right)P_p^{in}(t) \tag{17}$$

$$\frac{d\tilde{r}_2}{dt} = -\Gamma_{21}\tilde{r}_2 + \Gamma_{32}\tilde{r}_3 \tag{18}$$

$$\tilde{G}_k(t) = B_k\tilde{r}_2 - A_k. \tag{19}$$

Define

$$\begin{aligned} \tilde{e}_2(t) &= r_2(t) - \tilde{r}_2(t) \\ \tilde{e}_3(t) &= r_3(t) - \tilde{r}_3(t) \\ \tilde{e}_G(t) &= G_k(t) - \tilde{G}_k(t) \\ c(t) &= \sum_{k=1}^{N}\left(1-e^{G_k(t)}\right)P_k^{in}. \end{aligned} \tag{20}$$

Then, we obtain the following equations:

$$\frac{d\tilde{e}_3}{dt} = -\Gamma_{32}\tilde{e}_3 \tag{21}$$

$$\frac{d\tilde{e}_2}{dt} = -\Gamma_{21}\tilde{e}_2 + \Gamma_{32}\tilde{e}_3 + c(t) \tag{22}$$

$$\tilde{e}_G(t) = B_k\tilde{e}_2. \tag{23}$$

Notice that the system (21)-(23) has stable zero dynamics. So we have the following transfer function between the channel add/drop input $c(t) = \sum_{k=1}^{N}\left(1-e^{G_k(t)}\right)P_k^{in}$ and the output $\tilde{e}_G(t)$:

$$P(s) = \frac{L[\tilde{e}_G(t)]}{L\left[c(t)\right]} = \frac{B_k}{s+\Gamma_{21}} \tag{24}$$

where $L[\cdot]$ denotes the Laplace transform. Define a filter $Q(s)$ by

$$Q(s) = \frac{L[\hat{c}(t)]}{L\left[\tilde{e}_G(t)\right]} = \frac{A_D(s+\Gamma_{21})}{B_k(s+A_D)} \tag{25}$$

where $\hat{c}(t)$ is the estimated output of $c(t)$. We have the following relation between the channel add/drop signal $c(t)$ and its estimate $\hat{c}(t)$:

$$\frac{L[\hat{c}(t)]}{L\left[c(t)\right]} = \frac{A_D}{s+A_D} \tag{26}$$

where the positive constant A_D is to be chosen later. Here we use a first-order linear model for the resultant channel add/drop estimator for convenience, but any higher order model can be equally used.

2.4 Design of a state observer

In order to stabilize the error system given by (14) and (15) with error state feedback, we need to estimate the state variable $r_3(t)$. Usually a state estimator can be easily designed if the system is observable, but its design becomes nontrivial when the term $\sum_{k=1}^{N}\left(1-e^{G_k(t)}\right)P_k^{in}$ in the EDFA model given by (8)-(10) cannot be measured. However, it is possible to design a state estimator that guarantees asymptotic estimation performance even when an unknown term $\sum_{k=1}^{N}\left(1-e^{G_k(t)}\right)P_k^{in}$ is present, if we use the channel add/drop estimator proposed in the previous section. Now we propose the following state estimator for the system (8)-(10):

$$\frac{d\hat{r}_3}{dt} = -\Gamma_{32}\hat{r}_3 + \left(1-e^{G_P(t)}\right)P_p^{in}(t) + L_1(G_k(t)-\hat{G}_k(t)) \quad (27)$$

$$\frac{d\hat{r}_2}{dt} = -\Gamma_{21}\hat{r}_2 + \Gamma_{32}\hat{r}_3 + \hat{c}(t) + L_2\{G_k(t)-\hat{G}_k(t)\} \quad (28)$$

$$\hat{G}_k(t) = B_k\hat{r}_2 - A_k \quad (29)$$

where L_1 and L_2 are observer gains. The observer gains L_1 and L_2 are chosen such that $\begin{bmatrix} -\Gamma_{32} & L_1B_k \\ \Gamma_{32} & -\Gamma_{32}-L_2B_k \end{bmatrix}$ is stable,

3. Theoretical Analysis : A Singular Perturbation Approach

In this section, we introduce a singular perturbation approach to stability analysis, which provides a systematic procedure for analysis of multi-time scaled systems.

3.1 Reduced dynamics of time-scaled closed-loop system

Since the estimator should have a faster performance than the controller, the control system designed in the previous section is considered as a multi-time scaled system. So a singular perturbation method can be applied to the analysis of the EDFA gain control system designed in Section II.

From (14) - (16) and (27) - (29), we obtain the error equations of the closed loop system as follows:

$$\Sigma_1 : \frac{dr_3}{dt} = -(\Gamma_{32}+k_2)\left\{r_3 - \frac{\Gamma_{21}}{\Gamma_{32}}r_2^*\right\} - k_1e_2(t) + k_2\hat{e}_3 - k_3\hat{c}(t) \quad (30)$$

$$\frac{de_2}{dt} = -\Gamma_{21}e_2 + \Gamma_{32}r_3 - \Gamma_{21}r_2^* + c(t) \quad (31)$$

$$\frac{d\hat{e}_3}{dt} = -\Gamma_{32}\hat{e}_3 + L_1B_k\hat{e}_2 \quad (32)$$

$$\frac{d\hat{e}_2}{dt} = -\Gamma_{21}\hat{e}_2 + \Gamma_{32}\hat{e}_3 + \{c(t)-\hat{c}(t)\} + L_2B_k\hat{e}_2 \quad (33)$$

where

$$e_2 = r_2 - r_2^*,\ \hat{e}_2(t) = r_2(t) - \hat{r}_2(t),$$
$$\hat{e}_3(t) = r_3(t) - \hat{r}_3(t). \quad (34)$$

From (26), we obtain the following equation for the channel add/drop estimator:

$$\frac{d\hat{c}(t)}{dt} = -A_D\left\{\hat{c}(t) - c(t)\right\}. \quad (35)$$

If the design parameter A_D in (35) is chosen so that the dynamics given in (35) is faster than any other subsystems (30)-(33), then the system (35) is stabilized very fast and $\hat{c}(t)$ immediately converges to $c(t)$.

Since k_2 is chosen so that $(\Gamma_{32}+k_2)$ becomes much larger than Γ_{21}, a singular perturbation procedure is applied to (30)-(35) by letting $\dot{r}_3 \to 0$. Let k_3 be chosen as

$$k_3 = \frac{\Gamma_{32}+k_2}{\Gamma_{32}}. \quad (36)$$

Then we obtain the following reduced dynamics.

$$\Sigma_2 : \frac{d\bar{e}_2}{dt} = -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\bar{e}_2 + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\bar{e}}_3 + \bar{c}(t) - \hat{\bar{c}}(t) \quad (37)$$

$$\frac{d\hat{\bar{e}}_3}{dt} = -\Gamma_{32}\hat{\bar{e}}_3 + L_1B_k\hat{\bar{e}}_2 \quad (38)$$

$$\frac{d\hat{\bar{e}}_2}{dt} = -\Gamma_{21}\hat{\bar{e}}_2 + \Gamma_{32}\hat{\bar{e}}_3 + \{\bar{c}(t)-\hat{\bar{c}}(t)\} + L_2B_k\hat{\bar{e}}_2 \quad (39)$$

$$\frac{d}{dt}\hat{\bar{c}}(t) = -A_D\left\{\hat{\bar{c}}(t) - \bar{c}(t)\right\} \quad (40)$$

where

$$\bar{c}(t) = \sum_{k=1}^{N}\left(1-e^{\bar{G}_k(t)}\right)P_k^{in} \quad (41)$$

and

$$\bar{e}_2 = \bar{r}_2 - r_2^* \quad (42)$$

and $\bar{r}_2$ and $\bar{G}_k$ satisfy the following:

$$\frac{d\bar{r}_2}{dt} = -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\bar{r}_2 - r_2^*\right) + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\bar{e}}_3 + \bar{c}(t) - \hat{\bar{c}}(t)$$
$$\bar{G}_k(t) = B_k\bar{r}_2 - A_k. \quad (43)$$

The design parameter A_D in (35) or (40) is chosen so that the dynamics given by (35) or (40) is faster than the other subsystem dynamics (37)-(39). Then the system (40) is stabilized very fast and $\hat{\bar{c}}(t)$ immediately converges to $\bar{c}(t)$. So, if we apply a singular perturbation method again to (37)-(40) as $\hat{\bar{c}}(t) \to 0$, we have the following reduced dynamics:

$$\Sigma_3 : \frac{d\tilde{e}_2}{dt} = -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\tilde{e}_2 + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\tilde{e}}_3 \tag{44}$$

$$\frac{d}{dt}\hat{\tilde{e}}_3 = -\Gamma_{32}\hat{\tilde{e}}_3 + L_1 B_k \hat{\tilde{e}}_2 \tag{45}$$

$$\frac{d}{dt}\hat{\tilde{e}}_2 = -\Gamma_{21}\hat{\tilde{e}}_2 + \Gamma_{32}\hat{\tilde{e}}_3 + L_2 B_k \hat{\tilde{e}}_2. \tag{46}$$

Since the design parameters L_1 and L_2 of the state estimator are designed so that its performance is much faster than error state feedback control and stable, the estimation error states $\hat{\tilde{e}}_2$ and $\hat{\tilde{e}}_3$ of the reduced dynamics in (45) and (46) decay to zero much faster than $\tilde{e}_2$. So, as $\hat{\tilde{e}}_2(t) \to 0, \hat{\tilde{e}}_3(t) \to 0,$ we have the following reduced dynamics for the system Σ_3 :

$$\Sigma_4 : \frac{d\bar{\tilde{e}}_2}{dt} = -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\bar{\tilde{e}}_2. \tag{47}$$

From (47), it is obvious that

$$\left|\bar{\tilde{e}}_2(t)\right| \le \left|\bar{\tilde{e}}_2(0)\right| e^{-\lambda t}, \forall t \ge 0 \tag{48}$$

where

$$\lambda = \Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}. \tag{49}$$

The performance of the control system is determined by a desired bandwidth λ, and the controller gain k_1 is chosen as

$$k_1 = \frac{\Gamma_{32}+k_2}{\Gamma_{32}}\left(\lambda - \Gamma_{21}\right) \tag{50}$$

for given λ and k_2. Choice of the controller gain k_2 is discussed in the next section.

3.2 Stability analysis

Stability analysis is carried out by showing the asymptotic stability of each system $\Sigma_i, i = 1,2,3$ successively using asymptotic stability of systems $\Sigma_m, m = i+1,\cdots,4$. In order to show the asymptotic stability, we assume that channel add/drops are not persistent. That is, channel add/drops are assumed to occur finitely many times. We make the following assumption.

(A2) The number of Channel add/drops, M, is finite.

Define the instants at which channel add/drops occur by a time sequence $t_n, n = 1,\cdots,M$. So, if we define $t_{M+1} = \infty$, each channel input $p_k^{in}, k = 1,\cdots,N$ is zero or a positive constant for all $t \in [t_i, t_{i+1})$, $i = 1,\cdots,M$. So, the time derivative of each channel input is zero for all $t \in (t_i, t_{i+1}), i = 1,\cdots,M$ and the following equation holds for any $\varepsilon > 0$:

$$\frac{d}{dt}p_k^{in} = 0, k = 1,\cdots,N, \forall t \in [t_i + \varepsilon, t_{i+1}), i = 1,\cdots M. \tag{51}$$

Step 1. Stability analysis of Σ_3

In order to show that the system Σ_3 is asymptotically stable, we define the error variables between the system Σ_3 and the system Σ_4 as follows:

$$E(t) = \begin{bmatrix} E_2 & \hat{\tilde{e}}_2 & \hat{\tilde{e}}_3 \end{bmatrix}^T, E_2 = \tilde{e}_2 - \bar{\tilde{e}}_2. \tag{52}$$

Then,

$$\frac{dE_2}{dt} = -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)E_2 + \Gamma_{32}\hat{\tilde{e}}_3 \tag{53}$$

$$\frac{d}{dt}\hat{\tilde{e}}_3 = -\Gamma_{32}\hat{\tilde{e}}_3 + L_1 B_k \hat{\tilde{e}}_2 \tag{54}$$

$$\frac{d}{dt}\hat{\tilde{e}}_2 = -\Gamma_{21}\hat{\tilde{e}}_2 + \Gamma_{32}\hat{\tilde{e}}_3 + L_2 B_k \hat{\tilde{e}}_2. \tag{55}$$

It is obvious that the system (53)-(55) is asymptotically stable. So, there exist positive numbers α_3 and β_3 such that

$$\|E(t)\| \le \alpha_3 \|E(0)\| e^{-\beta_3 t}, \forall t \ge 0. \tag{56}$$

Step 2. Stability analysis of Σ_2 using the asymptotic stability of Σ_3 and Σ_4

Define an error vector $\bar{E}$ by

$$\bar{E}(t) = \begin{bmatrix} \bar{E}_2(t) & \hat{\bar{E}}_3(t) & \hat{\bar{E}}_2(t) \end{bmatrix}^T, \tag{57}$$

$$\bar{E}_C(t) = \bar{c}(t) - \hat{\bar{c}}(t).$$

$$\bar{E}_2(t) = \bar{e}_2(t) - \tilde{e}_2(t), \hat{\bar{E}}_3(t) = \hat{\bar{e}}_3(t) - \hat{\tilde{e}}_3(t),$$

$$\hat{\bar{E}}_2(t) = \hat{\bar{e}}_2(t) - \hat{\tilde{e}}_2(t).$$

From (37) - (40) and (44) - (46), the error system is described as follows:

$$\frac{d}{dt}\bar{E}_2 = -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\bar{E}_2 + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\bar{E}}_3 + \bar{E}_c \tag{58}$$

$$\frac{d}{dt}\hat{\bar{E}}_3 = -\Gamma_{32}\hat{\bar{E}}_3 + L_1 B_k \hat{\bar{E}}_2 \quad (59)$$

$$\frac{d}{dt}\hat{\bar{E}}_2 = -(\Gamma_{21} - L_2 B_k)\hat{\bar{E}}_2 + \Gamma_{32}\hat{\bar{E}}_3 + \bar{E}_c \quad (60)$$

Since the length of EDF, l is finite, the reservoir r_2, $G_k(t)$ and $\bar{c}(t)$ defined in (41) are bounded. So, $\bar{E}_C$ is also bounded from (40) such that there exists a positive constant B_C satisfying

$$\left|\bar{E}_C\right| < B_C, \forall t \geq 0. \quad (61)$$

Therefore, from (57), (58), (59), and (61), there exists a positive constant B_E such that the following inequality holds.

$$\left\|\bar{E}(t)\right\| \leq B_E, \forall t \geq 0 \quad (62)$$

From (41) , (43), (51) and (62),

$$\frac{d}{dt}\bar{c}(t) = \frac{d}{dt}\left\{\sum_{k=1}^{N}\left(1 - e^{\bar{G}_k(t)}\right)P_k^{in}\right\} = \sum_{k=1}^{N}\left(-e^{\bar{G}_k(t)}\dot{\bar{G}}_k(t)P_k^{in}\right)$$

$$= \sum_{k=1}^{N}\left(-e^{\bar{G}_k(t)}B_k P_k^{in}\right)\left(\dot{\bar{r}}_2(t)\right) \quad (63)$$

and

$$\frac{d\bar{r}_2}{dt} = -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\bar{r}_2 - r_2^*\right) + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\tilde{e}}_3 + \bar{E}_c$$

$$= -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\bar{E}_2 + E_2 + \tilde{\bar{e}}_2\right)$$

$$+ \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(\hat{\bar{E}}_3 + \hat{\tilde{e}}_3\right) + \bar{E}_c. \quad (64)$$

Thus, for all $t \in [t_i + \varepsilon, t_{i+1}), i = 1, \cdots, M$,

$$\frac{d\bar{E}_c}{dt} = -A\bar{E}_c + \dot{\bar{c}}$$

$$= -(A_D + \Pi_1)\bar{E}_c + \bar{\Pi}_1\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\bar{E}_2 + E_2 + \tilde{\bar{e}}_2\right)$$

$$-\bar{\Pi}_1\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(\hat{\bar{E}}_3 + \hat{\tilde{e}}_3\right) \quad (65)$$

where

$$\bar{\Pi}_1(t) = \sum_{k=1}^{N} B_k e^{\bar{G}_k(t)} P_k^{in}. \quad (66)$$

Meanwhile, the reservoir r_2 is bounded and $G_k(t)$ is also bounded since the length of EDF, l is finite. So, there exist positive constants Π_m and Π_M such that

$$\Pi_m \leq \bar{\Pi}_1(t) \leq \Pi_M, \forall t \geq 0. \quad (67)$$

Define a matrix A_3 by

$$A_3 = \begin{bmatrix} -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right) & \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2} & 0 \\ 0 & -\Gamma_{32} & L_1 B_k \\ 0 & \Gamma_{32} & -(\Gamma_{21} - L_2 B_k) \end{bmatrix}. \quad (68)$$

Since A_3 is stable for any positive numbers k_1 and k_2, there exist positive definite symmetric matrices P and Q satisfying the following Lyapunov equation

$$A_3^T P + PA_3 = -Q. \quad (69)$$

Define a Lyapunov-like function V_3 by

$$V_3(t) = X_3^T P_3 X_3 \quad (70)$$

where

$$X_3 = \left[\bar{E}^T \quad \bar{E}_c\right]^T, P_3 = diag(P, \rho_c), \rho_c > 0. \quad (71)$$

Then, from (58) and (65) , for all $t \in [t_i + \varepsilon, t_{i+1})$,

$$\frac{d}{dt}V_3(t) = \bar{E}^T\left(A_3^T P + PA_3\right)\bar{E} + 2\bar{E}^T PB_c\bar{E}_c$$

$$-2\rho_c\left(A + \bar{\Pi}_1\right)\bar{E}_c^2 - 2\rho_c\bar{\Pi}_1\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(\hat{\bar{E}}_3 + \hat{\tilde{e}}_3\right)\bar{E}_c$$

$$+2\rho_c\bar{\Pi}_1\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\bar{E}_2 + E_2 + \tilde{\bar{e}}_2\right)\bar{E}_c \quad (72)$$

where $B_c = [1\ 1\ 0]^T$. Since $\bar{\Pi}_1(t)$ is positive and bounded as in (67), we have the following inequality for all $t \in [t_i + \varepsilon, t_{i+1})$.

$$\frac{d}{dt}V_3(t) \leq -\bar{E}^T Q\bar{E} + 2\bar{E}^T PB_c\bar{E}_c$$

$$-2\rho_c\left(A_D + \Pi_m\right)\bar{E}_c^2 + 2\rho_c\bar{\Pi}_1\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\bar{E}_2\bar{E}_c$$

$$+2\rho_c\Pi_M\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\left|E_2\right| + \left|\tilde{\bar{e}}_2\right|\right)\left|\bar{E}_c\right|$$

$$+2\rho_c\bar{\Pi}_1\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\bar{E}}_3\bar{E}_c + 2\rho_c\Pi_M\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left|\hat{\tilde{e}}_3\right|\left|\bar{E}_c\right|$$

$$\leq -X_3^T \bar{Q}(t) X_3 + 2\rho_c \Pi_M \left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2} \right) \left(|E_2| + |\tilde{\bar{e}}_2| \right) |\bar{E}_c|$$
$$+2\rho_c \Pi_M \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2} \left(|\hat{\tilde{e}}_3| \right) |\bar{E}_c| \quad (73)$$

where

$$\bar{Q}(t) = \begin{bmatrix} \lambda_m(Q) I_{3\times 3} & -PB_c + \bar{\Pi}_1(t)\Lambda \\ (-PB_c + \bar{\Pi}_1(t)\Lambda)^T & 2\rho_c \left(A_D + \Pi_m \right) \end{bmatrix},$$
$$\Lambda = \begin{bmatrix} q_{14} & q_{24} & 0 \end{bmatrix}^T, \; q_{14} = -2\rho_c \left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2} \right), \quad (74)$$
$$q_{24} = 2\rho_c \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}.$$

For a positive number $0 < \bar{\lambda} < \lambda_m(Q)$, choose a design parameter A_D as follows

$$A_D > \frac{1}{2\rho_c} \left\{ \bar{\lambda} + \frac{\delta}{\lambda_m(Q) - \bar{\lambda}} \right\} - \Pi_m \quad (75)$$

where

$$\delta = \max_{\Pi_1(t)} \left\{ (PB_c - \bar{\Pi}_1(t)\Lambda)^T (PB_c - \bar{\Pi}_1(t)\Lambda) \right\}. \quad (76)$$

From (67), it follows that

$$\begin{aligned} \delta &= \max\{\delta_m, \delta_M\} \\ &= \max_{\Pi_1(t)} \left\{ (PB_c + \bar{\Pi}_1(t)\Lambda)^T (PB_c + \bar{\Pi}_1(t)\Lambda) \right\} \end{aligned} \quad (77)$$

where

$$\begin{aligned} \delta_m &= (PB_c - \Pi_m\Lambda)^T (PB_c - \Pi_m\Lambda), \\ \delta_M &= (PB_c - \Pi_M\Lambda)^T (PB_c - \Pi_M\Lambda). \end{aligned} \quad (78)$$

Then, $\bar{Q}$ is positive definite and satisfies

$$\bar{Q}(t) > \bar{\lambda} I_{4\times 4}. \quad (79)$$

So, the Lyapunov function V_3 satisfies the following inequality for all $t \in [t_i + \varepsilon, t_{i+1}), i = 1, \cdots, M$.

$$\frac{d}{dt} V_3(t) \leq -\frac{\lambda_m(\bar{Q}(t))}{\lambda_M(P)} V_3 + 2\rho_c \Pi_M \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2} \left(|\hat{\tilde{e}}_3| \right) \frac{1}{\sqrt{\lambda_m(P)}} V_3^{\frac{1}{2}}$$
$$+2\rho_c \Pi_M \left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2} \right) \left(|E_2| + |\tilde{\bar{e}}_2| \right) \frac{1}{\sqrt{\lambda_m(P)}} V_3^{\frac{1}{2}} \quad (80)$$

where $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ are respectively the smallest and the largest eigenvalue of the associated matrix. Dividing (80) by $V_3^{\frac{1}{2}}$ leads to the following inequality:

$$\begin{aligned} \frac{d}{dt} V_3^{\frac{1}{2}}(t) &\leq -\frac{\bar{\lambda}}{2\lambda_M(P)} V_3^{\frac{1}{2}} + \frac{\rho_c \Pi_M}{\sqrt{\lambda_m(P)}} \cdot \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2} |\hat{\tilde{e}}_3| \\ &+ \frac{\rho_c \Pi_M}{\sqrt{\lambda_m(P)}} \left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2} \right) \left(|E_2| + |\tilde{\bar{e}}_2| \right) \end{aligned} \quad (81)$$

for all $t \in [t_i + \varepsilon, t_{i+1}), i = 1, \cdots, M$. It follows from (48), (56), and (81) that

$$\begin{aligned} V_3^{\frac{1}{2}}(t) &\leq V_3^{\frac{1}{2}}(t_i + \varepsilon) e^{-\frac{\bar{\lambda}}{2\lambda_M(P)}(t - t_i - \varepsilon)} \\ &+ \frac{\rho_c \Pi_M}{\sqrt{\lambda_m(P)}} \cdot \left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2} + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2} \right) \\ &\times \frac{2\lambda_M(P)\alpha_3}{\bar{\lambda} - 2\lambda_M(P)\beta_3} \|E(0)\| \left\{ e^{-\beta_3(t - t_i - \varepsilon)} - e^{-\frac{\bar{\lambda}}{2\lambda_M(P)}(t - t_i - \varepsilon)} \right\} \\ &+ \frac{\rho_c \Pi_M}{\sqrt{\lambda_m(P)}} \cdot \left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2} \right) \\ &\times \frac{2\lambda_M(P)}{\bar{\lambda} - 2\lambda_M(P)\lambda} |\hat{\tilde{e}}_2(t_i + \varepsilon)| \left\{ e^{-\lambda(t - t_i - \varepsilon)} - e^{-\frac{\bar{\lambda}}{2\lambda_M(P)}(t - t_i - \varepsilon)} \right\} \end{aligned} \quad (82)$$

for all $t \in [t_i + \varepsilon, t_{i+1})$. So, there exist positive numbers $\bar{\alpha}_{3,i}$ and $\bar{\beta}_{3,i}$ for each $i \in [1, M]$ such that

$$\begin{aligned} &\|X_3(t)\| \leq \bar{\alpha}_{3,i} \|X_3(t_i + \varepsilon)\| e^{-\bar{\beta}_{3,i}(t - t_i - \varepsilon)}, \\ &\forall t \in [t_i + \varepsilon, t_{i+1}). \end{aligned} \quad (83)$$

From (70) and (71), there exists a positive constant B_{X3} such that the following inequality holds.

$$\|X_3(t)\| \leq B_{X3}, \forall t \geq 0 \quad (84)$$

Therefore, the asymptotic stability is satisfied since (83) holds for $t \in [t_M + \varepsilon, \infty)$.

Step 3. Stability Analysis of Σ_1 using the stability of Σ_2, Σ_3, and Σ_4.

Define an error vector $\tilde{E}$ and an error variable $\tilde{E}_3$ by

$$\begin{aligned} &\tilde{E}(t) = \begin{bmatrix} \tilde{E}_2(t) & \hat{\tilde{E}}_3(t) & \hat{\tilde{E}}_2(t) & \tilde{E}_C(t) \end{bmatrix}^T, \\ &\tilde{E}_3(t) = r_3(t) - \bar{r}_3(t) \end{aligned} \quad (85)$$

where

$$\begin{aligned}
&\bar{r}_3(t) = \frac{\Gamma_{21}}{\Gamma_{32}} r_2^* - \frac{1}{\Gamma_{32}+k_2}\left\{k_1 e_2(t) - k_2 \hat{e}_3(t) + k_3 \hat{c}(t)\right\},\\
&\tilde{E}_2(t) = e_2(t) - \bar{e}_2(t), \hat{\tilde{E}}_2(t) = \hat{e}_2(t) - \hat{\bar{e}}_2(t),\\
&\hat{\tilde{E}}_3(t) = \hat{e}_3(t) - \hat{\bar{e}}_3(t), \quad \tilde{E}_C(t) = c(t) - \hat{c}(t).
\end{aligned} \tag{86}$$

Then, we have the following error equations:

$$\frac{d\tilde{E}_3}{dt} = -(\Gamma_{32}+k_2)\tilde{E}_3 + \frac{1}{\Gamma_{32}+k_2}\left\{k_1 \dot{e}_2 - k_2 \dot{\hat{e}}_3 + k_3 \dot{\hat{c}}\right\} \tag{87}$$

$$\begin{aligned}
\frac{d\tilde{E}_2}{dt} &= -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\tilde{E}_2 + \Gamma_{32}\tilde{E}_3 \\
&\quad + \tilde{E}_C - \bar{E}_C + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\tilde{E}}_3
\end{aligned} \tag{88}$$

$$\frac{d\hat{\tilde{E}}_3}{dt} = -\Gamma_{32}\hat{\tilde{E}}_3 + L_1 B_k \hat{\tilde{E}}_2 \tag{89}$$

$$\frac{d\hat{\tilde{E}}_2}{dt} = -(\Gamma_{21} + L_2 B_k)\hat{\tilde{E}}_2 + \Gamma_{32}\hat{\tilde{E}}_3 + \tilde{E}_c - \bar{E}_c \tag{90}$$

Rewriting (31) and (32) , we obtain

$$\begin{aligned}
\frac{dr_2}{dt} &= \frac{de_2}{dt} = \frac{d\tilde{E}_2}{dt} + \frac{d\bar{e}_2}{dt}\\
&= -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\tilde{E}_2 + \Gamma_{32}\tilde{E}_3 + \tilde{E}_C + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\tilde{E}}_3(t)\\
&\quad -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\bar{e}_2 + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\bar{e}}_3(t)
\end{aligned} \tag{91}$$

$$\frac{d\hat{e}_3}{dt} = -\Gamma_{32}\left(\hat{\tilde{E}}_3 + \hat{\bar{e}}_3\right) + L_1 B_k\left(\hat{\tilde{E}}_2 + \hat{\bar{e}}_2\right). \tag{92}$$

It follows from (91) and (92) that (87) is described by

$$\begin{aligned}
\frac{d\tilde{E}_3}{dt} &= -(\Gamma_{32} + k_2 - \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2})\tilde{E}_3 - \frac{k_1}{\Gamma_{32}+k_2}\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\tilde{E}_2\\
&\quad - \frac{k_2}{\Gamma_{32}+k_2}L_1 B_k \hat{\tilde{E}}_2 + \left(1 + \frac{k_1}{\Gamma_{32}+k_2}\right)\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\hat{\tilde{E}}_3(t)\\
&\quad - \frac{k_1}{\Gamma_{32}+k_2}\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\bar{E}_2 + E_2 + \tilde{\bar{e}}_2\right)\\
&\quad + \frac{k_1 + k_3 A_D}{\Gamma_{32}+k_2}\tilde{E}_C - \frac{k_2}{\Gamma_{32}+k_2}L_1 B_k \hat{\bar{e}}_2\\
&\quad + \frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(1 + \frac{k_1}{\Gamma_{32}+k_2}\right)\hat{\bar{e}}_3
\end{aligned} \tag{93}$$

From the analysis in step 1 and step 2, $E_2, \bar{E}_2, \tilde{\bar{e}}_2$, $\hat{\bar{e}}_2, \hat{\bar{e}}_3(t)$ and $\bar{E}_C$ are bounded. Using the same arguments in step 2, we can also show that $\hat{c}$ and $\tilde{E}_C$ are bounded because $c(t)$ in (30) is bounded. Since $\Gamma_{21}, \Gamma_{32}, k_1, k_2$ and k_3 are positive constants and the observer gains L_1 and L_2 are chosen such that $\begin{bmatrix} -\Gamma_{32} & L_1 B_k \\ \Gamma_{32} & -\Gamma_{32} - L_2 B_k \end{bmatrix}$ is stable, the error equations given by (88)-(90) and (93) satisfy the BIBO stability. Therefore, the error state vector $\tilde{E}$ and $\tilde{E}_3$ defined by (85) are bounded and there exist positive constants $B_{\tilde{E}}$ and $B_{\tilde{E}3}$ such that

$$\left\|\tilde{E}(t)\right\| \le B_{\tilde{E}}, \left\|\tilde{E}_3(t)\right\| \le B_{\tilde{E}3}, \forall t \ge 0\,. \tag{94}$$

As in step 2, we now consider the performance for $t \in (t_i, t_{i+1})$, $i = 1, \cdots, M$. Since each channel input p_k^{in}, $k = 1, \cdots, N$ is zero or positive constant for all $t \in [t_i + \varepsilon, t_{i+1}), i = 1, \cdots, M$, the time derivative of $c(t)$ is given by

$$\begin{aligned}
\frac{d}{dt}c(t) &= \sum_{k=1}^{N}\left(-e^{G_k(t)} B_k P_k^{in}\right)\left(\dot{r}_2(t)\right)\\
&= \Pi_1\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\tilde{E}_2 + \bar{e}_2\right)\\
&\quad - \Pi_1\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(\hat{\tilde{E}}_3 + \hat{\bar{e}}_3\right) - \Pi_1\Gamma_{32}\tilde{E}_3 - \Pi_1\tilde{E}_C
\end{aligned} \tag{95}$$

where

$$\Pi_1(t) = \sum_{k=1}^{N} B_k e^{G_k(t)} P_k^{in}. \tag{96}$$

As in step 2, the following error equation holds for all $t \in [t_i + \varepsilon, t_{i+1}), i = 1, \cdots, M$.

$$\begin{aligned}
\frac{d\tilde{E}_c}{dt} &= -A_D\tilde{E}_c + \dot{c}.\\
&= -A_D\tilde{E}_c + \Pi_1\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left(\tilde{E}_2 + \bar{e}_2\right)\\
&\quad - \Pi_1\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(\hat{\tilde{E}}_3 + \hat{\bar{e}}_3\right) - \Pi_1\Gamma_{32}\tilde{E}_3 - \Pi_1\tilde{E}_C
\end{aligned} \tag{97}$$

Define a Lyapunov-like function V_2 by

$$V_2(t) = X_2^T P_2 X_2 \tag{98}$$

where

$$X_2 = \left[\tilde{E}^T \quad \tilde{E}_3\right]^T, P_2 = diag(P_3, \rho_3), \rho_3 > 0\,. \tag{99}$$

Then, it follows from (36), (49), (69), (71), (73), and (74) that for all $t \in [t_i + \varepsilon, t_{i+1}), i = 1, \cdots, M$,

$$\begin{aligned}\frac{d}{dt}V_2(t) &\le -\tilde{E}^T\bar{Q}(t)\tilde{E} + 2\tilde{E}^T P_3\bar{B}_3\tilde{E}_3 + 2\tilde{E}^T P_3\bar{B}_2\bar{E}_c \\ &-2\rho_3\left(\Gamma_{32}+k_2-\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\tilde{E}_3^2 + \frac{2k_2}{\Gamma_{32}+k_2}\rho_3 L_1 B_k \hat{\tilde{e}}_2\tilde{E}_3 \\ &-2\rho_3\frac{k_1}{\Gamma_{32}+k_2}\left(\Gamma_{21}+\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\tilde{E}_3\left(\bar{E}_2+E_2+\tilde{\bar{e}}_2\right) \\ &+2\rho_3\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(1+\frac{k_1}{\Gamma_{32}+k_2}\right)\hat{\tilde{e}}_3\tilde{E}_3 \\ &+2\rho_c\Pi_1\left(\Gamma_{21}+\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\tilde{E}_c\left(\bar{E}_2+E_2+\tilde{\bar{e}}_2\right) \\ &-2\rho_c\Pi_1\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\tilde{E}_c\hat{\tilde{e}}_3\end{aligned} \tag{100}$$

where

$$\begin{aligned}&\bar{B}_3 = B_{31} + B_{32}\frac{k_2}{\Gamma_{32}+k_2} + B_{33}\Pi_1 \\ &B_{31} = \begin{bmatrix}\delta_1 & 0 & 0 & \delta_4\end{bmatrix}^T,\ B_{32} = \begin{bmatrix}0 & \delta_3 & \delta_2 & 0\end{bmatrix}^T, \\ &B_{33} = \begin{bmatrix}0 & 0 & 0 & -\Gamma_{32}\end{bmatrix}^T,\ \bar{B}_2 = \begin{bmatrix}-1 & 0 & -1 & 0\end{bmatrix}^T, \\ &\delta_1 = -\frac{\lambda(\lambda-\Gamma_{21})}{\Gamma_{32}},\ \delta_2 = -L_1B_k, \\ &\delta_3 = \lambda-\Gamma_{21}+\Gamma_{32},\ \delta_4 = \frac{(\lambda-\Gamma_{21})}{\Gamma_{32}} - \frac{A_D}{\Gamma_{32}}.\end{aligned} \tag{101}$$

By (79), for all $t\in[t_i+\varepsilon, t_{i+1}), i=1,\cdots,M$,

$$\begin{aligned}\frac{d}{dt}V_2(t) &\le -\bar{\lambda}\tilde{E}^T\tilde{E} + 2\tilde{E}^T P_3 B_3\tilde{E}_3 + 2\tilde{E}^T P_3 B_2\bar{E}_c \\ &-2\rho_3\left(-\lambda+\Gamma_{21}+\Gamma_{32}+k_2\right)\tilde{E}_3^2 + \frac{2k_2}{\Gamma_{32}+k_2}\rho_3 L_1 B_k\left|\hat{\tilde{e}}_2\right|\left|\tilde{E}_3\right| \\ &+2\rho_3\frac{k_1}{\Gamma_{32}+k_2}\left(\Gamma_{21}+\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left|\tilde{E}_3\right|\left(\left|\bar{E}_2\right|+\left|E_2\right|+\left|\tilde{\bar{e}}_2\right|\right) \\ &+2\rho_3\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(1+\frac{k_1}{\Gamma_{32}+k_2}\right)\left|\tilde{E}_3\right|\left|\hat{\tilde{e}}_3\right| \\ &+2\rho_c\Pi_M\left(\Gamma_{21}+\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left|\tilde{E}_c\right|\left(\left|\bar{E}_2\right|+\left|E_2\right|+\left|\tilde{\bar{e}}_2\right|\right) \\ &+2\rho_c\Pi_M\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left|\tilde{E}_c\right|\left|\hat{\tilde{e}}_3\right| \\ &= -\tilde{E}^T\tilde{Q}\tilde{E} + \frac{2k_2}{\Gamma_{32}+k_2}\rho_3 L_1 B_k\left|\hat{\tilde{e}}_2\right|\left\|\tilde{E}\right\| + 2\left\|P_3B_2\right\|\left\|\bar{E}_c\right\|\left\|\tilde{E}\right\| \\ &+2\rho_3\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(1+\frac{k_1}{\Gamma_{32}+k_2}\right)\left|\tilde{E}_3\right|\left|\hat{\tilde{e}}_3\right| \\ &+2\rho_3\left(\Gamma_{21}+\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left\|\tilde{E}\right\|\left(\left|\bar{E}_2\right|+\left|E_2\right|+\left|\tilde{\bar{e}}_2\right|\right) \\ &+2\rho_c\Pi_M\left(\Gamma_{21}+\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left|\tilde{E}_c\right|\left(\left|\bar{E}_2\right|+\left|E_2\right|+\left|\tilde{\bar{e}}_2\right|\right) \\ &+2\rho_c\Pi_M\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left|\tilde{E}_c\right|\left|\hat{\tilde{e}}_3\right|\end{aligned} \tag{102}$$

where

$$\tilde{Q} = \begin{bmatrix}\bar{\lambda}I_{4\times4} & P_3\left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_1\right) \\ \left\{P_3\left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_1\right)\right\}^T & 2\rho_3\left(k_2-\lambda+\Gamma_{21}+\Gamma_{32}\right)\end{bmatrix} \tag{103}$$

Let $0<\tilde{\lambda}<\bar{\lambda}$. In order for $(\tilde{Q}(t)-\tilde{\lambda}I_{5\times5})$ to be positive definite for all $t\ge0$, the following inequality must be satisfied:

$$\begin{aligned}&2\rho_3\left(k_2-\lambda-\Gamma_{21}+\Gamma_{32}-\tilde{\lambda}\right)\left(\bar{\lambda}-\tilde{\lambda}\right) \\ &-\left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_1\right)^T \\ &P_3P_3\left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_1\right)>0\end{aligned} \tag{104}$$

Define σ_M by

$$\begin{aligned}\sigma_M = \max_{\Pi_1}&\left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_1\right)^T \\ &P_3P_3\left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_1\right)\end{aligned} \tag{105}$$

Since $\Pi_1(t)$ is positive and bounded as in (67), σ_M is obtained for each k_2 when $\Pi_1(t)=\Pi_m$ or $\Pi_1(t)=\Pi_M$. Let us define σ_1 and σ_2 as

$$\begin{aligned}\sigma_1(k_2) &= \left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_m\right)^T \\ &\quad P_3P_3\left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_m\right) \\ \sigma_2(k_2) &= \left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_M\right)^T \\ &\quad P_3P_3\left(B_{31}+B_{32}\frac{k_2}{\Gamma_{32}+k_2}+B_{33}\Pi_M\right)\end{aligned} \tag{106}$$

If k_2 is chosen such that

$$2\rho_3\left(k_2-\lambda+\Gamma_{21}+\Gamma_{32}-\tilde{\lambda}\right)\left(\bar{\lambda}-\tilde{\lambda}\right)-\sigma_1(k_2)>0 \tag{107}$$

and

$$2\rho_3\left(k_2-\lambda+\Gamma_{21}+\Gamma_{32}-\tilde{\lambda}\right)\left(\bar{\lambda}-\tilde{\lambda}\right)-\sigma_2(k_2)>0\,, \quad (108)$$

then $(\tilde{Q}(t)-\tilde{\lambda}I_{5\times5})$ is positive definite. Since the inequalities (107) and (108) are of third order in k_2 if $\left(\Gamma_{32}+k_2\right)^2$ is multiplied to both sides of the inequalities, there always exists a k_2 satisfying (107) and (108). Then, for all $t\in[t_i+\varepsilon,t_{i+1}), i=1,\cdots,M$,

$$\begin{aligned}\frac{d}{dt}V_2(t) &\le -\tilde{\lambda}\tilde{E}^T\tilde{E}\\ &+\frac{2k_2}{\Gamma_{32}+k_2}\rho_3 L_1 B_k\left|\hat{\bar{e}}_2\right|\left\|\tilde{E}\right\|+2\left\|P_3B_2\right\|\left\|\bar{E}_c\right\|\left\|\tilde{E}\right\|\\ &+2\rho_3\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left(1+\frac{k_1}{\Gamma_{32}+k_2}\right)\left\|\tilde{E}\right\|\left|\hat{\bar{e}}_3\right|\\ &+2\rho_3\left(\Gamma_{21}+\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left\|\tilde{E}\right\|\left(\left|\bar{E}_2\right|+\left|E_2\right|+\left|\tilde{\bar{e}}_2\right|\right)\\ &+2\rho_c\Pi_M\left(\Gamma_{21}+\frac{\Gamma_{32}k_1}{\Gamma_{32}+k_2}\right)\left\|\tilde{E}\right\|\left(\left|\bar{E}_2\right|+\left|E_2\right|+\left|\tilde{\bar{e}}_2\right|\right)\\ &+2\rho_c\Pi_M\frac{\Gamma_{32}k_2}{\Gamma_{32}+k_2}\left\|\tilde{E}\right\|\left|\hat{\bar{e}}_3\right|.\end{aligned} \quad (109)$$

Using the same arguments as in step 2, it can be shown from (51), (55), (56), (71), (82), and (109) that there exist positive numbers $\tilde{\alpha}_{2,i}$ and $\tilde{\beta}_{2,i}, i=1,\cdots,M$ satisfying

$$\begin{aligned}&\left\|X_2(t)\right\|\le\tilde{\alpha}_{2,i}\left\|X_2(t_i+\varepsilon)\right\|e^{-\tilde{\beta}_{2,i}(t-t_i-\varepsilon)},\\ &\forall t\in[t_i+\varepsilon,t_{i+1}).\end{aligned} \quad (110)$$

From (94), there exists a positive constant B_{X2} such that

$$\left\|X_2(t)\right\|\le B_{X2}, \forall t\ge 0 \quad (111)$$

Thus, asymptotic stability is achieved since (110) holds for $t\in[t_M+\varepsilon,\infty)$. Hence,

$$e_2=r_2-r_2^*=\tilde{E}_2+\bar{E}_2+E_2 \quad (112)$$

and from (55), (83), and (110)

$$e_2\to 0, r_2\to r_2^* \quad (113)$$

as $t\to\infty$. By (10) and (12),

$$G_k(t)\to G_k^C \quad (114)$$

as $t\to\infty$. This completes the stability analysis of Σ_1.

4. Simulations

In order to analyze the performance of the proposed method, computer simulations are carried out. In the simulations, the wavelength of the pump laser is 980nm. As for signals, two channel signals with 1552.4 nm and 1557.9 nm wavelengths are applied to the system. The signal power of each channel is 0.316mW. In the simulations, the desired channel 1 signal gain is set to 6.6897. The other EDFA system parameters in (8), (9), and (10) are given as follows.

$$\begin{aligned}&\Gamma_{21}=95.2381, \Gamma_{32}=9.5238\times10^{-5},\\ &A_1=5.0750, B_1=6.4998\times10^{-16},\\ &A_2=4.3750, B_2=5.9624\times10^{-16},\\ &A_p=8.9950, B_p=4.6900\times10^{-16}.\end{aligned} \quad (115)$$

Since the gain control is desired to be achieved within a microsecond, the controller gains k_1 and k_2 are chosen as follows

$$k_1=1.05\times10^8, k_2=1.9048\times10^7 \quad (116)$$

so that the natural frequency of the resultant second-order closed loop system may become $\omega_n=10^7(rad/\sec)$. Thus, k_3 is set to be 20.9999 from (36). Next, observer gains L_1 and L_2 are designed such that the bandwidth of the observer system is almost three times larger than that of state feedback control system. Observer gains L_1 and L_2 are given by

$$L_1=1.3630\times10^{24}, L_2=9.0845\times10^{22} \quad (117)$$

Finally, the channel add/drop estimator gain A_D in (25) or (26) is selected to be $A_D=1.5\times10^8$ in order for the channel add/drop estimation to be fast enough compared with other controller and observer. In this case, A_D = 1.5×10^8 is chosen such that the bandwidth is 5 times larger than that in state observer.

Firstly, we show that the performance of the proposed controller based on the three-level model is superior to that of the controller designed based on a simplified two-level model. In order to show this, we consider the following simple error state feedback controller including the same channel add/drop estimator.

$$P_p^{in}(t)=-\frac{1}{\left(1-e^{G_P(t)}\right)}\left[k_c e_2(t)-\hat{c}(t)-\Gamma_{21}r_2^*\right] \quad (118)$$

As in the selection of the gains of the proposed controller, the gain k_C in (114) is also chosen as large as possible so that the gain control of the resultant first order control system can be achieved within a microsecond. For

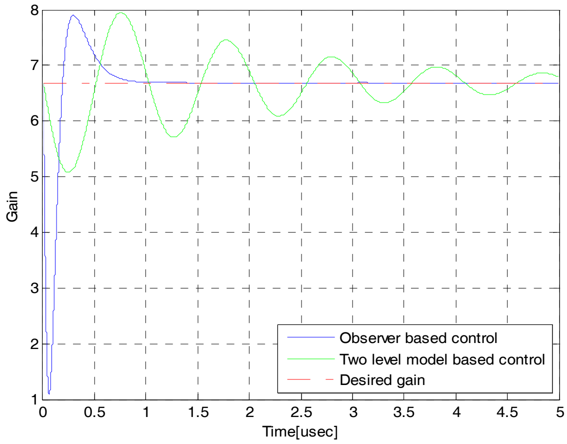


Fig. 2. Comparison of gain control performance

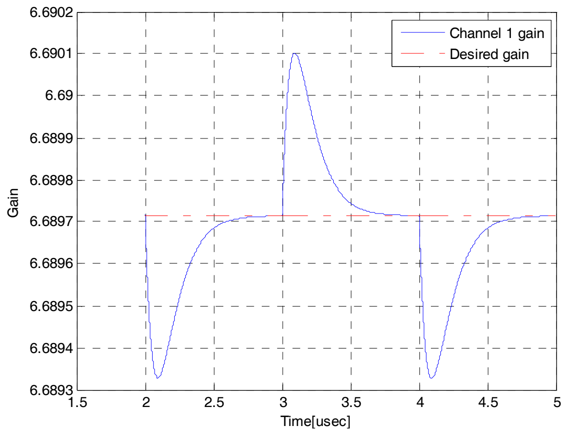


Fig. 3. Gain control performance over channel add/drops

example,

$$k_c = 1.0 \times 10^8 \tag{119}$$

Fig. 2 shows the graphs of the controlled gain of channel 1 signal when channel add/drop occurs at every microsecond as in Fig. 4. As expected, the proposed observer based controller designed based on the three-level model shows faster settling performance. The control based on the two-level model shows oscillation and longer settling performance because it considers the model simplified by ignoring the level three state.

The proposed controller guarantees the desired performance with 0.8 usec settling time, but the simplified control cannot guarantee the desired settling performance with a settling time longer than 1 usec. Fig. 3 is an enlarged version of Fig. 2 to show the gain control results and the influence on the gain due to channel add/drops was effectively compensated for within 1 usec as expected. The channel add/drop estimation performance is shown in Fig. 4. The channel add/drop estimation should be fastest compared with gain control and state observation and Fig. 4 shows that the estimation is done abruptly. In more details, Fig. 5 and Fig. 6 show the channel add/drop

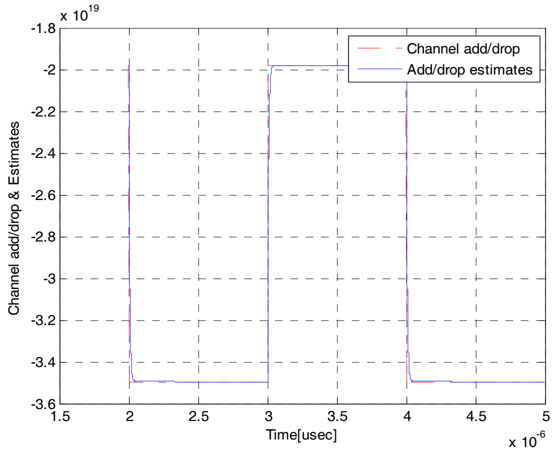
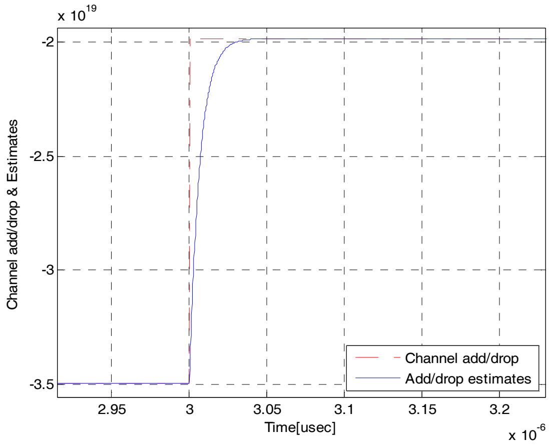
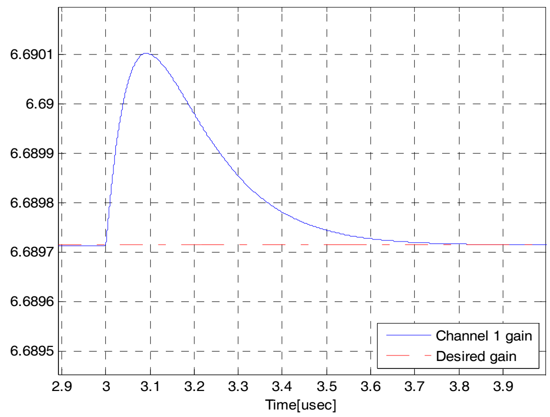


Fig. 4. Channel add/drop estimation



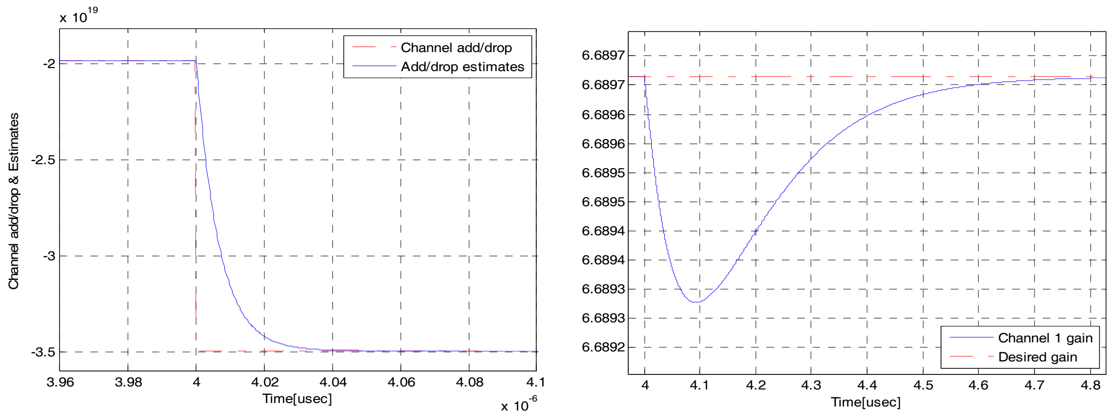
(a) Channel drop and estimation



(b) Gain control

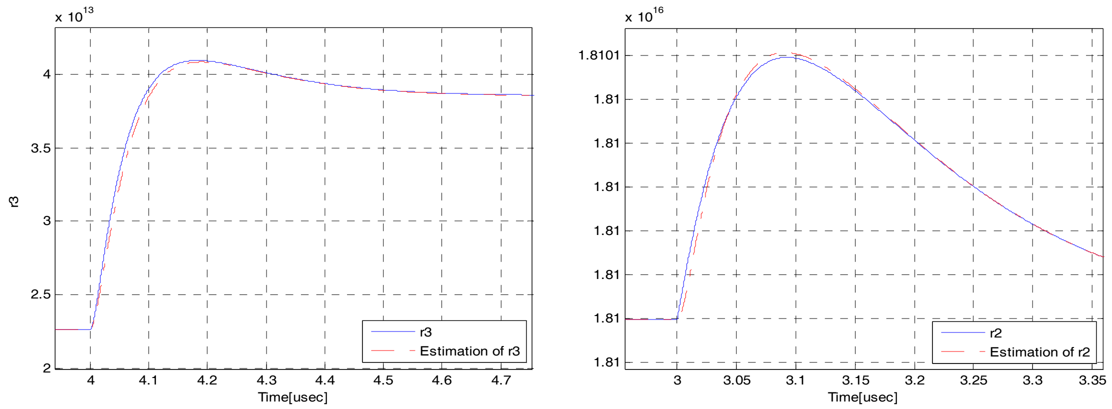
Fig. 5. Channel drop case

estimation results in case of channel drop and channel add. In both cases, the channel add/drop estimation is achieved in 0.03 microsecond and the channel gain is stabilized within 1 microsecond. Fig. 7 shows the results of state estimations. As we intended, the state estimation is done in 0.15 microsecond which is 5 times larger than channel add/drop estimation.



(a) Channel Add and Estimation (b) Gain Control

Fig. 6. Channel drop case.



(a) r_3 and $\hat{r}_3$ (b) r_2 and $\hat{r}_2$

Fig. 7. State estimation

5. Conclusion

In this paper, a systematic design methodology of an EDFA gain controller has been proposed based on singular perturbation and observer technique. The three-level EDFA model has been fully considered without any simplification, and time scaling design approach based on singular perturbation technique has been applied.

Theoretical stability analysis has been carried out thoroughly. Through computer simulation, it is shown that the performance of the proposed EDFA controller is superior to that of the controller designed based on the simplified two level model. The computer simulation also shows that the well-known disturbance observer technique plays an effective role in guaranteeing the desired performance when channel add/drops occur.

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