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h-STABILITY AND BOUNDEDNESS IN FUNCTIONAL PERTURBED DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate h-stability and boundedness for solutions of the functional perturbed differential systems using the notion of t_∞-similarity.

1. Introduction and Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, consider the functional perturbed differential systems of (1.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds + h(t, y(t), Ty(t)),\ y(t_0) = y_0, \tag{1.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator .

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t),\ v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t),\ z(t_0) = z_0. \tag{1.4}$$

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The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We recall some notions of h-stability [16].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 1.2. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called (hS) *h-stable* if there exists $\delta > 0$ such that (1.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Integral inequalities play a vital role in the study of boundedness and other qualitative properties of solutions of differential equations. In particular, Bihari's integral inequality continuous to be an effective tool to study sophisticated problems such as stability, boundedness, and uniqueness of solutions. The behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are three useful methods for showing the qualitative behavior of the solutions of perturbed nonlinear system : the use of integral inequalities, the method of variation of constants formula, and Lyapunov's second method.

The notion of h-stability (hS) was introduced by Pinto [15, 16] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Choi, Ryu [2] and Choi, Koo, and Ryu [3] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [7,8,9] and Goo et al. [11] investigated boundedness of solutions for nonlinear perturbed systems.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [5].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞*-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [5, 12].

In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of t_∞-similarity.

We give some related properties that we need in the sequal.

Lemma 1.4 ([16]). *The linear system*

$$x' = A(t)x, \;\; x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \;\; y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.5. *Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 1.6 ([2]). *If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.*

Theorem 1.7 ([3]). *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS, then the solution $z = 0$ of (1.4) is hS.*

Lemma 1.8 ([4]). (Bihari – type inequality) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \; t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big], \; t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 1.9 ([10]). *] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) u(\tau) d\tau ds.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) d\tau) ds\Big], \; t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) d\tau) ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 1.10 ([8]). *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) w(u(\tau)) d\tau ds.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) d\tau) ds\Big], \; t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

2. Main Results

In this section, we investigate hS and boundedness for solutions of the functional perturbed differential systems via t_∞-similarity.

Lemma 2.1. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C[\mathbb{R}^+, \mathbb{R}^+]$ and suppose that, for some $c \geq 0$ and $t \geq t_0$, we have*

(2.1)
$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \Big(\lambda_3(\tau)u(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(s)u(r)dr\Big)d\tau ds.$$

Then

$$u(t) \leq c \exp \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s)\Big(\int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau\Big)]ds, \ t \geq t_0. \tag{2.2}$$

Proof. Define a function $v(t)$ by the right member of (2.1). Then, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= \lambda_1(t)u(t) + \lambda_2(t)\Big(\int_{t_0}^{t} (\lambda_3(s)u(s) + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)u(\tau)d\tau)ds\Big) \\ &\leq \Big[\lambda_1(t) + \lambda_2(t)\Big(\int_{t_0}^{t} (\lambda_3(s) + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau)ds\Big)\Big]v(t), \ t \geq t_0, \end{aligned}$$

since $v(t)$ is nondecreasing and $u(t) \leq v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$v(t) \leq c \exp \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau\Big)ds. \tag{2.3}$$

Thus (2.3) yields the estimate (2.2). □

Theorem 2.2. *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1.1) is hS with the increasing function h, and g in (1.2) satisfies*

$$\Big|g(t, y(t))\Big| \leq a(t)|y(t)| + b(t) \int_{t_0}^{t} k(s)|y(s)|ds \tag{2.4}$$

and

$$|h(t, y(t), Ty(t))| \leq c(t)|y(t)|, \ t \geq t_0 \geq 0, \tag{2.5}$$

where $a, b, c, k, q \in C(\mathbb{R}^+)$, $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, $\int_{t_0}^{\infty} c(s)ds < \infty$, $\int_{t_0}^{\infty} k(s)ds < \infty$, $\int_{t_0}^{\infty} q(s)ds < \infty$, *and*

$$c = c_1 \exp\Big(c_2 \int_{t_0}^{\infty} [c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau]ds\Big) < \infty.$$

Then, any solution $y(t) = y(t, t_0, y_0)$ *of (1.2) is hS.*

Proof. Using the nonlinear variation of constants formula of Alekseev [1], any solution $y(t) = y(t, t_0, y_0)$ passing through (t_0, y_0) is given by

(2.6)

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))(\int_{t_0}^{s} g(\tau, y(\tau))d\tau + h(s, y(s), Ty(s)))ds.$$

By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. In view of Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.4),(2.5), and (2.6), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\
&\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(\int_{t_0}^{s} (a(\tau)|y(\tau)| \\
&\quad + b(\tau) \int_{t_0}^{\tau} k(r)|y(r)|dr)d\tau + c(s)|y(s)|\Big)ds \\
&\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(c(s)\frac{|y(s)|}{h(s)} \\
&\quad + \int_{t_0}^{s} (a(\tau)\frac{|y(\tau)|}{h(\tau)} + b(\tau) \int_{t_0}^{\tau} k(r)\frac{|y(r)|}{h(r)}dr)d\tau\Big)ds.
\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Now an application of Lemma 2.1 yields

$$\begin{aligned}
|y(t)| &\le c_1|y_0|h(t)\, h(t_0)^{-1} \exp\Big(c_2 \int_{t_0}^{t} [c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau]ds\Big) \\
&\le c|y_0|h(t)\, h(t_0)^{-1},
\end{aligned}$$

where $c = c_1 \exp\Big(c_2 \int_{t_0}^{\infty} [c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau]ds\Big)$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is hS, and so the proof is complete. □

Theorem 2.3. *Let* $a, b, c, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in* u *such that* $u \le w(u)$ *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$. *Suppose that* $f_x(t, 0)$ *is* t_∞*-similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \ge t_0 \ge 0$ *and* $|x_0| \le \delta$ *for some constant* $\delta > 0$, *the solution* $x = 0$ *of (1.1) is hS with the increasing function* h, *and* g *in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s))| ds \le a(t) w(|y(t))| + b(t) \int_{t_0}^{t} k(s) w(|y(s)|) ds,\ t \ge t_0 \ge 0, \tag{2.7}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)|y(t)|, \tag{2.8}$$

where $\int_{t_0}^{\infty} a(s) ds < \infty$, $\int_{t_0}^{\infty} b(s) ds < \infty$, $\int_{t_0}^{\infty} c(s) ds < \infty$, *and* $\int_{t_0}^{\infty} k(s) ds < \infty$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (1.2) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t) W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau) d\tau) ds\Big],\ t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} (a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau) d\tau) ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula (2.6), the hS condition of $x = 0$ of (1.1), (2.7), and (2.8), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| (\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))|) ds \\
&\le c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) h(s)^{-1} \Big(a(s) w(|y(s)|) + c(s)|y(s)| \\
&+ b(s) \int_{t_0}^{s} k(\tau) w(|y(\tau)|) d\tau\Big) ds \\
&\le c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) (c(s) \frac{|y(s)|}{h(s)} + a(s) w(\frac{|y(s)|}{h(s)})) ds \\
&+ \int_{t_0}^{t} c_2 h(t) b(s) \int_{t_0}^{s} k(\tau) w(\frac{|y(\tau)|}{h(\tau)}) d\tau ds.
\end{aligned}$$

Defining $u(t) = |y(t)||h(t)|^{-1}$, then, by Lemma 1.10, we have

$$|y(t)| \le h(t) W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau) d\tau) ds\Big].$$

Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. This completes the proof. □

Remark 2.4. Letting $c(t) = 0$ in Theorem 2.3, we obtain the same result as that of Theorem 3.2 in [7].

Theorem 2.5. *Let $a, b, c, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1.1) is hS with the increasing function h, and g in (1.2) satisfies*

$$\int_{t_0}^t |g(s,y(s))|ds \le a(t)|y(t)| + b(t)\int_{t_0}^t k(s)|y(s)|ds,\ t \ge t_0 \ge 0, \tag{2.9}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)w(|y(t)|), \tag{2.10}$$

where $\int_{t_0}^\infty a(s)ds < \infty$, $\int_{t_0}^\infty b(s)ds < \infty$, $\int_{t_0}^\infty c(s)ds < \infty$, and $\int_{t_0}^\infty k(s)ds < \infty$. Then, any solution $y(t) = y(t,t_0,y_0)$ of (1.2) is bounded on $[t_0,\infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t (a(s) + c(s) + b(s)\int_{t_0}^s k(\tau)d\tau)ds\Big],\ t_0 \le t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^t (a(s) + c(s) + b(s)\int_{t_0}^s k(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Applying Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.6), (2.9), and (2.10), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^t |\Phi(t,s,y(s))|(\int_{t_0}^s |g(\tau,y(\tau))|d\tau + |h(s,y(s),Ty(s))|)ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1}\Big(a(s)|y(s)| \\
&\quad + b(s)\int_{t_0}^s k(\tau)|y(\tau)|d\tau + c(s)w(|y(s)|)\Big)ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2h(t)(a(s)\frac{|y(s)|}{h(s)} + c(s)w(\frac{|y(s)|}{h(s)}))ds \\
&\quad + \int_{t_0}^t c_2h(t)b(s)\int_{t_0}^s k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau ds.
\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 1.9, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t (a(s) + c(s) + b(s)\int_{t_0}^s k(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\, h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. Hence, the proof is complete. □

Remark 2.6. Letting $c(t) = 0$ and $w(u) = u$ in Theorem 2.5, we obtain the same result as that of Theorem 3.1 in [6].

Lemma 2.7. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s)\int_{t_0}^s \lambda_3(\tau)u(\tau)d\tau ds \\ &+ \int_{t_0}^t \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)w(u(\tau))d\tau ds. \end{aligned} \tag{2.11}$$

Then

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) &+ \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau)d\tau \\ &+ \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)d\tau)ds\Big],\ t_0 \le t < b_1, \end{aligned} \tag{2.12}$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) &+ \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau)d\tau \\ &+ \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

Proof. Define a function $v(t)$ by the right member of (2.11). Then

$$v'(t) = \lambda_1(t)u(t) + \lambda_2(t)\int_{t_0}^t \lambda_3(s)u(s)ds + \lambda_4(t)\int_{t_0}^t \lambda_5(s)w(u(s))ds,$$

which implies

$$v'(t) \le \Big[\lambda_1(t) + \lambda_2(t)\int_{t_0}^t \lambda_3(s)ds + \lambda_4(t)\int_{t_0}^t \lambda_5(s)ds\Big]w(v(t))$$

since v and w are nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$v(t) \le c + \int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau)d\tau + \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)d\tau\Big)w(v(s))ds. \tag{2.13}$$

Then, by the well-known Bihari-type inequality, (2.13) yields the estimate (2.12). □

Theorem 2.8. *Let $a, b, c, k \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1.1) is hS with the increasing function h, and g in (1.2) satisfies*

$$\left|g(t, y(t))\right| \le a(t)|y(t)| \tag{2.14}$$

and

$$|h(t, y(t), Ty(t))| \le b(t)|y(t)| + c(t)\int_{t_0}^{t} k(\tau)w(|y(\tau)|)d\tau, \tag{2.15}$$

where $\int_{t_0}^\infty a(s)ds < \infty$, $\int_{t_0}^\infty b(s)ds < \infty$, $\int_{t_0}^\infty c(s)ds < \infty$, and $\int_{t_0}^\infty k(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(b(s) + \int_{t_0}^{s} a(\tau)d\tau + c(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big]$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(b(s) + \int_{t_0}^{s} a(\tau)d\tau + c(s)\int_{t_0}^{s} k(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula (2.6), the hS condition of $x = 0$ of (1.1), (2.14), and (2.15), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(\int_{t_0}^{s} a(\tau)|y(\tau)|d\tau \\
&+ b(s)|y(s)| + c(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau\Big)ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(b(s)\frac{|y(s)|}{h(s)} \\
&+ \int_{t_0}^{s} a(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau + c(s)\int_{t_0}^{s} k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.
\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, it follows from Lemma 2.7 that we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(b(s) + \int_{t_0}^{s} a(\tau)d\tau)ds + c(s)\int_{t_0}^{s} k(\tau)d\tau\Big],$$

where $c = c_1|y_0|\, h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. □

Lemma 2.9. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)w(u(\tau))d\tau ds + \int_{t_0}^{t} \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)w(u(\tau))d\tau ds. \tag{2.16}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau)ds\Big], \tag{2.17}$$

$t_0 \le t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Define a function $v(t)$ by the right member of (2.16) . Then

$$v'(t) = \lambda_1(t)u(t) + \lambda_2(t) \int_{t_0}^{t} \lambda_3(s)w(u(s))ds + \lambda_4(t) \int_{t_0}^{t} \lambda_5(s)w(u(s))ds,$$

which implies

$$v'(t) \le \Big[\lambda_1(t) + \lambda_2(t) \int_{t_0}^{t} \lambda_3(s)ds + \lambda_4(t) \int_{t_0}^{t} \lambda_5(s)ds\Big] w(v(t)),$$

since v and w are nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$v(t) \le c + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau\Big) w(v(s))ds. \tag{2.18}$$

Then, by the well-known Bihari-type inequality, (2.18) yields the estimate (2.17). □

Theorem 2.10. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1.1) is hS with the increasing function h, and g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \le a(t)|y(t)| + b(t) \int_{t_0}^{t} k(s)w(|y(s)|)ds, \tag{2.19}$$

and

$$(2.20)\qquad |h(t,y(t),Ty(t))| \le c(t)(|y(t)|+|Ty(t)|), |Ty(t)| \le \int_{t_0}^{t} q(s)w(|y(s)|)ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, $\int_{t_0}^{\infty} c(s)ds < \infty$, $\int_{t_0}^{\infty} k(s)ds < \infty$, *and* $\int_{t_0}^{\infty} q(s)ds < \infty$. *Then, any solution* $y = 0$ *of (1.2) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau ds + c(s)\int_{t_0}^{s}q(\tau)d\tau ds\Big],$$

$t_0 \le t < b_1$, *where* $c = c_1|y_0|\,h(t_0)^{-1}$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(a(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau ds \\ + c(s)\int_{t_0}^{s}q(\tau)d\tau ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula (2.6), the hS condition of $x = 0$ of (1.1), (2.19), and (2.20), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t}|\Phi(t,s,y(s))|(\int_{t_0}^{s}|g(\tau,y(\tau))|d\tau + |h(s,y(s),Ty(s))|)ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)h(s)^{-1}\Big((a(s)+c(s))|y(s)| \\
&\quad + b(s)\int_{t_0}^{s}k(\tau)w(|y(\tau)|)d\tau + c(s)\int_{t_0}^{s}q(\tau)w(|y(\tau)|)d\tau\Big)ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)(a(s)+c(s))\frac{|y(s)|}{h(s)}ds \\
&\quad + \int_{t_0}^{t}c_2h(t)b(s)\int_{t_0}^{s}k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau ds \\
&\quad + \int_{t_0}^{t}c_2h(t)c(s)\int_{t_0}^{s}q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau ds.
\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$ with $c = c|y_0|\,h(t_0)^{-1}$. Then, an application of Lemma 2.9 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s)+c(s)$$
$$+ b(s)\int_{t_0}^{s}k(\tau)d\tau ds + c(s)\int_{t_0}^{s}q(\tau)d\tau ds\Big],$$

where $t_0 \le t < b_1$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

Remark 2.11. Letting $c(t) = 0$ and $b(t) = a(t)$ in Theorem 2.10, we obtain the similar result as that of Theorem 3.3 in [11].

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