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STRONG CONVERGENCE IN NOOR-TYPE ITERATIVE SCHEMES IN CONVEX CONE METRIC SPACES

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Abstract. The author considers a Noor-type iterative scheme to approximate common fixed points of an infinite family of uniformly quasi-sup(f_n)-Lipschitzian mappings and an infinite family of g_n-expansive mappings in convex cone metric spaces. His results generalize, improve and unify some corresponding results in convex metric spaces [1, 3, 9, 16, 18, 19] and convex cone metric spaces [8].

1. Introduction

Recently, a class of three-step approximation schemes, which includes Mann and Ishikawa iterative schemes for solving general variational inequalities and related problems in Hilbert spaces, was considered by Noor [11]. And then, three-step methods (named as Noor methods by some authors) for solving various classes of variational inequalities and related problems were extensively studied by the same author in [12]. Since then Noor iteration schemes have been applied to study strong and weak convergences of nonexpansive mappings [2, 10, 13, 14, 16, 20, 21]. In 2002, Xu and Noor [20] considered a three-step iterative scheme with fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces. In 2007, Noor and Huang [13] analyzed three-step iteration methods for finding the common element of the set of fixed points of nonexpansive mappings and studied the convergence criteria for three-step iterative methods. In 2007, Nammanee and Suanti [8] considered the weak and strong convergences for asymptotically nonexpansive mappings for the modified Noor iteration schemes with errors in a uniformly convex Banach spaces. In 2008, Khan et al. [7] generalized the Noor-type iterative process considered in [20] to the case of a finite family of mappings.

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On the other hand, there have been many researches [1, 3, 5, 6, 9, 16-19] on iterative schemes for various kinds of nonexpansive mappings in convex metric spaces with convex structure [15] in the usual metric spaces. In 2010, Khan and Ahmed [6] introduced a generalized iterative scheme due to Khan et al. [7] in convex metric spaces and established a strong convergence to a unique common fixed point of a finite family of asymptotically quasi-nonexpansive mappings under the scheme. Very recently, Tian and Yang [17] gave some sufficient and necessary conditions for a new Noor-type iteration with errors to approximate a common fixed point for a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces.

A cone metric [4] in Banach spaces, which is a cone-version of the usual metric in $\mathbb{R}$ is very applicable in applied mathematics including nonlinear analysis by joining it with convex structures. Very recently, Lee [8] extended an Ishikawa type iterative scheme with errors to approximate a common fixed point of two sequences of uniformly quasi-Lipschitzian mappings on convex cone metric spaces.

Inspired by the works mentioned above, the author recalls some generalized nonexpansive mappings on cone metric spaces and gives some sufficient and necessary conditions for a Noor-type iteration to approximate a common fixed point of an infinite family of uniformly quasi-sup(f_n)-Lipschitzian mappings and an infinite family of g_n-expansive mappings in convex cone metric spaces. His results generalize and improve many corresponding results in convex metric spaces [1, 3, 9, 16, 18, 19] and convex cone metric spaces [8].

2. Preliminaries

Throughout this paper, E is a normed vector space with a normal solid cone P.

A nonempty subset P of E is called a cone if P is closed, $P \neq \{\theta\}$, for $a, b \in \mathbb{R}^+ = [0, \infty)$ and $x, y \in P$, $ax + by \in P$ and $P \cap \{-P\} = \{\theta\}$. We define a partial ordering $\preceq$ in E as $x \preceq y$ if $y - x \in P$. $x << y$ indicates that $y - x \in intP$ and $x \prec y$ means that $x \preceq y$ but $x \neq y$. A cone P is said to be solid if its interior $intP$ is nonempty. A cone P is said to be normal if there exists a positive number t such that for $x, y \in P$, $0 \preceq x \preceq y$ implies $\|x\| \leq t\|y\|$. The least positive number t is called the normal constant of P.

Let X be a nonempty set. A mapping $d : X \times X \to (E, P)$ is called a cone metric if (i) for $x, y \in X$, $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ iff $x = y$, (ii) for $x, y \in X$, $d(x, y) = d(y, x)$ and (iii) for $x, y, z \in X$, $d(x, y) \preceq d(x, z) + d(z, y)$. A nonempty set X with a cone metric $d : X \times X \to (E, P)$ is called a cone metric space denoted by

(X,d), where P is a solid normal cone.

A sequence $\{x_n\}$ in a cone metric space (X,d) is said to converge to $x \in (X,d)$ and denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ (as $n \to \infty$) if for any $c \in intP$, there exists a natural number N such that for all $n > N$, $c - d(x_n, x) \in intP$. A sequence $\{x_n\}$ in (X,d) is called a Cauchy sequence if for any $c \in intP$, there exists a natural number N such that for all $n, m > N$, $c - d(x_n, x_m) \in intP$. A cone metric space (X,d) is said to be complete if every Cauchy sequence converges.

Lemma 2.1 ([4]). *Let $\{x_n\}$ be a sequence in a cone metric space (X,d) and P be a normal cone with a normal constant t. Then*

(i) *$\{x_n\}$ converges to x in X if and only if $d(x_n, x) \to \theta$ (as $n \to \infty$) in E.*
(ii) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ (as $n, m \to \infty$) in E.*

We recall some generalized nonexpansive mappings and convex structures on cone metric spaces.

Definition 2.1. Let T be a self-mapping on a cone metric space (X,d) and $f : X \to (0,\infty)$ a function which is bounded above.

(i) T is *f-expansive*, if

$$d(Tx,Ty) \preceq \sup_{z\in X} f(z) \cdot d(x,y) \quad \text{for} \quad x,y \in X.$$

In particular, T is said to be *nonexpansive*, if $\sup_{z\in X} f(z) \equiv 1$, and T is said to be *contractive*, if $0 < \sup_{z\in X} f(z) < 1$.

(ii) T is *asymptotically f-expansive*, if there exists a sequence $\langle x_n\rangle_{n=0}^{\infty}$ in X such that $\lim_{n\to\infty} f(x_n) = 1$ satisfying

$$d(T^n x, T^n y) \preceq f(x_n) d(x,y) \quad \text{for} \quad x,y \in X,$$

in particular, T is *asymptotically nonexpansive*, if there exists a sequence $\langle k_n\rangle_{n=0}^{\infty}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ satisfying

$$d(T^n x, T^n y) \preceq k_n d(x,y) \quad \text{for} \quad x,y \in X \quad (n \in \mathbb{N} \cup \{0\}).$$

(iii) T is *asymptotically quasi-f-expansive*, if there exists a sequence $\langle x_n\rangle_{n=0}^{\infty}$ in X such that $\lim_{n\to\infty} f(x_n) = 1$ satisfying

$$d(T^n x, p) \preceq f(x_n) d(x,p) \quad \text{for} \quad x \in X \quad \text{and} \quad p \in F(T) \text{ a set of fixed points of } T,$$

in particular, T is *asymptotically quasi-nonexpansive*, if there exists a sequence $\langle k_n\rangle_{n=0}^{\infty}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ satisfying

$$d(T^n x, p) \preceq k_n d(x,p) \quad \text{for} \quad x\in X \quad \text{and} \quad p\in F(T) \quad (n\in\mathbb{N}\cup\{0\}).$$

(iv) T is *uniformly quasi*-sup(f)-*Lipschitzian*, if

$$d(T^n x, p) \preceq \sup_{x\in X} f(x) d(x,p) \quad \text{for} \quad x\in X \quad \text{and} \quad p\in F(T),$$

in particular, T is *uniformly quasi-L-Lipschitzian*, if there exists a constant $L>0$ such that

$$d(T^n x, p) \preceq L\cdot d(x,p) \quad \text{for} \quad x\in X \quad \text{and} \quad p\in F(T) \quad (n\in\mathbb{N}\cup\{0\}).$$

Definition 2.2. Let (X,d) be a cone metric space. A mapping $W : X^3\times I^3 \to X$ is a *convex structure* on X if $d(W(x,y,z,a_n,b_n,c_n),u) \preceq a_n\cdot d(x,u)+b_n\cdot d(y,u)+c_n\cdot d(z,u)$ for real sequences $\{a_n\},\{b_n\}$ and $\{c_n\}$ in $I=[0,1]$ satisfying $a_n+b_n+c_n=1$ $(n\in\mathbb{N})$ and x,y,z and $u\in X$. A cone metric space (X,d) with a convex structure W is called a *convex cone metric space* and denoted as (X,d,W). A nonempty subset C of a convex cone metric space (X,d,W) is said to be *convex* if $W(x,y,z,a,b,c)\in C$ for all $x,y,z\in C$ and $a,b,c\in I$.

In this paper, we consider the following Noor-type three-step iteration in convex cone metric spaces ;

For given $x_0\in C$, define a sequence $\{x_n\}$ in C as follows;

$$\begin{cases} x_{n+1} = W(S_n x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n = W(S_n x_n, T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n = W(S_n x_n, T_n^n x_n, w_n; d_n, e_n, l_n), \end{cases} \tag{2.1}$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are any sequences in X, C is a nonempty convex subset of (X,d,W), $T_n : C\to C$ is a uniformly quasi-sup(f_n)-Lipschitzian mapping, $S_n : C\to C$ is a g_n-expansive mapping and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{l_n\}$ are sequences in I such that $a_n+b_n+c_n=1$, $\alpha_n+\beta_n+\geq_n=1$ and $d_n+e_n+l_n=1$ $(n\in\mathbb{N})$.

Remark 2.1. (i) We have the following iteration by putting $S_n x_n = x_n$ in $y_n = W(S_n x_n, T_n^n x_n, q : a_n, b_n, c_n)$ of (2.1)

$$\begin{cases} x_{n+1} = W(S_n x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n = W(x_n, T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n = W(x_n, T_n^n x_n, w_n; d_n, e_n, l_n). \end{cases} \tag{2.2}$$

(ii) By putting $S_nx_n = x_n$, we obtain the following iteration from (2.2) ;

$$\begin{cases} x_{n+1} = W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n = W(x_n, T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n = W(x_n, T_n^n x_n, w_n; d_n, e_n, l_n). \end{cases} \tag{2.3}$$

which generalizes many kinds of Ishikawa-type iterations.

(iii) The following Ishikawa-type iteration is a special case of (2.1);

$$\begin{cases} x_{n+1} = W(S_nx_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n = W(S_nx_n, T_n^n x_n, v_n; a_n, b_n, c_n). \end{cases} \tag{2.4}$$

which was considered in [1].

3. Main Results

Throughout this paper, f_n, $g_n : X \to (0, \infty)$ are functions, which are bounded above, and whose least upper bounds are M_n and N_n, respectively ($n \in \mathbb{N}$).

Lemma 3.1 ([5, 9])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be nonnegative real sequences satisfying the following conditions;*

(i) $a_{n+1} \le (1 + b_n)a_n + c_n$ $(n \in \mathbb{N})$,

(ii) $\sum_{n=1}^{\infty} b_n$ *and* $\sum_{n=1}^{\infty} c_n$ *are finite,*

then the following hold;

(1) $\lim_n a_n$ *exists.*

(2) *If* $\underline{\lim}_n a_n = 0$, *then* $\lim_n a_n = 0$.

The following lemma considered in convex cone metric space (X, d, W) is a result on the properties of an iteration sequence $\{x_n\}$ defined as (2.1) for an infinite family $\{T_n\}$ of uniformly quasi-sup(f_n)-Lipschitzian mappings and an infinite family $\{S_n\}$ of g_n-expansive mappings.

Lemma 3.2. *Let $d : X \times X \to (E, P)$ be a cone metric, where P is a solid normal cone with the normal constant t. Let C be a nonempty convex subset of a convex cone metric space (X, d, W). Let $T_n : C \to C$ be uniformly quasi-*sup(f_n)*-Lipschitzian mappings and $S_n : C \to C$ be g_n-expansive mappings ($n \in \mathbb{N}$) with a nonempty set*

$$D := \left(\bigcap_{n=1}^{\infty} F(T_n) \right) \bigcap \left(\bigcap_{n=1}^{\infty} F(S_n) \right).$$

Let $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *be bounded sequences in* C *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ *and* $\{l_n\}$ *be sequences in* I *such that* $\alpha_n+\beta_n+\gamma_n = a_n + b_n + c_n = d_n + e_n + l_n = 1$ $(n \in \mathbb{N})$.

Assume that the following conditions hold;

(i) $f_n > 1$ *and* $g_n > 1$ $(n \in \mathbb{N})$,

(ii) $M = \sup_{n\in\mathbb{N}} M_n$ *and* $N = \sup_{n\in\mathbb{N}} N_n$ *are finite,*

(iii) $L = (1 + M + M^2) \cdot L_0$, *where*

$$L_0 = \max\big\{ \sup_{\substack{p\in D\\ n\geq 1}} d(u_n, p), \sup_{\substack{p\in D\\ n\geq 1}} d(v_n, p), \sup_{\substack{p\in D\\ n\geq 1}} d(w_n, p)\big\}$$

is finite.

Let $\{x_n\}$ *be the iteration defined as* (2.1), *then the following holds.*

$$\begin{aligned} d(x_{n+1}, p) &\preceq (1 + \eta_n(M + M^2 + M^3)) \cdot N \cdot d(x_n, p) + L \cdot \delta_n, \\ &\preceq N \cdot e^{\eta_n(M+M^2+M^3)} \cdot d(x_n, p) + L \cdot \delta_n \end{aligned} \tag{1}$$

for $p \in D$, *where* $\eta_n = \alpha_n + \beta_n$ *and* $\delta_n = \beta_n + \gamma_n$ $(n \in \mathbb{N})$.

(2) *there exists a constant* $K > 0$ *such that*

$$d(x_{n+m}, p) \preceq K \cdot d(x_n, p) + K \cdot L \cdot \sum_{k=n}^{n+m-1} \delta_k \text{ for } p \in D.$$

Proof. For any $p \in D$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W_q(S_n x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), p) \\ &\preceq \alpha_n \cdot d(S_n x_n, p) + \beta_n \cdot d(T_n^n y_n, p) + \gamma_n \cdot d(u_n, p) \\ &\preceq \alpha_n \cdot g_n(x_n) \cdot d(x_n, p) + \beta_n \cdot M_n \cdot d(y_n, p) + \gamma_n \cdot d(u_n, p) \\ &\preceq \alpha_n \cdot N_n \cdot d(x_n, p) + \beta_n \cdot M_n \cdot d(y_n, p) + \gamma_n \cdot d(u_n, p), \end{aligned} \tag{3.1}$$

$$\begin{aligned} d(y_n, p) &= d(W_q(S_n x_n, T_n^n x_n, v_n; a_n, b_n, c_n), p) \\ &\preceq a_n \cdot d(S_n x_n, p) + b_n \cdot d(T_n^n z_n, p) + c_n \cdot d(v_n, p) \\ &\preceq a_n \cdot g_n(x_n) \cdot d(x_n, p) + b_n \cdot M_n \cdot d(x_n, p) + c_n \cdot d(v_n, p) \\ &\preceq a_n \cdot N_n \cdot d(x_n, p) + b_n \cdot M_n \cdot d(z_n, p) + c_n \cdot d(v_n, p). \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} d(z_n, p) &= d(W_q(S_n x_n, T_n^n x_n, w_n; d_n, e_n, l_n), p) \\ &\preceq d_n \cdot d(S_n x_n, p) + e_n \cdot d(T_n^n x_n, p) + l_n \cdot d(w_n, p) \\ &\preceq d_n \cdot g_n(x_n) \cdot d(x_n, p) + e_n \cdot M_n \cdot d(x_n, p) + l_n \cdot d(w_n, p) \\ &\preceq d_n \cdot N_n \cdot d(x_n, p) + e_n \cdot M_n \cdot d(x_n, p) + l_n \cdot d(w_n, p). \end{aligned} \tag{3.3}$$

From (3.1) to (3.3), by the fact that $1 + x \le e^x$ for $x \ge 0$, we have

$$
\begin{aligned}
(1)\ \ d(x_{n+1},p) &\preceq \alpha_n \cdot N_n \cdot d(x_n,p) + \beta_n \cdot M_n \cdot d(y_n,p) + \gamma_n \cdot d(u_n,p) \\
&\preceq \alpha_n \cdot N_n \cdot d(x_n,p) + \beta_n \cdot M_n \cdot \{a_n \cdot N_n \cdot d(x_n,p) \\
&\quad + b_n \cdot M_n \cdot d(z_n,p) + c_n \cdot d(v_n,p)\} + \gamma_n \cdot d(u_n,p) \\
&\preceq \alpha_n \cdot N_n \cdot d(x_n,p) + \beta_n \cdot M_n \cdot \big[a_n \cdot N_n \cdot d(x_n,p) \\
&\quad + b_n \cdot M_n \cdot \{d_n \cdot N_n \cdot d(x_n,p) + e_n \cdot M_n \cdot d(x_n,p) \\
&\quad + l_n \cdot d(w_n,p)\} + c_n \cdot d(v_n,p)\big] + \gamma_n \cdot d(w_n,p) \\
&= (\alpha_n \cdot N_n + \beta_n \cdot a_n \cdot M_n \cdot N_n + \beta_n \cdot b_n \cdot d_n \cdot M_n^2 \cdot N_n \\
&\quad + \beta_n \cdot b_n \cdot e_n \cdot M_n^3) \cdot d(x_n,p) + \beta_n \cdot b_n \cdot l_n \cdot M_n^2 \cdot d(w_n,p) \\
&\quad + \beta_n \cdot c_n \cdot M_n \cdot d(v_n,p) + \gamma_n \cdot d(w_n,p) \qquad (3.4) \\
&\preceq \big[(\alpha_n + \beta_n) + (\alpha_n + \beta_n)(M_n + M_n^2 + M_n^3)\big] \cdot N_n \cdot d(x_n,p) \\
&\quad + (\beta_n + \gamma_n)(1 + M_n + M_n^2) \cdot L_0 \\
&\preceq \big[1 + \eta_n(M + M^2 + M^3)\big] \cdot N \cdot d(x_n,p) + L \cdot \delta_n \\
&\preceq N \cdot e^{\eta_n(M+M^2+M^3)} \cdot d(x_n,p) + L \cdot \delta_n.
\end{aligned}
$$

Moreover, we obtain the following result (2) from the result (1).

$$
\begin{aligned}
(2)\ \ d(x_{n+m},p) &\preceq N \cdot e^{\eta_{n+m-1}\cdot(M+M^2+M^3)} \cdot d(x_{n+m-1},p) + L \cdot \delta_{n+m-1} \\
&\preceq N \cdot e^{\eta_{n+m-1}\cdot(M+M^2+M^3)} \cdot \big(N \cdot e^{\eta_{n+m-2}\cdot(M+M^2+M^3)} \\
&\quad \cdot d(x_{n+m-2},p) + L \cdot \delta_{n+m-2}\big) + L \cdot \delta_{n+m-1} \\
&= N^2 \cdot e^{(\eta_{n+m-1}+\eta_{n+m-2})\cdot(M+M^2+M^3)} \cdot d(x_{n+m-2},p) \\
&\quad + N \cdot e^{\eta_{n+m-1}\cdot(M+M^2+M^3)} \cdot L \cdot \delta_{n+m-2} + L \cdot \delta_{n+m-1} \qquad (3.5) \\
&\preceq N^2 \cdot e^{(\eta_{n+m-1}+\eta_{n+m-2})\cdot(M+M^2+M^3)} \cdot \big(N \cdot e^{\eta_{n+m-3}\cdot(M+M^2+M^3)} \\
&\quad \cdot d(x_{n+m-3},p) + L \cdot \delta_{n+m-3}\big) \\
&\quad + N \cdot e^{\eta_{n+m-1}\cdot(M+M^2+M^3)} \cdot L \cdot \delta_{n+m-2} + L \cdot \delta_{n+m-1} \\
&= N^3 \cdot e^{(\eta_{n+m-1}+\eta_{n+m-2}+\eta_{n+m-3})\cdot(M+M^2+M^3)} \cdot d(x_{n+m-3},p) \\
&\quad + N^2 \cdot e^{(\eta_{n+m-1}+\eta_{n+m-2})\cdot(M+M^2+M^3)} \cdot L \cdot \delta_{n+m-3} \\
&\quad + N \cdot e^{\eta_{n+m-1}\cdot(M+M^2+M^3)} \cdot L \cdot \delta_{n+m-2} + L \cdot \delta_{n+m-1} \\
&\ \ \vdots
\end{aligned}
$$

$$\begin{aligned}
&\preceq N^m \cdot e^{\sum_{k=n}^{n+m-1} \eta_k \cdot (M+M^2+M^3)} \cdot d(x_n, p) \\
&\quad + N^{m-1} \cdot e^{\sum_{k=n+1}^{n+m-1} \eta_k \cdot (M+M^2+M^3)} \cdot L \cdot \delta_n \\
&\quad \vdots \\
&\quad + N^3 \cdot e^{\sum_{k=n+m-3}^{n+m-1} \eta_k \cdot (M+M^2+M^3)} \cdot L \cdot \delta_{n+m-4} \\
&\quad + N^2 \cdot e^{(\eta_{n+m-1}+\eta_{n+m-2}) \cdot (M+M^2+M^3)} \cdot L \cdot \delta_{n+m-3} \\
&\quad + N \cdot e^{\eta_{n+m-1} \cdot (M+M^2+M^3)} \cdot L \cdot \delta_{n+m-2} \\
&\quad + L \cdot \delta_{n+m-1} \\
&\preceq K \cdot d(x_n, p) + K \cdot L \cdot \sum_{k=n}^{n+m-1} \delta_k,
\end{aligned}$$

where $K = N^m \cdot e^{\sum_{k=n}^{n+m-1} \eta_k \cdot (M+M^2+M^3)}$. □

Remark 3.1. If $f_n \geq 1$ and $g_n \leq 1$ $(n \in \mathbb{N})$, then

$$N_n = \sup_{x \in X} g_n(x) = 1 \text{ and } N = \sup_{n \in \mathbb{N}} N_n(x) = 1.$$

Hence from (3.4)

$$d(x_{n+1}, p) \preceq e^{\eta_n \cdot (M+M^2+M^3)} \cdot d(x_n, p) + L \cdot \delta_n$$

and from (3.5)

$$d(x_{n+m}, p) \preceq K_1 \cdot d(x_n, p) + K_1 \cdot L \cdot \sum_{k=n}^{n+m-1} \delta_k,$$

where $K_1 = e^{\sum_{k=n}^{n+m-1} \eta_k \cdot (M+M^2+M^3)}$.

Remark 3.2. If $f_n \leq 1$ and $g_n \geq 1$ $(n \in \mathbb{N})$, then

$$M_n = \sup_{x \in X} f_n(x) = 1 \text{ and } M = \sup_{n} M_n = 1.$$

Hence from (3.4)

$$d(x_{n+1}, p) \preceq N \cdot e^{\eta_n} \cdot d(x_n, p) + L_0 \cdot \delta_n,$$

and from (3.5)

$$d(x_{n+m}, p) \preceq K_2 \cdot d(x_n, p) + K_2 \cdot L_0 \cdot \sum_{k=n}^{n+m-1} \delta_k,$$

where $K_2 = N^m \cdot e^{\sum_{k=n}^{n+m-1} \eta_k}$.

Remark 3.3. If $f_n \leq 1$ and $g_n \leq 1$, then from (3.4) and (3.5), respectively,

$$d(x_{n+1}, p) \preceq e^{\eta_n} \cdot d(x_n, p) + L_0 \cdot \delta_n,$$

and

$$d(x_{n+m}, p) \preceq e^{\sum_{k=n}^{n+m-1} \eta_k} \cdot \left(d(x_n, p) + L_0 \cdot \sum_{k=n}^{n+m-1} \delta_k \right).$$

Remark 3.4. Lemma 2 in [1] dealt with the Ishikawa-type iteration (2.4) for an infinite family of uniformly quasi-L_n-Lipschitzian mappings T_n with $L = \max\{1, \sup_{n\geq 1} L_n\} < \infty$ and an infinite family of nonexpansive mappings S_n, i.e., $L \geq 1$ and $g_n = 1$ ($n \in \mathbb{N}$). Hence Lemma 2 in [1] is a corollary of Lemma 3.1.

Now we introduce our main result for the iteration defined as (2.1) with an infinite family $\{T_n\}$ of uniformly quasi-sup(f_n)-Lipschitzian mappings and an infinite family $\{S_n\}$ of g_n-expansive mappings in convex cone metric spaces (X, d, W).

Theorem 3.1. *Let C be a nonempty closed convex subset of a convex cone metric space (X, d, W) with a solid normal cone P with a normal constant t. Let $T_n : C \to C$ be uniformly quasi-*sup(f_n)*-Lipschitzian mappings and $S_n : C \to C$ be g_n-expansive mappings with a nonempty set $D := \left(\bigcap_{n=1}^{\infty} F(T_n) \right) \bigcap \left(\bigcap_{n=1}^{\infty} F(S_n) \right)$. Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{l_n\}$ be real sequences in I such that $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + l_n = 1$ ($n \in \mathbb{N}$).*

Assume that the following conditions hold;

(i) *$f_n > 1$ and $g_n > 1$ ($n \in \mathbb{N}$),*

(ii) *$M = \sup_{n\in\mathbb{N}} M_n$ and $N = \sup_{n\in\mathbb{N}} N_n$ are finite,*

(iii) *$L = (1 + M + M^2) \cdot L_0$, where*

$$L_0 = \max\left\{ \sup_{\substack{p\in D \\ n\geq 1}} d(u_n, p), \sup_{\substack{p\in D \\ n\geq 1}} d(v_n, p), \sup_{\substack{p\in D \\ n\geq 1}} d(w_n, p) \right\}$$

is finite,

(iv) *$\sum_{n=1}^{\infty} (\alpha_n + \beta_n)$ and $\sum_{n=1}^{\infty} (\beta_n + \gamma_n)$ are finite.*

Let $\{x_n\}$ be the iteration defined as (2.1).

We have the following equivalent result;

(1) $\{x_n\}$ *converges to a common fixed point* $p \in D$.

(2) $\varliminf_{n\to\infty} d(x_n, D) = \theta$.

Proof. We only consider the case of $f_n(x) > 1$ and $g_n(x) > 1$ for $x \in X$ $(n \in \mathbb{N})$. Obviously the statement (1) implies the statement (2).

Now we show that the statement (2) implies the statement (1). From (3.4), we have

$$d(x_{n+1}, D) \preceq (1 + \eta_n(M + M^2 + M^3)) \cdot N \cdot d(x_n, D) + L \cdot \delta_n,$$

where $\eta_n = \alpha_n + \beta_n$ and $\delta_n = \beta_n + \gamma_n$ for $n \in \mathbb{N}$.

Thus the normality of P implies that
$\|d(x_{n+1}, D)\| \leq t \cdot (1 + \eta_n(M + M^2 + M^3)) \cdot N \cdot \|d(x_n, D)\| + t \cdot \|L\| \cdot \delta_n$,
where t is the normal constant of P.

By the condition (iv) that $\sum_{n=1}^{\infty} \eta_n$ and $\sum_{n=1}^{\infty} \delta_n$ are finite, $\lim_n \|d(x_n, D)\|$ exists from Lemma 3.1. Since $\varliminf_n \|d(x_n, D)\| = 0$ by the hypothesis, we have $\lim_n \|d(x_n, D)\| = 0$. Now, we show that the sequence $\{x_n\}$ is convergent. For any given $\varepsilon > 0$, take a positive integer N_0 such that

$$\begin{cases} \|d(x_n, D)\| < \dfrac{\varepsilon}{4(t^2K_3 + t)} \\ \sum\limits_{k=N_0}^{\infty} \delta_k < \dfrac{\varepsilon}{2t^2K_3 \cdot \|L\|} \end{cases}$$

where $K_3 = N^n \cdot e^{\sum_{k=N_0}^{N_0+n-1} \eta_k(M+M^2+M^3)}$ for $n \geq N_0$.

By using a property of infimum for $\|d(x_n, D)\|$, we take a positive integer $N_1 \geq N_0$ and $p_0 \in D$ such that

$$\|d(x_{N_1}, p_0)\| < \frac{\varepsilon}{2(t^2K_3 + t)}.$$

Hence for any positive integer $n > N_1$, from (3.5) by the normality of P we have

$$\begin{aligned} \|d(x_{n+N_1}, x_{N_1})\| &\leq t(\|d(x_{N_1+n}, p_0)\| + \|d(x_{N_1}, p_0)\|) \\ &\leq t^2\big(K_4 \cdot \|d(x_{N_1}, p_0)\| + K_4 \cdot \|L\| \cdot \sum_{k=N_1}^{n+N_1-1} \delta_k\big) + t \cdot \|d(x_{N_1}, p_0)\| \\ &\leq (t^2K_4 + t)\|d(x_{N_1}, p_0)\| + t^2 \cdot K_4 \cdot \|L\| \sum_{k=N_1}^{n+N_1-1} \delta_k \end{aligned}$$

$$\begin{aligned}
&\leq (t^2K_4+t)\cdot\frac{\varepsilon}{2(t^2K_3+t)}+t^2\cdot K_4\cdot\|L\|\cdot\frac{\varepsilon}{2t^2K_3\cdot\|L\|}\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\
&=\varepsilon,\ \text{where } K_4=N^n\cdot e^{\sum\limits_{k=N_1}^{N_0+n-1}\eta_k(M+M^2+M^3)}.
\end{aligned}$$

Hence $\lim\limits_{n\to\infty}\|d(x_{n+N_1},x_{N_1})\|=0$, which shows that $\lim\limits_{n\to\infty}d(x_n,x_{N_1})=0$, i.e., $\lim\limits_{n\to\infty}x_n=x_{N_1}$ by Lemma 2.1(i). Since C is closed, $p^*:=x_{N_1}\in C$.

Now, we show that the set D is closed. In fact, let $\{p_n\}$ a sequence in D converging to p in C. Then

$$\begin{aligned}
d(p,T_ip)&\leq d(p,p_n)+d(p_n,T_ip)\\
&\leq d(p,p_n)+d(T_ip_n,T_ip)\\
&\leq d(p,p_n)+M\cdot d(p_n,p)\quad\text{for all } i\in\mathbb{N}.
\end{aligned}\tag{3.6}$$

Letting $n\to\infty$ in (3.6), we have

$$p\in F(T_i)\ \text{for all}\ i\in\mathbb{N}.$$

Similarly,

$$\begin{aligned}
d(p,S_ip)&\leq d(p,p_n)+d(p_n,S_ip)\\
&\leq d(p,p_n)+d(S_ip_n,S_ip)\\
&\leq d(p,p_n)+N\cdot d(p_n,p)\ \text{for all } i\in\mathbb{N}.
\end{aligned}\tag{3.7}$$

Letting also $n\to\infty$ in (3.7), we have

$$p\in F(S_i)\ \text{for all}\ i\in\mathbb{N}.$$

Hence $p\in D$, which implies that D is closed.
Moreover, we have $p^*\in D$. In fact, since $d(p^*,D)=d(\lim\limits_{x\to\infty}x_n,D)=\lim\limits_{x\to\infty}d(x_n,D)=\theta$ by Lemma 2.1 (i), we have $p^*\in D$, which says that $\{x_n\}$ converges to a common fixed point in D. □

Remark 3.5. In Theorem 3.1, the full space need not to be complete.

Remark 3.6. We obtain the same results for iterations defined as (2.2) and (2.3) with an infinite family $\{T_n\}$ of uniformly quasi-sup(f_n)-Lipschitzian mappings and an infinite family $\{S_n\}$ of g_n-expansive mappings in convex metric spaces (X,d,W).

Remark 3.7. The main result of [1] considered the Ishikawa-type iteration (2.4) for an infinite family of uniformly quasi-L_n-Lipschitzian mappings T_n with $L=$

$\max\{1, \sup_{n\geq 1} L_n\} < \infty$ and an infinite family of nonexpansive mappings S_n, i.e., $L \geq 1$ and $g_n = 1$ $(n \in \mathbb{N})$ in convex metric spaces. Hence Theorem 1 in [1] is a corollary of Theorem 3.1.

Remark 3.8. By putting $f_n(x) = 1$ and $g_n(x) = 1$ for $x \in X$ in Theorem 3.1, we have the corresponding results in convex metric spaces and convex cone metric spaces.

Remark 3.9. We obtain the same results as Theorem 3.1 by replacing the uniform quasi-sup(f)-Lipschitzian of T_n $(n \in \mathbb{N})$ with the asymptotical quasi-f-expansiveness of T_n $(n \in \mathbb{N})$.

Remark 3.10. Theorem 3.1 generalizes, improves and unifies the corresponding results in convex metric spaces [1, 3, 9, 16, 18, 19].

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