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A FUNCTIONAL APPROACH TO d-ALGEBRAS

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Abstract. In this paper we discuss a functional approach to obtain a lattice-like structure in d-algebras, and obtain an exact analog of De Morgan law and some other properties.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([8, 9]). BCK-algebras have some connections with other areas: D. Mundici [13] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng [11] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. It is well known that bounded commutative BCK-algebras, D-posets and MV-algebras are logically equivalent each other (see [4, p. 420]). We refer useful textbooks for BCK/BCI-algebra to [4, 6, 7, 12, 17]. J. Neggers and H. S. Kim ([14]) introduced the notion of d-algebras which is a useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs. J. S. Han et al. ([5]) defined a variety of special d-algebras, such as strong d-algebras, (weakly) selective d-algebras and others. The main assertion is that the squared algebra $(X;\square,0)$ of a d-algebra is a d-algebra if and only if the root $(X;*,0)$ of the squared algebra $(X;\square,0)$ is a strong d-algebra. Recently, the present author with H. S. Kim and J. Neggers ([10]) explored properties of the set of d-units of a d-algebra. It was noted that many d-algebras are weakly associative, and the existence of non-weakly associative d/BCK-algebras was demonstrated. Moreover, they discussed the notions of a d-integral domain and a left-injectivity in d/BCK-algebras. We refer to [1, 2, 15, 16] for more information on d-algebras.

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In this paper we discuss a functional approach to obtain a lattice-like structure in d-algebras, and obtain an exact analog of De Morgan law and some other properties.

2. PRELIMINARIES

An (*ordinary*) *d-algebra* ([14]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(D1) $x * x = 0$,

(D2) $0 * x = 0$,

(D3) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK-algebra is a d-algebra X satisfying the following additional axioms:

(D4) $(x * y) * (x * z)) * (z * y) = 0$,

(D5) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Example 2.1 ([14])**.** Consider the real numbers $\mathbf{R}$, and suppose that $(\mathbf{R}; *, \mathbf{e})$ has the multiplication

$$x * y = (x - y)(x - e) + e$$

Then $x*x = e; e*x = e; x*y = y*x = e$ yields $(x-y)(x-e) = 0$, $(y-x)(y-e) = e$ and $x = y$ or $x = e = y$, i.e., $x = y$, i.e., $(\mathbf{R}; *, e)$ is a d-algebra.

3. A FUNCTIONAL APPROACH TO d-ALGEBRAS

Let $(X, *, 0)$ be a d-algebra. A map $\varphi : X \to X$ is said to be *order reversing* if $x * y = 0$ then $\varphi(y) * \varphi(x) = 0$ for all $x, y \in X$; *self-inverse* if $\varphi(\varphi(x)) = x$ for all $x \in X$; an *anti-homomorphism* if $\varphi(x * y) = \varphi(y) * \varphi(x) = 0$ for all $x, y \in X$; a *homomorphims* if $\varphi(x * y) = \varphi(x) * \varphi(y)$ for all $x, y \in X$.

Example 3.1. Consider $X := \{0, a, 1\}$ with

$*$	0	a	1
0	0	0	0
a	a	0	0
1	1	a	0

Then $(X; *, 0)$ is a d-algebra. If we define a map $\varphi : X \to X$ by $\varphi(0) = 1, \varphi(a) = a$ and $\varphi(1) = 0$, then it is easy to see that φ is both self-inverse and order reversing, but it is not an anti-homomorphism, since $\varphi(a*1) = \varphi(0) = 1$ and $\varphi(1)*\varphi(a) = 0*a = 0$.

Moreover, it is not a homomorphism, since $\varphi(0 * a) = \varphi(0) = 1 \neq a = 1 * a = \varphi(0) * \varphi(a)$.

Proposition 3.2. *Let $(X, *, 0)$ be a d-algebra. If $\varphi : X \to X$ is a (anti-) homomorphism, then $\varphi(0) = 0$.*

Proof. Since X is a d-algebra, by $(D1)$, we obtain $\varphi(0) = \varphi(x * x) = \varphi(x) * \varphi(x) = 0$. □

Proposition 3.3. *If $(X, *, 0)$ is a d-algebra, then every anti- homomorphism is order reversing.*

Proof. Let $\varphi : X \to X$ be an anti-homomorphism. If we assume that $x * y = 0$, then $\varphi(y) * \varphi(x) = \varphi(x * y) = \varphi(0) = 0$ by Proposition 3.2. This proves the proposition. □

Remark. The converse of Proposition 3.3 need not be true in general. In Example 3.1, the mapping φ is an order reversing, but not an anti-homomorphism.

Let $(X, *, 0)$ be a d-algebra and let $\varphi : X \to X$ be a map. We denote by $1 := \varphi(0)$.

Proposition 3.4. *Let $(X, *, 0)$ be a d-algebra and let $\varphi : X \to X$ be both order reversing and self-inverse. Then $(X, *, 0)$ is bounded.*

Proof. Given $x \in X$, we have

$$\begin{aligned} x * 1 &= x * \varphi(0) && [1 = \varphi(0)] \\ &= \varphi(\varphi(x)) * \varphi(0) && [\varphi\text{: self-inverse}] \\ &= 0 && [\varphi\text{: order reversing}] \end{aligned}$$

□

Let $(X, *, 0)$ be a d-algebra. We define a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$. Note that the relation $\leq$ need not be a partial order on X. We define a relation "$\wedge$ on X by $x \wedge y := x * (x * y))$ for all $x, y \in X$.

Proposition 3.5. *Let $(X, *, 0)$ be a d-algebra. If $\varphi : X \to X$ is self-inverse, then $\varphi(1) = 0$.*

Proof. It follows from φ is self-inverse that $0 = \varphi(\varphi(0)) = \varphi(1)$. □

Theorem 3.6. *Let $(X, *, 0)$ be a d-algebra and let $\varphi : X \to X$ be a self-inverse* *map. If we define* $x \vee y := \varphi[\varphi(y) \wedge \varphi(x)]$*, then*

$$\varphi(x \wedge y) = \varphi(y) \vee \varphi(x)$$

for all $x, y \in X$.

Proof. Given $x, y \in X$, we have

$$\begin{aligned} \varphi(x \wedge y) &= \varphi[\varphi(\varphi(x)) \wedge \varphi(\varphi(y))] && [\varphi\text{: self-inverse}] \\ &= \varphi[\varphi(a)) \wedge \varphi(b)] && [a = \varphi(x), b = \varphi(y)] \\ &= b \vee a \\ &= \varphi(y) \vee \varphi(x) \end{aligned}$$

□

Theorem 3.6 shows that the first De Morgan's law implies the analog of the second De Morgan's law and conversely, since $x \vee y \neq y \vee x$ in general. Moreover, it follows that $x \wedge y = \varphi(\varphi(x \wedge y)) = \varphi[\varphi(y) \vee \varphi(x)]$ for all $x, y \in X$.

Theorem 3.7. *Let* $(X, *, 0)$ *be a d-algebra with*

(1) $$x * 0 = x$$

for all $x \in X$*. If* $\varphi : X \to X$ *is a self-inverse map, then* $x \vee x = x$ *and* $x \wedge x = x$ *for all* $x \in X$.

Proof. (i). Given $x \in X$, we have

$$\begin{aligned} x \vee x &= \varphi[\varphi(x) \wedge \varphi(x)] && [\text{Theorem 3.6}] \\ &= \varphi[\varphi(x) * (\varphi(x) * \varphi(x)] \\ &= \varphi(\varphi(x) * 0) && [(\text{D1})] \\ &= \varphi(\varphi(x)) && [(1)] \\ &= x && [\varphi\text{: self-inverse}] \end{aligned}$$

(ii). $x \wedge x = x * (x * x) = x * 0 = x$. □

Proposition 3.8. *Let* $(X, *, 0)$ *be a d-algebra with*

(2) $$(x * y) * z = (x * z) * y$$

for all $x, y, z \in X$*. Then* $x \wedge y \leq x$ *and* $x \wedge y \leq y$ *for all* $x, y \in X$.

Proof. (i). Given $x, y \in X$, by applying (2), we obtain

$$
\begin{aligned}
(x \wedge y) * a &= (x * (x * y)) * a \\
&= (x * x) * (x * y) \\
&= 0 * (x * y) \\
&= 0
\end{aligned}
$$

(ii). Given $x, y \in X$, we have $(x \wedge y) * y = (x * (x * y)) * y = (x * y) * (x * y) = 0$. □

Theorem 3.9. *Let $(X, *, 0)$ be a d-algebra with the condition (2). If $\varphi : X \to X$ is a self-inverse anti-homomorphism, then $x * (x \vee y) = 1$ and $y * (x \vee y) = 1$ for all $x, y \in X$.*

Proof. (i). Since $\varphi : X \to X$ is a self-inverse anti-homomorphism, for all $x, y \in X$, we obtain

$$
\begin{aligned}
x * (x \vee y) &= x * \varphi(\varphi(y) \wedge \varphi(x)) \\
&= x * \varphi[\varphi(y) * (\varphi(y) * \varphi(x))] \\
&= \varphi(\varphi(x)) * \varphi[\varphi(y) * (\varphi(y) * \varphi(x))] \\
&= \varphi[[\varphi(y) * (\varphi(y) * \varphi(x))] * \varphi(x)] \\
&= \varphi[[(\varphi(y) * \varphi(x)) * (\varphi(y) * \varphi(x))]] \\
&= \varphi(0) \\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
y * (x \vee y) &= \varphi(\varphi(x)) * \varphi[\varphi(y) * (\varphi(y) * \varphi(x))] \\
&= \varphi[[\varphi(y) * (\varphi(y) * \varphi(x))] * \varphi(y)] \\
&= \varphi[(\varphi(y) * \varphi(y)) * (\varphi(y) * \varphi(x))] \\
&= \varphi(0) \\
&= 1
\end{aligned}
$$

□

Conclusion

Whether such functions exists or not depends on the special properties of the d-algebras. BCK-algebras have the partial order structure, but d-algebras have no such a structure and so we need to seek another conditions for obtaining the analog of structures in d-algebras. This kind of functional approach can be connected with mirror d-algebras discussed in [3] in a new direction.

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